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# A monolithic FE formulation for the analysis of membranes in fluids

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**ABSTRACT:** We propose here an efficient approach for treating the interaction between membranes and fluids. Slight compressibility of the fluid is assumed. Classical total Lagrangian formulation including wrinkling is adopted for the membrane representation, whereas fluid is treated in an updated Lagrangian manner, developed in the current work. Assumption of slight compressibility of the fluid enables one to define the monolithic fluid-membrane system in a natural way. The displacements are the primary variables of both the fluid and the membrane domains. The formulation adopts the Particle Finite Element Method (PFEM) philosophy for free-surface identification and mesh regeneration. Three examples illustrate the functionality of the formulation in application to FSI problems involving motion of membranes in water.

**Key Words:** Membranes, fluids, particle finite element method (PFEM), wrinkles, floating ball.

## 1. THE PARTICLE FINITE ELEMENT METHOD CONCEPT

In recent years fluid-structure interaction (FSI) has become one of the 'hot-spots' in the field of computational mechanics. One of the techniques developed by Oñate, Idelsohn et al. [1], that has proven its competitiveness is the so-called Particle Finite Element method. The Particle Finite Element method adopts lagrangian framework for the descritption of the fluid, where the mesh nodes are treated as particles that can freely move and even separate from the main fluid domain. The main idea of the PFEM is that the variables of interest are stored at the nodes instead of Gauss points. This results in a hybrid between standard FEM and meshless methods. The finite element mesh is created at every time step of the dynamic problem and the solution is afterwards stored at the nodes. The nodes move according to their velocity obtaining their new position and then the finite element mesh is recreated using Delaunay triangulation (i.e. connectivities are updated in the best possible manner). It is important to keep in mind that the convective terms of the momentum equation disappear in the Lagrangian description, and thus the numerical problems arising for convection dominated flows disappear. The PFEM has especially proven its applicability and robustness in treating FSI problems involving free surface ows. The boundary at every solution step is determined using the so-called 'alpha-shape' technique, which can be seen as a geometrical criterion, that enables one to decide wether or not an element needs to be preserved. Highly distorted elements are thus eliminated by a geometrical filter.

Current work adopts the basic concepts of the PFEM and the reader is re-ferred to [1] for the detailed description of the method and [2] for its validation.

Here we propose an extension of the PFEM which consists in modeling the incompressible fluid as the limiting case of the compressible formulation. This is often used in conjunction with explicit SPH methods, but is utilized here in an implicit framework. Introduction of the heuristic constitutive pressure-displacement relation permits the definition of the monolithic FSI systems formulated in terms of displacements only. This enables effective treatement of the FSI problems involving large deformations,

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where usual partitioned approaches suffer convergence problems. The layout of the paper is the following: first the advantages of monolithic approach in application to fluid-membrane interactions are discussed, then the membrane and the fluid formulations are presented and the coupling issues are addressed, finally three examples are provided to illustrate the functionality of the formulation.

## 2. MONOLITHIC VS. PARTITIONED

Generally there exist two ways of treating FSI-problems - the partitioned approach and the monolithic one. Partitioned approaches rely on the independent solution of the fluid and the structural domain so that the best available solvers for each sub-problem can be chosen. The data is interchanged between the subdomains across the interface boundary. Though this approach is computationally cheaper (per iteration) than the monolithic one, it suffers from stability problems for light weight structures, or to be precise once the ratio between the density of the fluid and the structure approaches the unity. In the monolithic approach the whole fluid-structure domain is discretized at once and the total system is built and solved at every time step. The monolithic solution is the correct one for the original problem, and thus no additional modeling error is introduced. Therefore it provides the same "guarantee" as the standard FE. Nevertheless, the discrete equation system describing the problem is larger than the subsystems of the partitioned approach. Another complication that can arise in the monolithic approach in FSI comes from the fact that the variables describing the fluid and the solid are of different nature. Even if a possibility exists to define a monolithic system in terms of velocities (or displacements) and pressure, this would result generally in badly conditioned system matrices. In the present work we strive to derive a formulation with all the advantages of the monolithic approach, but leading to a well-conditioned global system.

In order to describe the fluid in terms of displacements without using the pressure as a primary variable, one needs to allow a slight compressibility. This enables one to relate the fluid pressure to the displacements via a constitutive law. In fact many engineering fluids, such as water, are compressible, but their compressibility coefficient, or the so-called "bulk-modulus" $K$ is very high ~ $1E9N/m^2$. Nevertheless, water can be modelled in a realistic way with a smaller bulk modulus, as long as the associated volume variation is admissible, and the speed of wave propogation in the medium $c = \sqrt{\frac{K}{\rho}}$, where $\rho$ is the fluid density and $c$ is the wave speed), is large enough to represent desired phenomena. So, our objective is to derive a compressible formulation that will allow us to set up the global system in displacements only, but to be able to represent the quasi-incompressible fluid behavior by the choice of a large bulk modulus. We propose to condense pressure at the level of element, and thus to obtain the purely displacement formulation. This is a computationally efficient procedure, and the only limitation is the value of the compressibility constant. In the current formulation, the sensibility to the volumetric locking is diminished by a pressure smoothing procedure. This allows us to use "acceptable" values for the bulk modulus. Note however that the physical value of the bulk modulus for water still leads to locking.

## 3. MEMBRANE FORMULATION AND WRINKLING

A membrane is a 2D solid that lives in a 3D environment. Due to its low thickness it lacks bending stiffness and can withstand normal loads only by adjusting its shape. From the numerical point of view, this results in an intrinsic geometrical non-linearity, that has to be taken into account in the FE model. In the current work a standard total Lagrangian formulation is used. The detailed description of the membrane element used can be found in [3] together with the validation of the model.

Dou to their very low flexural stiffness, membranes easily undergo local buckling phenomena leading to "wrinkles" in the presence of compressive stresses. Such a situation makes appealing the possibility of enriching the element to introduce wrinkling effects in the formulation. Large effort was spent in developing simplified wrinkling models over the past decade.

In his landmark paper Roddeman [4], [5] observed how the formation of wrinkles could be introduced through an additional inelastic deformation gradient describing the contraction of the average plane in presence of compressive stresses. The additional term leads to the introduction of an inelastic strain component in the FE model [6]. Other more general wrinkling models were developed in recent years. A simple explicit wrinkling model was presented in [7]. In our work a simple implicit wrinkling model is used. The model features a consistent linearization and is suitable for the isotropic case. The entire derivation is provided in [3]. Only the basic concepts are discussed here.

When the membrane is subjected to a biaxial tension state, its behavior is purely elastic. This state is

known as taut. The situation in which the membrane is subjected to multi-axial compression is known as slack. In between, when tension remains in one direction but the membrane buckles in the other, the state is known as wrinkled.

Over the years different methods were proposed to assess the state of a membrane subjected to a given deformation. Early methods were based either on the evaluation of the principal stresses or on the assessment of the principal strains. Here a mixed method is used, based both on stresses and strains, as described in [7] and [8]. By considering the principal elastic stresses (PK2) S , S and the principal strains $E_I$, $E_{II}$ the assessment is as follows

- $S_{II} > 0 \rightarrow$ "**TAUT**"
- $S_{II} < 0$ & $E_I > 0 \rightarrow$ "**WRINKLED**"
- else "**SLACK**"

The stresses to be used in the assessment are the ones obtained through the elastic constitutive law from the given strains.

As stated before wrinkling effect can be included by introducing a further strain $E_w$ which describes the behavior of the membrane outside of the taut domain.

The detailed derivation of the implicit wrinkling model can be found in [3]. The fundamental concepts are explained next. The elastic strain vector coming directly from the FE discretization assumes a diagonal form in a base identified by two vectors $\mathbf{v}_I^{\epsilon}$, $\mathbf{v}_{II}^{\epsilon}$ . The corresponding elastic stress can be diagonalized, in a base that is generally different from the one of the strain, identified by vectors $\mathbf{v}_I^s$, $\mathbf{v}_{II}^s$. When the constitutive law exhibits an isotropic behavior, the principal direction of strains and stresses coincide with $\mathbf{v}_I$, $\mathbf{v}_{II}$. Wrinkling effects can be taken into account by considering a modified constitutive tensor $\mathbf{C}_{mod}$ which relates the "wrinkled" stress tensor $\tilde{\mathbf{S}}$ and the total strain tensor $\mathbf{E}$ as $\tilde{\mathbf{S}} = \mathbf{C}_{mod} : \mathbf{E}$. This can be achieved by defining

$$\mathbf{C}_{mod} = \Bigg( \mathbf{C} - \mathbf{C} : (\mathbf{v_{II}} \otimes \mathbf{v_{II}}) \otimes \left( \frac{(\mathbf{v_{II}} \otimes \mathbf{v_{II}}) : \mathbf{C}}{(\mathbf{v_{II}} \otimes \mathbf{v_{II}}) : \mathbf{C} : (\mathbf{v_{II}} \otimes \mathbf{v_{II}})} \right) \Bigg) \tag{1}$$

Or in practice

$$\mathbf{C}_{mod} = E(\mathbf{v_I} \otimes \mathbf{v_I}) \otimes (\mathbf{v_I} \otimes \mathbf{v_I}) + \frac{E}{2(1+\nu)} (\mathbf{v_I} \otimes \mathbf{v_{II}} + \mathbf{v_{II}} \otimes \mathbf{v_I}) \tag{2}$$

where $E$ is the Young's modulus and $\mathbf{v}$ - the Poisson's ratio.

In order to guarantee quadratic convergence, we need to perform consistent linearization. The resulting algorithmic tangent operator in Voigt format reads

$$[\mathbf{C_{tan}}] = [\mathbf{C_{mod}}] + EE_I \, [\mathbf{A}^*] \, [\mathbf{B}] \tag{3}$$

where A and B are defined as

$$[\mathbf{A}^*] := \begin{pmatrix} 2v_{I1} & 0 \\ 0 & 2v_{I2} \\ v_{I2} & v_{I1} \end{pmatrix}$$

and

$$[\mathbf{B}] := \frac{1}{E_{11}^2 - 2E_{11}E_{22} + E_{22}^2 + E_{12}^2} \begin{pmatrix} v_{I2}E_{12} & -v_{I2}E_{12} & -v_{I2}(E_{11} - E_{22}) \\ -v_{I1}E_{12} & v_{I1}E_{12} & v_{I1}(E_{11} - E_{22}) \end{pmatrix}$$

where $v_{Ii}$ stands for the $i$ -spatial component of the vector $\mathbf{v}_I$ . Such an operator can then be used to form the material part of the stiffness matrix for the membrane element [3].

## 4. COMPRESSIBLE FLUID FORMULATION

We start by deriving the FE equations of the slightly compressible fluid by discretization of the momentum equations. In the Lagrangian framework it can be written as

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \sigma + \rho \mathbf{b} \tag{4}$$

where $\rho$ is the fluid density, $\sigma$ - the Cauchy stress tensor and $\mathbf{b}$ - the body force, $p$ is the fluid pressure. Instead of the incompressibility condition $\nabla \cdot v = 0$ we use the modified mass conservation equation:

$$\delta p = K \frac{\delta V}{V} = K \delta t \nabla \cdot \mathbf{v} \tag{5}$$

where $V$ and $\delta V$ are the volume and differential volume change respectively, $K$ is the bulk modulus, v is the velocity and $t$ - the time. Splitting the total stress in the fluid into its volumetric and deviatoric parts reads $\sigma = -p\mathbf{I} + 2\mu \, \nabla^s \mathbf{v}$, where $\nabla^s \mathbf{v}$ is the symmetric velocity gradient tensor, $I$ - the identity tensor, and $\mu$ - the dynamic viscosity. Substitution into the momentum equation results in

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla\cdot(-p\mathbf{I} + 2\mu\,\nabla^s\mathbf{v}) + \rho\mathbf{b} \tag{6}$$

### 4.1. Discretization of the momentum equations

The momentum equation discretizied in space reads

$$\mathbf{M}\dot{\mathbf{v}} = -\mathbf{f}_{vol}^{int} + \mathbf{A}\mathbf{v} + \mathbf{f}^{\mathbf{ext}} \tag{7}$$

where

$$\mathbf{M} = \int_\Omega \rho\boldsymbol{N}^T\boldsymbol{N}d\Omega \tag{8}$$

$$\mathbf{A} = \int_\Omega \boldsymbol{B}^T\boldsymbol{C}_\mu\boldsymbol{B}\,d\Omega \tag{9}$$

$$\mathbf{f}_{vol}^{int} = \int_\Omega \boldsymbol{B}^T\,p\,d\Omega \tag{10}$$

$$\mathbf{f}^{ext} = \int_\Omega \rho\boldsymbol{N}\mathbf{b}d\Omega + \int_{\Gamma_t}\boldsymbol{N}\mathbf{t}d\Gamma \tag{11}$$

$C_\mu$ is the viscous part of the constitutive tensor

$$\mathbf{C}_\mu = \mu\begin{pmatrix} 4/3 & -2/3 & -2/3 & 0 & 0 & 0\\ -2/3 & 4/3 & -2/3 & 0 & 0 & 0\\ -2/3 & -2/3 & 4/3 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

where $\Gamma_t$ is the Neumann boundary, **t** - is the surface traction and **b** is the body force and **B** - is the standard linear strain-displacement matrix. Note that since the equations are written in the current configuration, matrix **B** depends on the unknown position **x** and thus Eq. 7 defines a geometrically non-linear problem. On the other hand, it is important to note that in contrast to solid mechanics the relationship between the strains and the primary variable (velocity) is linear even for large deformations of the fluid domain. In order to write the equations in terms of displacements instead of velocities, we substitute the time derivatives of the displacements in a symbolic way and obtain the following form of the momentum equation:

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{f}_{vol}^{int} + \mathbf{A}\dot{\mathbf{x}} + \mathbf{f}^{\mathbf{ext}} \tag{13}$$

Time integration of the Eq.7 is performed with the Newmark scheme, where the finite difference formulae for displacments and velocities respectively are as follows ($h$ denotes the time step, i.e. $h = t_{n+1} - tn$):

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\dot{\mathbf{x}}_\mathbf{n} + h^2(\frac{1-2\beta}{2}\ddot{\mathbf{x}}_\mathbf{n} + \beta\ddot{\mathbf{x}}_{\mathbf{n+1}}) \tag{14}$$

$$\dot{\mathbf{x}}_{\mathbf{n+1}} = \ddot{\mathbf{x}}_\mathbf{n} + h((1-\gamma)\ddot{\mathbf{x}}_\mathbf{n} + \gamma\ddot{\mathbf{x}}_{\mathbf{n+1}}) \tag{15}$$

From above equations we obtain the expressions for the velocities and accelerations in terms of the current positions $\mathbf{x}_{n+1}$ (that are unknown) and the velocities and accelerations of the previous time step, $v_n = \dot{\mathbf{x}}_n$ and $a_n = \ddot{\mathbf{x}}n$ respectively :

$$\ddot{\mathbf{x}}_{\mathbf{n+1}} = \frac{\mathbf{x}_{n+1}-\mathbf{x}_n}{h^2\beta} - \frac{1}{h\beta}\dot{\mathbf{x}}_\mathbf{n} - \frac{1}{2\beta}\ddot{\mathbf{x}}_\mathbf{n} \tag{16}$$

$$\begin{aligned}\dot{\mathbf{x}}_{\mathbf{n+1}} &= \gamma\frac{\mathbf{x}_{n+1}-\mathbf{x}_n}{h\beta}\\ &+ (1-\frac{\gamma}{\beta})\dot{\mathbf{x}}_\mathbf{n} + (h(1-\gamma) - (\frac{h\gamma}{2\beta} - h\gamma))\ddot{\mathbf{x}}_\mathbf{n}\end{aligned} \tag{17}$$

Using the above formulae for the time derivatives and applying the standard Newton-Raphson scheme for the linearization of the geometrically non-linear problem, one arrives to the following definitions of dynamic residual **RHS**

$$\mathbf{RHS} = \mathbf{f_{ext}} - \mathbf{f_{int}} \tag{18}$$

Pressure in the residual is calculated from the elemental volume changes using the discrete form of Eq.5 and the tangent stiffness matrix$\mathbf{H}_{\text{dyn}}$:

$$\mathbf{H}_{dyn} = \frac{1}{\beta h^2}\mathbf{M} + \frac{\gamma}{\beta h}\mathbf{A} + \mathbf{K_p} \tag{19}$$

where

$$\mathbf{K_p} = \int_\Omega \boldsymbol{B}^T C_{vol}\boldsymbol{B}\,d\Omega \tag{20}$$

and

$$\mathbf{C_{vol}} = \begin{pmatrix} K & K & K & 0 & 0 & 0\\ K & K & K & 0 & 0 & 0\\ K & K & K & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

where $\mathbf{K}_\text{p}$ is the linearization of the pressure term $\mathbf{f}_{vol}^{int} = \int_\Omega \boldsymbol{B}^T p\,d\Omega$ of the fluid and $\mathbf{C}_\text{vol}$ is the volumetric constitutive tensor.

### 4.2. Coupling with the structure and solution algorithm

On the structural side, the dynamic tangent stiffness is described by a relation similar to Eq. 19 in the form:

$$\mathbf{H}_{dyn} = \frac{1}{\beta h^2}\mathbf{M} + \mathbf{K}_{tan} \tag{22}$$

where $\mathbf{K}_{tan}$ is the tangent stiffness matrix of the membrane. This matrix and the residual terms for the membrane are calculated in the standard manner as described in [3] or [7]. The residuals represent the unbalanced forces at the nodes. A standard FE assembly process runs over all the fluid and structural elements and leads naturally to a monolithic formulation. The solution of the resulting system guarantees the equilibrium of forces at the level of each node in the same way as in a single-domain problem. In this approach the pressure is calculated at the level of Gauss points from the consitutive equations in the standard manner. To allow fast remeshing, a continuum pressure distribution at the nodes is computed as

$$\mathbf{M_l p_{n+1}} = \mathbf{M_{cons} p_n} + \mathbf{M_l \Delta p} \tag{23}$$

where $M_l$ and $M_{cons}$ are the lumped and consistent "pressure" mass matrices, $p_n$ and $p_{n+1}$ - the nodal pressures at time steps $n$ and $n + 1$ respectively. The pressure increment is computed from Eq. 5, by the variation of volume around each node. The impact of this procedure will not be discussed here in detail. Nevertheless we would like to remark that this procedure acts as a rudimental pressure stabilization due to the difference between the consistent and lumped mass matrices. Consequently it affects beneficially the sensitivity to locking. The entire solution strategy can be summarized as follows:

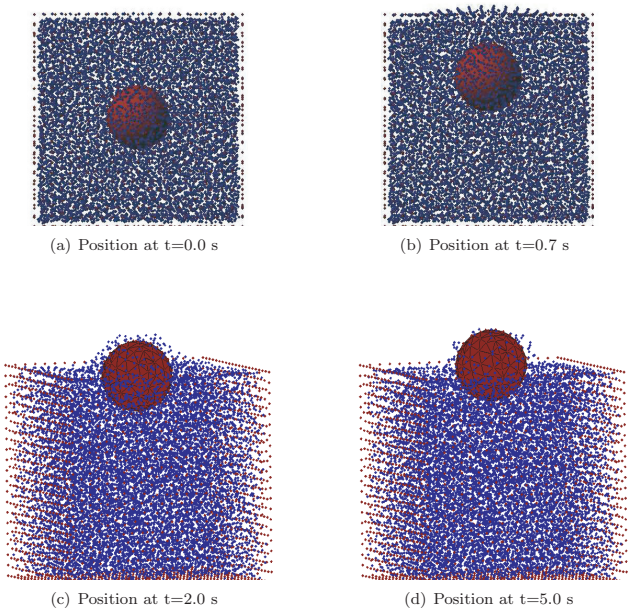
- assemble the monolithic system
- start the non-linear solution loop
- solve the momentum equations for displacements
- update the position of mesh nodes
- recover the pressures for the fluid elements
- move the nodes
- remesh

## 5. EXAMPLES

In previous sections, the fundamentals of the fluid and membrane formulations used and the coupling strategy were presented. In this section we illustrate the functionality of the proposed monolithic method in application to several test-cases.

### 5.1. Floating ball

First, the motion of the inflatable within a fluid volume is obtained. A ball of 0.15 m radius is immersed in the middle of the cube with the edge length of 1m. The ball is modelled as a membrane with 0.003 m thickness, Poisson's ratio of 0.1, Young's modulus of 0.81 MP*a*, and density of 800 $kg/m^3$. The internal pressure of 15 *KPa* is applied to the sphere. The total fluid-membrane domain is meshed with 80.000 linear tetrahedra. Fig. 1 illustrates the ball motion. Initially the ball expands, until the equilibrium with the external pressure is achieved and then moves up towards to the surface. First two figures illustrate a 2D cut and the last two show a 3D view.



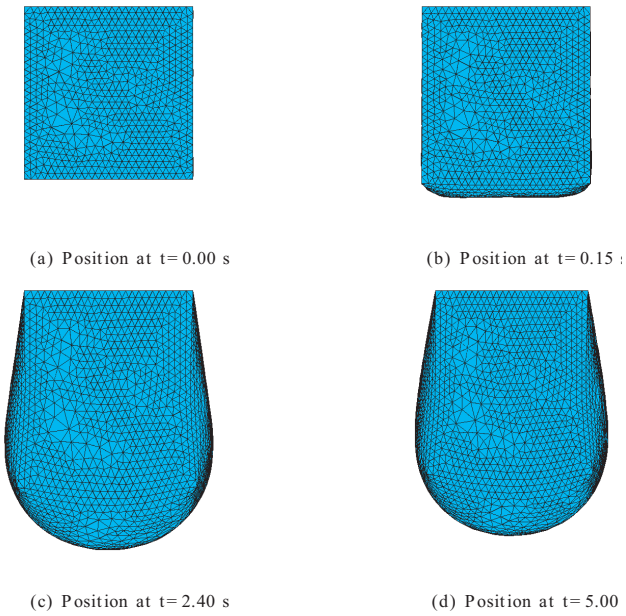
(a) Position at t=0.0 s (b) Position at t=0.7 s

(c) Position at t=2.0 s (d) Position at t=5.0 s

Figure 1. Motion of an in ated ball in water

### 5.2. Membrane cube filled with water

In this example a membrane initially having cubical chape (edge size of 1m) is loaded with water subjected to gravity. Membrane properties were chosen as follows: 4 mm thickness, density of 1800 *kg/m*), Young's modulus 2100000 Pa, and the Poisson's ratio of 0.2. On Fig.2 the deformed membrane at four different time instances is shown. The analysis was performed on two different meshes. The coarser discretization had 5300 tetrahedral fluid elements, and 3500 triangular membrane elements. The finer discretization consisted of 32.000 tetrahedra and 13000 triangles for the fluid and the membrane respectively. Fig. 3 depicts the displacement of the membrane measured at the middle point of the lower face of the cube. The curves exhibit good agreement.



(a) Position at t= 0.00 s (b) Position at t= 0.15 s

(c) Position at t= 2.40 s (d) Position at t= 5.00

Figure 2. Deformation of a membrane filled with water

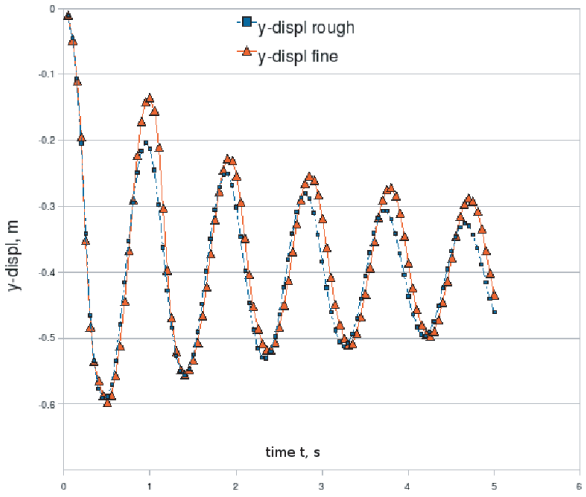
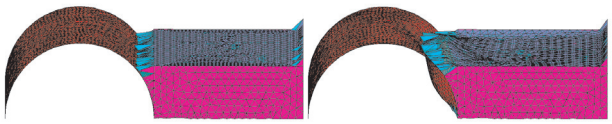



Figure 3. Y-displacement of the lower edge center obtained with two different meshes

### 5.3. Membrane dam

Much interest exist in in atable structures for water contention. A practical application of membranes is the construction of in atable dams. An analysis of the interaction between the dam and water reservoir is presented below. We recall that the example is purely academic, i.e. the dimensions do not correspond to any real structure. Nevertheless it shows the potential of the PFEM for this kind of problem. Here a tube (modelled as a membrane of 2 mm thickness) is in ated with internal pressure of 100Pa and subjected to the water load. The radius of the tube is 0.25m and the water level is 0.2 m. Fig. 4 shows the original and deformed shapes of the membrane.



(a) Position at t= 0.0 s (b) Position at t= 0.4 s

Figure 4. Interaction between an in atable dam and a water reservoir

## 6. CONLUSIONS

The proposed methodology allows treating strongly coupled fluid-membrane systems. In general the fluid formulation described here permits a natural definition of a monolithic system with any structural element having displacements as a primary variable. It is especially advantageous in application to problems where free surface ows are involved. The pressure smoothing procedure leads to a reduction of locking phenomena.

## REFERENCES

[1] Oñate E., Idelsohn S., Del Pin F., and Aubri R. The particle finite element method. an overview. *International Journal of Computational Methods*, 1:267–307, 2004.

[2] Larese A., Rossi R., Oñate E., and Idelsohn S.R. Validation of the particle finite element method (pfem) for simulation of free surface ows. *In press: Engineering Computations*, (2008).

[3] Rossi R. *Light weight Structures:Structural Analysis and Coupling Issues*. PhD thesis, University of Padova, 2005.

[4] Roddeman D.G. and Drukker J. et al. The wrinkling of thin membranes: Part 1 - theory. *Journal of Applied Mechanics*, 54:884–887, 1987.

[5] Roddeman D.G. and Drukker J. et al. The wrinkling of thin membranes: Part 2 - numerical analysis. *Journal of Applied Mechanics*, 54:888–892, 1987.

[6] Accorsi M, Lu K., and Leonard J. finite element analysis of membrane wrinkling. *IJNME*, 50:1017–1038, 2001.

[7] Rossi R., Lazzari M., Vitaliani R., and Oñate E. Simulation of light-weight membrane structures by wrinkling model. *International Journal For Numerical Methods In Engineering*, 62:2127–2153, 2005.

[8] Rossi R. A finite element formulation for 3d membrane structures including a wrinkling modified material model. Technical Report 226, CIMNE, 2003.