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Space and time error estimates for a first order, pressure stabilized finite element method for the incompressible Navier–Stokes equations

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Abstract

In this paper we analyze a pressure stabilized, finite element method for the unsteady, incompressible Navier–Stokes equations in primitive variables; for the time discretization we focus on a fully implicit, monolithic scheme. We provide some error estimates for the fully discrete solution which show that the velocity is first order accurate in the time step and attains optimal order accuracy in the mesh size for the given spatial interpolation, both in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$; the pressure solution is shown to be order $\frac{1}{2}$ accurate in the time step and also optimal in the mesh size. These estimates are proved assuming only a weak compatibility condition on the approximating spaces of velocity and pressure, which is satisfied by equal order interpolations. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this paper is to provide some error estimates for a pressure stabilized, finite element method for the numerical solution of the unsteady, incompressible Navier–Stokes equations in the primitive variables velocity and pressure. The method was introduced in [7] as an extension to the transient case of a technique initially developed for the Stokes problem [5] and then extended to the steady, incompressible Navier–Stokes equations [6].

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The stabilization of the pressure in incompressible flow problems has received much attention in the last decades. Numerical schemes have been developed which bypass the need for the approximating spaces of velocity and pressure to satisfy the compatibility condition met when using standard Galerkin methods. Stabilized formulations were first introduced under the idea of *Petrov–Galerkin* methods [17], which then led to *Galerkin Least Squares* (GLS) techniques. These were first developed in the context of advection–diffusion equations [18], and then extended to the linearized, steady incompressible Navier–Stokes equation in [9] (see also [10] and the references therein). More recently, the GLS technique has evolved into the idea of *subgrid-scale models* (see [4,16]). All these techniques have been analyzed in the literature for steady problems using arbitrary finite element interpolations. Error analysis both in space and time for stabilized formulations of transient problems have been given in [19], for advection–diffusion problems, and [13], for the incompressible Navier–Stokes equations. In this last reference, the analysis was based on the assumption that the time step δt is of the same order as the mesh size h: $\delta t \simeq h$. Moreover, it was restricted to the case of piecewise linear elements.

On the other hand, some combinations of finite element spaces which satisfy the discrete compatibility condition have been analyzed for the Stokes problem and proven to be stable and yield optimal order accuracy of the solution (see, for instance, [3,22]). Assuming a (mixed) finite element pair which satisfies the discrete compatibility condition, some analysis of methods for the unsteady problem have been given: Heywood and Rannacher [14,15] proved second order error estimates in the time step and optimal order accuracy in the mesh size for a mixed method using a Crank–Nicholson time integration scheme; Boukir et al. [1] also proved second order estimates in time and optimal order in space for a characteristic-based method under a stability restriction on the time step of the form $\delta t \leqslant h^{d/6}$, where d is the dimension of space; finally, Guermond and Quartapelle [12] analyzed the classical fractional-step projection method of A.J. Chorin and R. Temam in its incremental form, which yields a first order scheme which is also optimal in space (a second order method can also be developed). Their analysis is based on the satisfaction of the LBB condition, which has traditionally been considered unnecessary in projection methods based on a Poisson equation for the pressure. This condition can be avoided assuming $\delta t \geqslant h^{l+1}$, where l is the order of the spatial interpolation, in the stability analysis of the non-incremental form of the method, but not in the convergence one.

We analyze here a stabilized formulation of the unsteady problem which employs a finite element, *pressure gradient projection* technique [6] and a fully implicit, backward Euler scheme for the time integration. We show that first order accuracy in time is maintained in the fully discrete method, which attains optimal order accuracy in space for the given interpolation. The analysis is carried out assuming only a weak compatibility condition on the approximating spaces of velocity and pressure, which was proven to be satisfied by simplicial equal order finite element interpolations in [5]. The error estimates obtained are given in terms of a certain norm of the velocity in $L^2(\Omega)$ and $H_0^1(\Omega)$ and the pressure and its gradient in $L^2(\Omega)$. We first analyze the temporal error by considering a semidiscrete approximation of the problem, and then study the fully discrete method, with both a linearized and a nonlinear approximation of the convective term.

It has to be remarked that the purpose of the technique employed here is to stabilize the pressure solution; the instabilities due to the convective term at high cell Reynolds numbers are not addressed at with this formulation. Moreover, the interest here relies on showing how the technique that we use to stabilize the pressure, with respect to the spatial interpolation, can be analyzed in transient problems, regardless of the particular time integration method employed. We concentrate on a fully implicit, backward Euler scheme, which, although being only first order accurate, is unconditionally stable;

however, other methods could also be considered (see [7]). The resulting scheme is computationally feasible (see [7]), and also suitable as an iterative method to reach steady states.

Our presentation is split into two sections. In Section 2 we state the problem to solve, recall some known properties of its solution and introduce some notation; we then present the semidiscrete approximation considered and finally the fully discrete, stabilized finite element method. In Section 3 we state and prove our error estimates, first for the semidiscrete and then for the fully discrete problems. We first recall a stability estimate which was proven in [7] under weak assumptions on the continuous solution; then we prove some optimal order error estimates for the velocity, from which we obtain an improved stability estimate as a side product. We finally analyze the pressure solution, for which we also obtain optimal order error estimates.

2. Description of the method

2.1. Problem statement

The evolution of viscous, incompressible fluid flow in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is governed, in the primitive variable formulation, by the unsteady, incompressible Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \times (0, T) \tag{2}$$

on $\Omega \times (0, T)$ (with $T > 0$ a given final time), where $\boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^d$ is the fluid velocity at position $\boldsymbol{x} \in \Omega$ and time $t \in (0, T)$, $p(\boldsymbol{x}, t) \in \mathbb{R}$ is the fluid kinematic pressure, $\nu > 0$ is the kinematic viscosity, $\boldsymbol{f}(\boldsymbol{x}, t)$ is an external force, ∇ is the gradient operator, $\nabla\cdot$ is the divergence operator and Δ is the Laplacian operator (here, and in what follows, boldface characters denote vector quantities). Boundary conditions have to be given to complete the equation system (1)–(2). For the sake of simplicity, only homogeneous Dirichlet type boundary conditions are considered here:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times (0, T), \tag{3}$$

where $\Gamma = \partial\Omega$. An initial condition must also be specified for the velocity:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \Omega. \tag{4}$$

The treatment of the above equations of motion requires of the usual Sobolev spaces $H^m(\Omega)$, $m \geqslant 0$, consisting of functions with distributional derivatives up to order m belonging to $L^2(\Omega)$. The scalar product in $H^m(\Omega)$ is denoted by $(u, v)_m$ (the subscript m may be omitted when it equals 0) and its norm by $||u||_m$. The closed subspaces $H_0^1(\Omega)$, consisting of functions in $H^1(\Omega)$ with zero trace on Γ, and $L_0^2(\Omega)$, made up with functions in $L^2(\Omega)$ with zero mean on Ω, will also be needed. Also, let $H^{-1}(\Omega)$ denote the dual space of $H_0^1(\Omega)$, the duality between these two spaces being denoted by $\langle\, ,\, \rangle$, and let:

$$W = \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega) \mid \nabla \cdot \boldsymbol{u} = 0\}.$$

Assuming $\boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{f} \in L^2(0, T; \boldsymbol{H}^{-1}(\Omega))$, and if Ω is bounded and Lipschitz continuous, problem (1)–(4) has at least one solution $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^2(\Omega)) \cap L^2(0, T; \boldsymbol{H}_0^1(\Omega))$ (see [24]).

Uniqueness and more regularity of the solution can be achieved by assuming more regularity on $\boldsymbol{f}$, $\boldsymbol{u}_0$ and Ω. In particular, we assume hereafter that the continuous solution $(\boldsymbol{u}, p)$ of (1)–(4) is unique and satisfies:

(R1) $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}^2(\Omega)) \cap C^0(0, T; W),\ p \in L^\infty(0, T; H^1(\Omega)) \cap C^0(0, T; L^2_0(\Omega))$,
(R2) $\boldsymbol{u}_t \in L^2(0, T; \boldsymbol{L}^2(\Omega))$,
(R3) $\int_0^T t\|\boldsymbol{u}_{tt}(t)\|^2_{-1}\,\mathrm{d}t \leqslant C$,
(R4) $\int_0^T \|\boldsymbol{u}_{tt}(t)\|^2_{W'}\,\mathrm{d}t \leqslant C$.

The subscript t is employed hereafter for $\partial/\partial t$, and we use C as a generic constant depending of $\boldsymbol{f}$, $\boldsymbol{u}_0$, Ω and ν, but not on the time step δt nor on the mesh size h; also, W' is the dual space of W. Sufficient conditions for (R1)–(R3) to hold can be found [14]; for (R4), see [20,21]. In particular, it is required that $\boldsymbol{f} \in L^2(0, T; \boldsymbol{L}^2(\Omega))$, which we assume from now on.

Let us call $V = \boldsymbol{H}^1_0(\Omega)$ and $Q = L^2_0(\Omega)$. In what follows the following notation will be used for the weak form of the different terms in Eqs. (1)–(2):

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}), && \boldsymbol{u}, \boldsymbol{v} \in V,\\ b(q, \boldsymbol{v}) &= -(q, \nabla\cdot\boldsymbol{v}), && \boldsymbol{v} \in V,\ q \in Q,\\ c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}\big) + \tfrac{1}{2}\big((\nabla\cdot\boldsymbol{u})\boldsymbol{v}, \boldsymbol{w}\big), && \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \end{aligned}$$

All these forms are continuous on the specified spaces, and the expression taken for the trilinear form c arising from the convective term in (1) is skew-symmetric in its last two arguments (see [24]); under the incompressibility condition (2), this expression is equivalent to that obtained from the original convective term in (1). Besides, a is coercive as a consequence of the Poincaré–Friedrics inequality, that is, there exists a constant $K_a > 0$ such that:

$$a(\boldsymbol{u}, \boldsymbol{u}) = \nu\,\|\nabla\boldsymbol{u}\|_0^2 \geqslant K_a\|\boldsymbol{u}\|_1^2, \quad \forall\boldsymbol{u} \in V.$$

and b satisfies the (continuous) *inf–sup* condition, that is, there exists a constant $K_b > 0$ such that:

$$\inf_{q\in Q}\left(\sup_{\boldsymbol{v}\in V}\frac{b(q, \boldsymbol{v})}{||\boldsymbol{v}||_1||q||_0}\right) \geqslant K_b > 0 \tag{5}$$

(infima and suprema are always taken with respect to nonzero functions). Condition (5) is usually referred to as the *inf–sup* or LBB condition, after the work of O.A. Ladyzhenskaya, I. Babuška and F. Brezzi. Finally, c satisfies other continuity properties, some of which are (see, e.g., [8]):

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \leqslant \begin{cases} C||\boldsymbol{u}||_1||\boldsymbol{v}||_1||\boldsymbol{w}||_1\\ C||\boldsymbol{u}||_0||\boldsymbol{v}||_2||\boldsymbol{w}||_1\\ C||\boldsymbol{u}||_2||\boldsymbol{v}||_1||\boldsymbol{w}||_0\\ C||\boldsymbol{u}||_0||\boldsymbol{v}||_1||\boldsymbol{w}||_{L^\infty(\Omega)}. \end{cases}$$

2.2. Finite element approximation

The numerical approximation of problem (1)–(2) that we analyze here was introduced in [7] as an extension to the transient case of a finite element method originally developed for steady problems. It is well known that discrete approximations of incompressible flow problems in primitive variables are restricted by the discrete *inf–sup* condition, that is, the discrete counterpart of condition (5); this prevents the use of many simple finite element combinations for the discrete spaces of the velocity and the pressure, such as equal order ones. The methods based on a pressure gradient projection circumvent this restriction by introducing the projection of the gradient of the discrete pressure onto the space of discrete velocities as a new variable of the problem; this allows, in particular, the use of equal order interpolations.

In the transient case, this methodology can be applied together with different time integration schemes; we concentrate here on an implicit, monolithic scheme using the trapezoidal rule, but extensions to other schemes such as fractional-step or multistep methods can be derived in a similar way (see [7] for a description of some of them).

2.2.1. *Semidiscrete problem*

We consider a parameter $\theta \in (0,1]$ and discretize Eqs. (1)–(2) in time first by the following implicit scheme, which we write in variational form: given a time step size $\delta t > 0$, let $N = [T/\delta t] - 1$; for $n \in \{0, \ldots, N\}$, let $t_n = n\delta t$; given $\boldsymbol{u}^n \in V$ and $p^n \in Q$, approximations of $\boldsymbol{u}(t_n)$ and $p(t_n)$, respectively, find $\boldsymbol{u}^{n+1} \in V$ and $p^{n+1} \in Q$ such that:

$$\left(\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t}, \boldsymbol{v}\right) + c\big(\boldsymbol{u}^{n+\varepsilon\theta}, \boldsymbol{u}^{n+\theta}, \boldsymbol{v}\big) + a\big(\boldsymbol{u}^{n+\theta}, \boldsymbol{v}\big) + \big(\nabla p^{n+\theta}, \boldsymbol{v}\big), = \big(\boldsymbol{f}^{n+\theta}, \boldsymbol{v}\big), \tag{6}$$

$$b\big(q, \boldsymbol{u}^{n+1}\big) = 0 \tag{7}$$

for all $(\boldsymbol{v}, q) \in V \times Q$, where for a given function g the notation $g^{n+\theta}$ stands for:

$$g^{n+\theta} = \theta g^{n+1} + (1-\theta) g^n.$$

The parameter ε which appears in the approximation of the nonlinear term in (6) can take the values 0 and 1, corresponding to a linearized and a nonlinear approximation of convection, respectively. The first option is suitable in the first order, backward Euler case $\theta = 1$, since the approximation it provides is also first order accurate and it results in a lower computational cost of the fully discrete problem (which is then linear in each time step); the second option, however, enhances stability for highly convective flows and is compulsory in the Crank–Nicholson case $\theta = \frac{1}{2}$ to maintain second order accuracy. In this sense, we use the expression $c(\boldsymbol{u}^{n+\theta}, \boldsymbol{u}^{n+\theta}, \boldsymbol{v})$ which differs from $(c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}))^{n+\theta}$ (the form to which strict application of the trapezoidal rule would lead) by a second order term and is computationally simpler. Moreover, we assume that the semidiscrete pressures satisfy $\nabla p^{n+1} \in \boldsymbol{L}^2(\Omega)$; conditions on $\boldsymbol{f}$ and Ω for this assumption to hold can be found, for instance, in [11].

2.2.2. *Fully discrete method*

We now proceed to introduce a spatial approximation of the semidiscrete problem (6)–(7). Let Θ_h denote a finite element partition of the domain Ω of diameter h. We assume that all the element domains $K \in \Theta_h$ are the image of a reference element $\widehat{K}$ through polynomial mappings $\boldsymbol{F}_K$, affine for simplicial elements, bilinear for quadrilaterals and trilinear for hexahedra. On $\widehat{K}$ we define the polynomial spaces $R_k(\widehat{K})$ where, as usual, $R_k = P_k$ for simplicial elements and $R_k = Q_k$ for quadrilaterals and hexahedra. The finite element spaces we need are:

$$Q_h = \big\{q_h \in C^0(\Omega) \cap L_0^2(\Omega) \mid q_{h|K} = \hat{q} \circ \boldsymbol{F}_K^{-1}, \hat{q} \in R_{k_q}\big(\widehat{K}\big),\ K \in \Theta_h\big\},$$

$$V_h = \big\{\boldsymbol{v}_h \in \big(C^0(\Omega)\big)^d \mid \boldsymbol{v}_{h|K} = \hat{\boldsymbol{v}} \circ \boldsymbol{F}_K^{-1},\ \hat{\boldsymbol{v}} \in \big(R_{k_v}\big(\widehat{K}\big)\big)^d,\ K \in \Theta_h\big\},$$

$$V_{h,0} = \big\{\boldsymbol{v}_h \in V_h | \boldsymbol{v}_{h|\Gamma} = \boldsymbol{0}\big\}.$$

Notice that both the velocity and pressure finite element spaces $V_{h,0}$ and Q_h are referred to the same partition and both are made up with continuous functions. These finite element spaces satisfy the following approximating properties (see, e.g., [23]): given $\boldsymbol{v} \in \boldsymbol{H}^r(\Omega)$, $r \geqslant 2$, and $q \in H^s(\Omega)$, $s \geqslant 1$, there exist $\Pi_{h,1}(\boldsymbol{v}) \in V_{h,0}$, $\Pi_{h,2}(q) \in Q_h$ and $\Pi_{h,3}(\nabla q) \in V_h$ such that:

$$\begin{aligned}
&\left\|\boldsymbol{v} - \Pi_{h,1}(\boldsymbol{v})\right\|_{m_1} \leqslant C_1 h^{k_1 - m_1} \|\boldsymbol{v}\|_{k_1},\\
&\left\|q - \Pi_{h,2}(q)\right\|_{m_2} \leqslant C_2 h^{k_2 - m_2} \|q\|_{k_2},\\
&\left\|\nabla q - \Pi_{h,3}(\nabla q)\right\|_{m_3} \leqslant C_3 h^{k_3 - m_3} \|\nabla q\|_{k_3},
\end{aligned}$$

for $0 \leqslant m_i \leqslant k_i$ $(i = 1, 2, 3)$, where:

$$k_1 = \min\{r, k_{\mathrm{v}} + 1\}, \qquad k_2 = \min\{s, k_{\mathrm{q}} + 1\}, \qquad k_3 = \min\{s - 1, k_{\mathrm{v}} + 1\}.$$

Let now $\alpha > 0$ be a given parameter. Given $(\boldsymbol{u}_h^n, p_h^n, \boldsymbol{\xi}_h^n) \in V_{h,0} \times Q_h \times V_h$, approximations of $(\boldsymbol{u}^n, p^n, \nabla p^n)$, we discretize (6)–(7) in space by finding $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\xi}_h^{n+1}) \in V_{h,0} \times Q_h \times V_h$ such that:

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\delta t}, \boldsymbol{v}_h\right) + c\big(\boldsymbol{u}_h^{n+\varepsilon\theta}, \boldsymbol{u}_h^{n+\theta}, \boldsymbol{v}_h\big) + a\big(\boldsymbol{u}_h^{n+\theta}, \boldsymbol{v}_h\big) + \big(\nabla p_h^{n+\theta}, \boldsymbol{v}_h\big) = \big(\boldsymbol{f}^{n+\theta}, \boldsymbol{v}_h\big), \tag{8}$$

$$-b\big(q_h, \boldsymbol{u}_h^{n+1}\big) + \alpha\big(\big(\nabla p_h^{n+1}, \nabla q_h\big) - \big(\boldsymbol{\xi}_h^{n+\beta}, \nabla q_h\big)\big) = 0, \tag{9}$$

$$-\big(\nabla p_h^{n+1}, \boldsymbol{\eta}_h\big) + \big(\boldsymbol{\xi}_h^{n+1}, \boldsymbol{\eta}_h\big) = 0 \tag{10}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$, where again either $\beta = 0$ or $\beta = 1$. Eq. (10) says that $\boldsymbol{\xi}_h^{n+1}$ is the L^2-projection of ∇p_h^{n+1} onto the space V_h; thus, the cases $\beta = 0$ and $\beta = 1$ correspond to an explicit and an implicit approximation of the pressure gradient projection in the modified continuity equation (9), respectively (see [7]).

In the formulation (8)–(10) we have used a 'global' parameter α, with the same value on all the element domains; the numerical analysis of this method then requires of some regularity properties of the finite element mesh such as its *quasi-uniformity*. However, this restriction can be relaxed by considering a set of elemental parameters α_K, $K \in \Theta_h$, and replacing the L^2-scalar products appearing in (9)–(10) by a sum of products weighed in each element by α_K. This extension to local parameters was analyzed in [6] for the steady, incompressible Navier–Stokes equations, and the analysis given there can be readily applied to the transient case. We restrict our attention here to the global parameter case to simplify the presentation.

3. Stability and error analysis

We now present a numerical analysis of the finite element method (8)–(10). For the time approximation, we restrict to the fully implicit, backward Euler case $\theta = 1$, which is first order accurate in the time step. We split the errors of the method into a temporal error, due to the semidiscretization (6)–(7), and a spatial error, due to the stabilized, fully discrete method (8)–(10). In the case of study $\theta = 1$, first order accuracy in the time step for the semidiscrete velocity solution can be shown by standard arguments; we include a proof of this result for completeness. We consider both the linearized method $\varepsilon = 0$ and the fully nonlinear scheme $\varepsilon = 1$. We then concentrate on the spatial approximation in the implicit pressure gradient case $\beta = 1$.

3.1. Error estimates for the semidiscrete solution

Let us define the *continuous* errors (as for the spatial variables) as

$$\boldsymbol{e}_c^{n+1} = \boldsymbol{u}(t_{n+1}) - \boldsymbol{u}^{n+1},$$
$$r_c^{n+1} = p(t_{n+1}) - p^{n+1},$$
$$\boldsymbol{g}_c^{n+1} = \nabla r_c^{n+1}.$$

We then have:

Theorem 1. *Assume* (R1), (R2) *and* (R4) *hold. Then, there is a constant* C *independent of* δt *such that*:

$$\|\boldsymbol{e}_c^{N+1}\|_0^2 + \nu\delta t \sum_{n=0}^{N} \|\boldsymbol{e}_c^{n+1}\|_1^2 \leqslant C\delta t^2. \tag{11}$$

If $\varepsilon = 1$, (11) *holds for sufficiently small* δt.

Proof. We call $\boldsymbol{R}^n$ the truncation error defined by

$$\frac{1}{\delta t}\big(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)\big) - \nu\Delta\boldsymbol{u}(t_{n+1}) + \big(\boldsymbol{u}(t_{n+1})\cdot\nabla\big)\boldsymbol{u}(t_{n+1}) + \nabla p(t_{n+1}) = \boldsymbol{f}(t_{n+1}) + \boldsymbol{R}^n \tag{12}$$

so that

$$\boldsymbol{R}^n = \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} (t - t_n)\boldsymbol{u}_{tt}(t)\,\mathrm{d}t.$$

Multiplying (12) by $\boldsymbol{v} \in V$ and (2) (at $t = t_{n+1}$) by $q \in Q$, and subtracting (6) (with $\theta = 1$) and (7) from them, respectively, we find

$$\left(\frac{\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n}{\delta t}, \boldsymbol{v}\right) + \nu\big(\nabla\boldsymbol{e}_c^{n+1}, \nabla\boldsymbol{v}\big) + \big(\nabla r_c^{n+1}, \boldsymbol{v}\big)$$
$$= \big\langle \boldsymbol{R}^n, \boldsymbol{v}\big\rangle + c\big(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{v}\big) - c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{v}\big), \tag{13}$$
$$b\big(q, \boldsymbol{e}_c^{n+1}\big) = 0. \tag{14}$$

Taking $\boldsymbol{v} = 2\delta t\boldsymbol{e}_c^{n+1}$ in (13) and $q = r_c^{n+1}$ in (14), and using the identity $(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2$, we get

$$\|\boldsymbol{e}_c^{n+1}\|_0^2 - \|\boldsymbol{e}_c^n\|_0^2 + \|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\|_0^2 + 2\delta t\nu\|\nabla\boldsymbol{e}_c^{n+1}\|_0^2 = 2\delta t\big\langle\boldsymbol{R}^n, \boldsymbol{e}_c^{n+1}\big\rangle + 2\delta t\,\mathrm{NLT}$$

where NLT stands for

$$\mathrm{NLT} = c\big(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big).$$

For the Taylor residual term, one has (see, e.g., [20])

$$2\delta t\big\langle\boldsymbol{R}^n, \boldsymbol{e}_c^{n+1}\big\rangle \leqslant \frac{\delta t\nu}{3}\|\nabla\boldsymbol{e}_c^{n+1}\|_0^2 + C\delta t^2\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}_{tt}\|_{W'}^2\,\mathrm{d}t.$$

The treatment of the NLT is different in the cases $\varepsilon = 0$ and $\varepsilon = 1$.

Linearized case. When $\varepsilon = 0$, we have

$$\begin{aligned}
2\delta t\mathrm{NLT} &= 2\delta t\big(c\big(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big)\big) \\
&= 2\delta t\big(-c\big(\boldsymbol{u}^n, \boldsymbol{e}_c^{n+1}, \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{e}_c^n, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big)\big) \\
&= T_1 + T_2 + T_3,
\end{aligned}$$

where $T_1 = 0$ due to the skew-symmetry of the trilinear form c, and, due to its continuity properties and the regularity property (R1) of $\boldsymbol{u}$:

$$\begin{aligned}
T_2 &= -2\delta t c\big(\boldsymbol{e}_c^n, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big) \\
&\leqslant C\delta t\big\|\boldsymbol{e}_c^n\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant \frac{\delta t\nu}{3}\big\|\nabla\boldsymbol{e}_c^{n+1}\big\|_0^2 + C\delta t\big\|\boldsymbol{e}_c^n\big\|_0^2, \\
T_3 &= -2\delta t c\big(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big) \\
&\leqslant C\delta t\big\|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant \frac{\delta t\nu}{3}\big\|\nabla\boldsymbol{e}_c^{n+1}\big\|_0^2 + C\delta t^2\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2 - \big\|\boldsymbol{e}_c^n\big\|_0^2 + \big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\big\|_0^2 + \delta t\nu\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 \\
&\quad\leqslant C\delta t^2\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_{tt}\|_{W'}^2\,\mathrm{d}t + C\delta t^2\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t + C\delta t\big\|\boldsymbol{e}_c^n\big\|_0^2. \qquad (15)
\end{aligned}$$

Adding up (15) for $n = 0, \ldots, N$, and using the regularity properties (R2) and (R4) of the continuous solution, we get

$$\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\big\|_0^2 + \delta t\nu\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 \leqslant C\delta t^2 + C\delta t\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^n\big\|_0^2.$$

Applying the discrete Gronwall inequality, this implies

$$\big\|\boldsymbol{e}_c^{N+1}\big\|_0^2 + \sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1} - \boldsymbol{e}_c^n\big\|_0^2 + \delta t\nu\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2 \leqslant C\delta t^2 \qquad (16)$$

and (11) follows.

Nonlinear case. When $\varepsilon = 1$ we have

$$\begin{aligned}
2\delta t\mathrm{NLT} &= 2\delta t\big(c\big(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big)\big) \\
&= 2\delta t\big(-c\big(\boldsymbol{u}^{n+1}, \boldsymbol{e}_c^{n+1}, \boldsymbol{e}_c^{n+1}\big) - c\big(\boldsymbol{e}_c^{n+1}, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big)\big) \\
&= T_1 + T_2,
\end{aligned}$$

where again $T_1 = 0$ due to the skew-symmetry of the trilinear form c, and

$$\begin{aligned}
T_2 &= -2\delta t c\big(\boldsymbol{e}_c^{n+1}, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_c^{n+1}\big) \\
&\leqslant C\delta t\big\|\boldsymbol{e}_c^{n+1}\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\big\|\boldsymbol{e}_c^{n+1}\big\|_1 \leqslant \frac{\delta t\nu}{3}\big\|\nabla\boldsymbol{e}_c^{n+1}\big\|_0^2 + C\delta t\big\|\boldsymbol{e}_c^{n+1}\big\|_0^2.
\end{aligned}$$

Therefore,

$$\|e_c^{n+1}\|_0^2 - \|e_c^n\|_0^2 + \|e_c^{n+1} - e_c^n\|_0^2 + \delta t\nu\|e_c^{n+1}\|_1^2 \leqslant C\delta t^2 \int_{t_n}^{t_{n+1}} \|u_{tt}\|_{W'}^2 \,\mathrm{d}t + C\delta t^2 \int_{t_n}^{t_{n+1}} \|u_t\|_0^2 \,\mathrm{d}t + C\delta t\|e_c^{n+1}\|_0^2 \tag{17}$$

and

$$\|e_c^{N+1}\|_0^2 + \sum_{n=0}^{N}\|e_c^{n+1} - e_c^n\|_0^2 + \delta t\nu\sum_{n=0}^{N}\|e_c^{n+1}\|_1^2 \leqslant C\delta t^2 + C\delta t\sum_{n=0}^{N}\|e_c^{n+1}\|_0^2.$$

Applying the discrete Gronwall inequality, this implies, for sufficiently small δt,

$$\|e_c^{N+1}\|_0^2 + \sum_{n=0}^{N}\|e_c^{n+1} - e_c^n\|_0^2 + \delta t\nu\sum_{n=0}^{N}\|e_c^{n+1}\|_1^2 \leqslant C\delta t^2 \tag{18}$$

and (11) follows again. □

Remark 1. The error estimates proved in Theorem 1 ensure that the semidiscrete velocities u^{n+1} are first order accurate in the time step, in the following sense: given a Banach space $(X, \|z\|)$, for $s > 0$ let $l^s(X)$ denote the space of finite sequences $Z = \{z^{n+1}\}_{n=0}^N \subset X$ equipped with the norm

$$|Z|_s = \left(\frac{1}{N}\sum_{n=0}^{N}\|z^{n+1}\|^s\right)^{1/s}$$

for $s < \infty$ and $|Z|_\infty = \max_{n=0,\dots,N}\|z^{n+1}\|$. Then, u^{n+1} is first order accurate in $l^\infty(L^2(\Omega))$ and in $l^2(H_0^1(\Omega))$. This result proves, in particular, that these semidiscrete velocities are bounded in $l^\infty(H_0^1(\Omega))$ by a constant independent of δt, since:

$$\|u^{n+1}\|_1 \leqslant \|u(t_{n+1})\|_1 + \|e_c^{n+1}\|_1 \leqslant \|u(t_{n+1})\|_1 + (C\delta t)^{1/2} \leqslant C$$

due to Theorem 1 and the regularity assumed on the continuous solution. Moreover, we also have $\|e_c^{n+1}\|_1 \leqslant C\delta t^{1/2}$. We will use these results later on.

We also have an error estimate for the semidiscrete pressure p^{n+1}:

Proposition 1. *Let* (R1)–(R4) *hold. Then, there is a constant C independent of δt such that*

$$\delta t\sum_{n=0}^{N}\|r_c^{n+1}\|_0^2 \leqslant C\delta t. \tag{19}$$

If $\varepsilon = 1$, (19) *holds for sufficiently small δt.*

Proof. By the continuous inf–sup condition (5), we have, using (13),

$$\|r_c^{n+1}\|_0 \leqslant C\sup_{v\in V}\frac{(\nabla r_c^{n+1}, v)}{\|v\|_1} = C\sup_{v\in V}\frac{1}{\|v\|_1}\left\{\frac{-1}{\delta t}(e_c^{n+1} - e_c^n, v) - \nu(\nabla e_c^{n+1}, \nabla v) + \langle R^n, v\rangle\right\} + \text{NLT}.$$

We bound each term as follows (for the Taylor residual term, see [21]):

$$\frac{1}{\delta t\|\boldsymbol{v}\|_1}\big(\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^n,\boldsymbol{v}\big)\leqslant\frac{C}{\delta t}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^n\big\|_0,$$
$$\frac{\nu}{\|\boldsymbol{v}\|_1}\big(\nabla\boldsymbol{e}_c^{n+1},\nabla\boldsymbol{v}\big)\leqslant C\nu^{1/2}\big\|\boldsymbol{e}_c^{n+1}\big\|_1,$$
$$\frac{1}{\|\boldsymbol{v}\|_1}\langle\boldsymbol{R}^n,\boldsymbol{v}\rangle\leqslant\big\|\boldsymbol{R}^n\big\|_{-1}\leqslant C\left(\int_{t_n}^{t_{n+1}}t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t\right)^{1/2}.$$

The treatment of the NLT is again different for $\varepsilon=0$ and 1.

Linearized case. Using the continuity properties of the trilinear form c, the regularity property (R1) of $\boldsymbol{u}$ and the results of Theorem 1 and Remark 1, we have

$$\begin{aligned}
\mathrm{NLT}&=\frac{1}{\|\boldsymbol{v}\|_1}\big(c\big(\boldsymbol{u}^n,\boldsymbol{u}^{n+1},\boldsymbol{v}\big)-c\big(\boldsymbol{u}(t_{n+1}),\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\big)\big)\\
&=\frac{1}{\|\boldsymbol{v}\|_1}\big(-c\big(\boldsymbol{u}^n,\boldsymbol{e}_c^{n+1},\boldsymbol{v}\big)-c\big(\boldsymbol{e}_c^n,\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\big)-c\big(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n),\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\big)\big)\\
&=T_1+T_2+T_3,\\
T_1&=\frac{-1}{\|\boldsymbol{v}\|_1}c\big(\boldsymbol{u}^n,\boldsymbol{e}_c^{n+1},\boldsymbol{v}\big)\leqslant C\big\|\boldsymbol{u}^n\big\|_1\big\|\boldsymbol{e}_c^{n+1}\big\|_1\leqslant C\nu^{1/2}\big\|\boldsymbol{e}_c^{n+1}\big\|_1,\\
T_2&=\frac{-1}{\|\boldsymbol{v}\|_1}c\big(\boldsymbol{e}_c^n,\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\big)\leqslant C\big\|\boldsymbol{e}_c^n\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\leqslant C\big\|\boldsymbol{e}_c^n\big\|_0\leqslant C\delta t,\\
T_3&=\frac{-1}{\|\boldsymbol{v}\|_1}c\big(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n),\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\big)\\
&\leqslant C\big\|\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\leqslant C\delta t^{1/2}\left(\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\big\|r_c^{n+1}\big\|_0^2\leqslant C\left(\frac{1}{\delta t^2}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^n\big\|_0^2+\nu\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2+\int_{t_n}^{t_{n+1}}t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t+\delta t^2+\delta t\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t\right).$$

Finally,

$$\begin{aligned}
\delta t\sum_{n=0}^{N}\big\|r_c^{n+1}\big\|_0^2&\leqslant C\left(\frac{1}{\delta t}\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}-\boldsymbol{e}_c^n\big\|_0^2+\nu\delta t\sum_{n=0}^{N}\big\|\boldsymbol{e}_c^{n+1}\big\|_1^2\right.\\
&\qquad\left.+\delta t\int_0^T t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t+\delta t^2+\delta t^2\int_0^T\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t\right)\\
&\leqslant C\delta t+C\delta t\int_0^T t\|\boldsymbol{u}_{tt}\|_{-1}^2\,\mathrm{d}t+C\delta t^2\int_0^T\|\boldsymbol{u}_t\|_0^2\,\mathrm{d}t
\end{aligned}$$

due to (16). Estimate (19) follows from the regularity properties (R2) and (R3) of $\boldsymbol{u}$.

Nonlinear case. This time we have:

$$\begin{aligned}
\mathrm{NLT} &= \frac{1}{\|\boldsymbol{v}\|_1}\bigl(c\bigl(\boldsymbol{u}^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{v}\bigr) - c\bigl(\boldsymbol{u}(t_{n+1}),\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\bigr)\bigr)\\
&= \frac{1}{\|\boldsymbol{v}\|_1}\bigl(-c\bigl(\boldsymbol{u}^{n+1},\boldsymbol{e}_c^{n+1},\boldsymbol{v}\bigr) - c\bigl(\boldsymbol{e}_c^{n+1},\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\bigr)\bigr)\\
&= T_1 + T_2,\\
T_1 &= \frac{-1}{\|\boldsymbol{v}\|_1} c\bigl(\boldsymbol{u}^{n+1},\boldsymbol{e}_c^{n+1},\boldsymbol{v}\bigr) \leqslant C\nu^{1/2}\bigl\|\boldsymbol{e}_c^{n+1}\bigr\|_1,\\
T_2 &= \frac{-1}{\|\boldsymbol{v}\|_1} c\bigl(\boldsymbol{e}_c^{n+1},\boldsymbol{u}(t_{n+1}),\boldsymbol{v}\bigr) \leqslant C\bigl\|\boldsymbol{e}_c^{n+1}\bigr\|_0 \leqslant C\delta t
\end{aligned}$$

and (19) follows again. □

3.2. A priori stability estimate

We begin the analysis of the discrete problem recalling a stability estimate which was proven in [7] under weak regularity assumptions on the continuous solution. When studying pressure-gradient-projection methods for steady, incompressible flow problems, the following assumptions are encountered (see [5]), all of which carry over to the unsteady case:

H1. *There exist $\alpha_- > 0$ and $\alpha_+ > 0$ independent of h such that*:

$$\alpha_- h^2 \leqslant \alpha \leqslant \alpha_+ h^2. \tag{20}$$

This assumption dictates the behaviour of the numerical parameter α.

H2. *The family of finite element partitions Θ_h is quasi-uniform, that is, there exists a constant $\sigma > 0$ independent of h such that, for all $h > 0$*:

$$\min\bigl\{\mathrm{diam}(B_K) \mid K \in \Theta_h\bigr\} \geqslant \sigma \max\bigl\{\mathrm{diam}(B_K) \mid K \in \Theta_h\bigr\}, \tag{21}$$

where B_K is the largest ball contained in K. Condition (21) *is needed in order to have the following inverse estimate* (*see* [2]):

$$\|\boldsymbol{v}_h\|_1 \leqslant \frac{C}{h}\|\boldsymbol{v}_h\|_0, \quad \forall \boldsymbol{v}_h \in V_h. \tag{22}$$

This assumption can be weakened by using local parameters α_K (see [6]).

H3. *As in* [5,6], *let ∇Q_h denote the space*

$$\nabla Q_h = \bigl\{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{v}_h = \nabla q_h, q_h \in Q_h\bigr\}$$

and define the space E_h by

$$E_h = V_h + \nabla Q_h \subset \boldsymbol{L}^2(\Omega).$$

We consider three mutually orthogonal subspaces $E_{h,i}$ *of* E_h *defined by*

$$E_{h,1} = V_{h,0}, \qquad E_{h,2} = V_{h,0}^{\perp} \cap V_h, \qquad E_{h,3} = V_h^{\perp} \cap E_h$$

so that

$$E_h = E_{h,1} \oplus E_{h,2} \oplus E_{h,3}.$$

For $i = 1, 2, 3$, *we call* $P_{h,i}$ *the* L^2*-projection of* E_h *onto* $E_{h,i}$, *and for* $i \neq j$, $P_{h,ij} = P_{h,i} + P_{h,j}$ *and* $E_{h,ij} = E_{h,i} \oplus E_{h,j}$. *In this notation,* $\boldsymbol{\xi}_h^{n+1} = P_{h,12}(\nabla p_h^{n+1})$. *We assume that there is a constant* β_0 *independent of h such that*

$$\|\nabla q_h\|_0 \leqslant \beta_0 \big\| P_{h,13}(\nabla q_h)\big\|_0, \tag{23}$$

that is to say, that the second component of the decomposition of every ∇q_h *in* E_h *can be bounded in terms of the other two. This condition can also be written in the form*

$$\inf_{q_h \in Q_h} \left(\sup_{\boldsymbol{v}_h \in E_{h,13}} \frac{(\nabla q_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_0 \|\nabla q_h\|_0} \right) \geqslant \beta_1 > 0, \tag{24}$$

in a similar way to the classical inf–sup condition; *condition* (24), *however, is weaker since the space where the supremum is taken,* $E_{h,13}$, *is larger than in the classical case,* $V_{h,0} = E_{h,1}$. *Condition* (24) *was analyzed in* [5], *where it was shown to be satisfied by equal order simplicial finite element interpolations.*

The scheme analyzed in [7] differs slightly from (8)–(10) in the interpretation of the parameter α and the pressure gradient projection $\boldsymbol{\xi}_h^{n+1}$; moreover, it is restricted to the case $\varepsilon = 1$. However, a straightforward extension of the proofs in [7] leads to the following stability result:

Theorem 2. *Assume* H1–H3 *hold*; *then, there exists a constant* $C > 0$ *independent of* δt *and h such that, for small enough* δt:

$$\big\|\boldsymbol{u}_h^{N+1}\big\|_0^2 + \nu \delta t \sum_{n=0}^{N} \big\|\boldsymbol{u}_h^{n+1}\big\|_1^2 + \delta t h \sum_{n=0}^{N} \big\|\nabla p_h^{n+1}\big\|_0 \leqslant C. \tag{25}$$

Remark 2. This theorem proves that the discrete velocities are stable in $l^\infty(\boldsymbol{L}^2(\Omega))$ and $l^2(\boldsymbol{H}_0^1(\Omega))$, while the discrete pressure gradients (scaled by h) are stable in $l^1(\boldsymbol{L}^2(\Omega))$; this proves, in particular, that the discrete problem is always well-posed. The result for the pressure can be improved to $l^2(\boldsymbol{L}^2(\Omega))$ in 2D flows or for the linear Stokes case (see [7]). We improve this estimates later on to $l^\infty(\boldsymbol{H}_0^1(\Omega))$ for the velocity and $l^2(\boldsymbol{L}^2(\Omega))$ for the pressure as a consequence of the error estimates of the next section.

3.3. Error estimates for the velocity

We now proceed to obtain error estimates for the fully discrete velocity solution $\boldsymbol{u}_h^{n+1}$ as an approximation of the semidiscrete solution $\boldsymbol{u}^{n+1}$ under stronger regularity assumptions on the continuous problem. For simplicity, we assume that the domain Ω is polyhedral, so that it can be exactly covered by triangulations. We define and split the *errors of the method* as

$$\boldsymbol{e}^{n+1} = \boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_h^{n+1} = \boldsymbol{e}_c^{n+1} + \boldsymbol{e}_d^{n+1},$$
$$r^{n+1} = p(t_{n+1}) - p_h^{n+1} = r_c^{n+1} + r_d^{n+1},$$
$$\boldsymbol{g}^{n+1} = \nabla p(t_{n+1}) - \boldsymbol{\xi}_h^{n+1} = \boldsymbol{g}_c^{n+1} + \boldsymbol{g}_d^{n+1},$$

where the *discrete* errors are defined as

$$\boldsymbol{e}_d^{n+1} = \boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}, \qquad r_d^{n+1} = p^{n+1} - p_h^{n+1}, \qquad \boldsymbol{g}_d^{n+1} = \nabla p^{n+1} - \boldsymbol{\xi}_h^{n+1}.$$

Subtracting (8) (with $\theta = 1$) from (6) and (9) (with $\beta = 1$) from (7), it can be seen that these discrete errors satisfy the following equations, which hold for any $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$:

$$\left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, \boldsymbol{v}_h\right) + a\left(\boldsymbol{e}_d^{n+1}, \boldsymbol{v}_h\right) + \left(\nabla r_d^{n+1}, \boldsymbol{v}_h\right) - c\left(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + c\left(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right) = 0, \quad (26)$$

$$\left(\nabla \cdot \boldsymbol{e}_d^{n+1}, q_h\right) + \alpha\left(\left(\nabla r_d^{n+1}, \nabla q_h\right) - \left(\boldsymbol{g}_d^{n+1}, \nabla q_h\right)\right) = 0, \quad (27)$$

$$-\left(\nabla r_d^{n+1}, \boldsymbol{\eta}_h\right) + \left(\boldsymbol{g}_d^{n+1}, \boldsymbol{\eta}_h\right) = 0. \quad (28)$$

We also introduce the following notation. Given $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$ arbitrary, we call:

$$\begin{aligned} I_0\left(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right) &= \left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right\|_0, & I_1\left(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right) &= \left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right\|_1, \\ I_0\left(p^{n+1}, q_h\right) &= \left\|p^{n+1} - q_h\right\|_0, & I_1\left(p^{n+1}, q_h\right) &= \left\|\nabla p^{n+1} - \nabla q_h\right\|_0, \\ I_0\left(\nabla p^{n+1}, \boldsymbol{\eta}_h\right) &= \left\|\nabla p^{n+1} - \boldsymbol{\eta}_h\right\|_0, & G_{n+1} &= \left\|\boldsymbol{\xi}_h^{n+1} - \nabla p_h^{n+1}\right\|_0 \end{aligned}$$

and

$$\begin{aligned} E_n(h) &= \inf_{\boldsymbol{v}_h \in V_{h,0}} \left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right\|_1 + \frac{1}{h} \inf_{\boldsymbol{v}_h \in V_{h,0}} \left\|\boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right\|_0 + \inf_{q_h \in Q_h} \left\|p^{n+1} - q_h\right\|_0 \\ &\quad + h \inf_{q_h \in Q_h} \left\|\nabla p^{n+1} - \nabla q_h\right\|_0 + h \inf_{\boldsymbol{\eta}_h \in V_h} \left\|\nabla p^{n+1} - \boldsymbol{\eta}_h\right\|_0, \\ E(h) &= \max_{n=0,\ldots,N} E_n(h). \end{aligned} \quad (29)$$

We begin with a rather technical lemma:

Lemma 1. *Assume* H2 *and* H3 *hold; then, for $n = 0, \ldots, N$, for any $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$ and for small enough h*:

$$\begin{aligned} \left\|\nabla r_d^{n+1}\right\|_0 \leqslant C\bigg\{ & I_0\left(\nabla p^{n+1}, \boldsymbol{\eta}_h\right) + I_1\left(p^{n+1}, q_h\right) + \frac{1}{\delta t}\left\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\right\|_0 \quad (30) \\ & + G_{n+1} + \frac{\nu^{1/2}}{h}\left(\left\|\boldsymbol{e}_d^{n+1}\right\|_1 + \left\|\boldsymbol{e}_d^n\right\|_1\right) \\ & + \frac{\nu}{h}\left(\left\|\boldsymbol{e}_d^{n+1}\right\|_1^2 + \left\|\boldsymbol{e}_d^n\right\|_1^2 + \left\|\boldsymbol{e}_c^{n+1}\right\|_1^2 + \left\|\boldsymbol{e}_c^n\right\|_1^2\right)\bigg\}. \quad (31) \end{aligned}$$

Proof. By the triangle inequality and the previous definitions, we have

$$\begin{aligned} \left\|\nabla r_c^{n+1}\right\|_0 &\leqslant \left\|\nabla p^{n+1} - P_{h,12}(\nabla q_h)\right\|_0 + \left\|P_{h,1}(\nabla q_h) - P_{h,1}\left(\nabla p_h^{n+1}\right)\right\|_0 \\ &\quad + \left\|P_{h,2}(\nabla q_h) - P_{h,2}\left(\nabla p_h^{n+1}\right)\right\|_0 + \left\|P_{h,3}\left(\nabla p_h^{n+1}\right)\right\|_0 \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

We bound each term separately. For the first term, we use a similar argument to that of [5] for the corresponding term in the analysis of an approximation of the Stokes problem, to get

$$T_1 = \left\|\nabla p^{n+1} - P_{h,12}(\nabla q_h)\right\|_0 \leqslant I_0\left(\nabla p^{n+1}, \boldsymbol{\eta}_h\right) + I_1\left(p^{n+1}, q_h\right).$$

For the second term, we have, due to the orthogonality of the projection $P_{h,1}$,

$$\begin{aligned} T_2^2 &= \|P_{h,1}(\nabla q_h) - P_{h,1}(\nabla p_h^{n+1})\|_0^2 \\ &= (P_{h,1}(\nabla q_h - \nabla p_h^{n+1}), P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &= (\nabla q_h - \nabla p^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) + (\nabla r_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &= T_{2,a} + T_{2,b} \end{aligned}$$

so that

$$T_{2,a} \leqslant I_1(p^{n+1}, q_h)\|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0.$$

Moreover, taking $\boldsymbol{v}_h = P_{h,1}(\nabla q_h - \nabla p_h^{n+1}) \in V_{h,0}$ in (26), we get

$$\begin{aligned} T_{2,b} &= -\left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\right) - \nu(\nabla \boldsymbol{e}_d^{n+1}, \nabla P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &\quad + c(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) - c(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &= -\left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\right) - \nu(\nabla \boldsymbol{e}_d^{n+1}, \nabla P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &\quad - c(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) - c(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})). \end{aligned}$$

Then,

$$\begin{aligned} -\left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\right) &\leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0, \\ -\nu(\nabla \boldsymbol{e}_d^{n+1}, \nabla P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) &\leqslant C\nu^{1/2}\|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_1 \\ &\leqslant C\frac{\nu^{1/2}}{h}\|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0, \\ -c(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) &\leqslant C\|\boldsymbol{e}_d^{n+\varepsilon}\|_1 \|\boldsymbol{u}^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_1 \\ &\leqslant C\frac{\nu^{1/2}}{h}\|\boldsymbol{e}_d^{n+\varepsilon}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0, \end{aligned}$$

$$\begin{aligned} &-c(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &\quad = c(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) + c(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) \\ &\qquad - c(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})), \end{aligned}$$

$$\begin{aligned} c(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) &\leqslant C\|\boldsymbol{e}_d^{n+\varepsilon}\|_1 \|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_1 \\ &\leqslant C\frac{\nu}{h}\|\boldsymbol{e}_d^{n+\varepsilon}\|_1 \|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0 \\ &\leqslant C\frac{\nu}{h}(\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + \|\boldsymbol{e}_d^{n+1}\|_1^2)\|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0, \\ c(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})) &\leqslant C\|\boldsymbol{e}_c^{n+\varepsilon}\|_1 \|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_1 \\ &\leqslant C\frac{\nu}{h}\|\boldsymbol{e}_c^{n+\varepsilon}\|_1 \|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0 \\ &\leqslant C\frac{\nu}{h}(\|\boldsymbol{e}_c^{n+\varepsilon}\|_1^2 + \|\boldsymbol{e}_d^{n+1}\|_1^2)\|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0, \end{aligned}$$

$$-c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\big) \leqslant C\|\boldsymbol{u}(t_{n+\varepsilon})\|_2 \|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0$$
$$\leqslant C\|\boldsymbol{e}_d^{n+1}\|_1 \|P_{h,1}(\nabla q_h - \nabla p_h^{n+1})\|_0$$

due to Remark 1 and the regularity of the continuous velocity. Assuming $h \leqslant C\nu^{1/2}$ in the last term, we get

$$T_2 \leqslant C\bigg(I_1(p^{n+1}, q_h) + \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0 + \frac{\nu^{1/2}}{h}\big(\|\boldsymbol{e}_d^{n+1}\|_1 + \|\boldsymbol{e}_d^n\|_1\big) + \frac{\nu}{h}\big(\|\boldsymbol{e}_d^{n+1}\|_1^2 + \|\boldsymbol{e}_d^n\|_1^2 + \|\boldsymbol{e}_c^{n+1}\|_1^2 + \|\boldsymbol{e}_c^n\|_1\big)\bigg).$$

Moreover, due to condition (23) and since $P_{h,3} = Id - P_{h,12}$ and $\boldsymbol{\xi}_h^{n+1} = P_{h,12}(\nabla p_h^{n+1})$, we have

$$\begin{aligned}
T_3 &= \|P_{h,2}(\nabla q_h) - P_{h,2}(\nabla p_h^{n+1})\|_0 \\
&\leqslant C\big(\|P_{h,1}(\nabla q_h) - P_{h,1}(\nabla p_h^{n+1})\|_0 + \|P_{h,3}(\nabla q_h) - P_{h,3}(\nabla p_h^{n+1})\|_0\big) \\
&\leqslant C\big(T_2 + \|P_{h,3}(\nabla q_h)\|_0 + \|P_{h,3}(\nabla p_h^{n+1})\|_0\big) \\
&\leqslant C\big(T_2 + \|\nabla q_h - \nabla p^{n+1}\|_0 + \|\nabla p^{n+1} - P_{h,12}(\nabla q_h)\|_0 + G_{n+1}\big) \\
&= C\big(T_2 + I_1(p^{n+1}, q_h) + T_1 + G_{n+1}\big).
\end{aligned}$$

Finally,

$$T_4 = \|P_{h,3}(\nabla p_h^{n+1})\|_0 = G_{n+1}$$

and (30) follows. □

In our convergence analysis we will also need the following assumption:

H4. *There exists $C > 0$ independent of h and δt such that*:

$$\delta t \geqslant Ch^2. \tag{32}$$

This condition does not impose an upper bound on the time step, so that the method remains unconditionally stable (see also Remark 5). Our main result of this section is the following:

Theorem 3. *Assume* (R1), (R2), (R4) *and* H1–H4 *hold*; *then, there exists a constant $C > 0$ independent of δt and h such that, for small enough h and, if $\varepsilon = 1$, small enough δt,*

$$\|\boldsymbol{e}_d^{N+1}\|_0^2 + \nu\delta t\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2 \leqslant C\big(\big(E(h)\big)^2 + E(h)\delta t^2\big). \tag{33}$$

Proof. Let us call

$$\begin{aligned}
A &= \left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, \boldsymbol{e}_d^{n+1}\right) + \nu(\nabla\boldsymbol{e}_d^{n+1}, \nabla\boldsymbol{e}_d^{n+1}) + (\nabla r_d^{n+1}, \boldsymbol{e}_d^{n+1}) + (\nabla\cdot\boldsymbol{e}_d^{n+1}, r_d^{n+1}) \\
&\quad + \alpha(\nabla r_d^{n+1}, \nabla r_d^{n+1}) - \alpha(\boldsymbol{g}_d^{n+1}, \nabla r_d^{n+1}) - \alpha(\boldsymbol{g}_d^{n+1}, \nabla r_d^{n+1}) + \alpha(\boldsymbol{g}_d^{n+1}, \boldsymbol{g}_d^{n+1}) \\
&= \frac{1}{2\delta t}\big(\|\boldsymbol{e}_d^{n+1}\|_0^2 - \|\boldsymbol{e}_d^n\|_0^2 + \|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2\big) + \nu\|\boldsymbol{e}_d^{n+1}\|_1^2 + \alpha\|\boldsymbol{\xi}_h^{n+1} - \nabla p_h^{n+1}\|_0^2.
\end{aligned}$$

Given $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$ arbitrary, we take $\boldsymbol{v}_h - \boldsymbol{u}_h^{n+1}$, $q_h - p_h^{n+1}$ and $\boldsymbol{\eta}_h - \boldsymbol{\xi}_h^{n+1}$ as test functions in (26), (27) and (28), respectively, to get

$$
\begin{aligned}
A = {} & \left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right) + \nu\big(\nabla \boldsymbol{e}_d^{n+1}, \nabla(\boldsymbol{u}^{n+1} - \boldsymbol{v}_h)\big) \\
& + \big(\nabla r_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) + c\big(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) - c\big(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) \\
& + \big(\nabla \cdot \boldsymbol{e}_d^{n+1}, p^{n+1} - q_h\big) + \alpha\big(\nabla r_d^{n+1} - \boldsymbol{g}_d^{n+1}, \boldsymbol{\eta}_h - \nabla q_h\big).
\end{aligned}
$$

We bound each term as follows:

$$
\begin{aligned}
& \left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\right) \leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0 I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \leqslant \frac{1}{4\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \frac{C}{\delta t} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2, \\
& \nu\big(\nabla \boldsymbol{e}_d^{n+1}, \nabla(\boldsymbol{u}^{n+1} - \boldsymbol{v}_h)\big) \leqslant C\nu\|\boldsymbol{e}_d^{n+1}\|_1 I_1(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \leqslant \frac{\nu}{10}\|\boldsymbol{e}_d^{n+1}\|_1^2 + C I_1(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2, \\
& \big(\nabla \cdot \boldsymbol{e}_d^{n+1}, p^{n+1} - q_h\big) \leqslant \frac{\nu}{10}\|\boldsymbol{e}_d^{n+1}\|_1^2 + C I_0(p^{n+1}, q_h)^2, \\
& \alpha\big(\nabla r_d^{n+1} - \boldsymbol{g}_d^{n+1}, \boldsymbol{\eta}_h - \nabla q_h\big) = \alpha\big(\nabla p_h^{n+1} - \boldsymbol{\xi}_h^{n+1}, \boldsymbol{\eta}_h - \nabla q_h\big) \\
& \qquad \leqslant \alpha G_{n+1}\big(I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h) + I_1(p^{n+1}, q_h)\big) \\
& \qquad \leqslant C h^2 G_{n+1}\big(I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h) + I_1(p^{n+1}, q_h)\big) \\
& \qquad \leqslant \frac{\alpha_- h^2}{3} G_{n+1}^2 + C h^2 I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h)^2 + C h^2 I_1(p^{n+1}, q_h)^2,
\end{aligned}
$$

where α_- was defined in (20). Moreover, due to Lemma 1 we have

$$
\begin{aligned}
\big(\nabla r_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) \leqslant {} & \|\nabla r_d^{n+1}\|_0 I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
\leqslant {} & C\bigg(I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h) + I_1(p^{n+1}, q_h) \\
& + \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0 + G_{n+1} + \frac{\nu^{1/2}}{h}\big(\|\boldsymbol{e}_d^{n+1}\|_1 + \|\boldsymbol{e}_d^n\|_1\big) \\
& + \frac{\nu}{h}\big(\|\boldsymbol{e}_d^{n+1}\|_1^2 + \|\boldsymbol{e}_d^n\|_1^2 + \|\boldsymbol{e}_c^{n+1}\|_1^2 + \|\boldsymbol{e}_c^n\|_1\big)\bigg) I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
\leqslant {} & C\bigg(h^2 I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h)^2 + h^2 I_1(p^{n+1}, q_h)^2 + \frac{1}{h^2} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2\bigg) \\
& + \frac{1}{4\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \frac{C}{\delta t} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2 + \frac{\alpha_- h^2}{3} G_{n+1}^2 \\
& + \frac{\nu}{10}\|\boldsymbol{e}_d^{n+1}\|_1^2 + C\frac{\nu^{1/2}}{h}\|\boldsymbol{e}_d^n\|_1 I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
& + \nu\big(\|\boldsymbol{e}_d^{n+1}\|_1^2 + \|\boldsymbol{e}_d^n\|_1^2 + \|\boldsymbol{e}_c^{n+1}\|_1^2 + \|\boldsymbol{e}_c^n\|_1^2\big)\frac{C}{h} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h).
\end{aligned}
$$

We split the convective terms the following way:

$$
\begin{aligned}
& c\big(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) - c\big(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) \\
& \quad = c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) + c\big(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) \\
& \quad = -c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_c^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) + c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big)
\end{aligned}
$$

$$
\begin{aligned}
&- c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) - c\big(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) \\
&+ c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, \boldsymbol{u}_h^{n+1} - \boldsymbol{v}_h\big) \\
= {} & c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_c^{n+1}, \boldsymbol{e}_d^{n+1}\big) - c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_c^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) \\
&- c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_d^{n+1}\big) + c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) \\
&+ c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) - c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) \\
&+ c\big(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) - c\big(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) \\
&- c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) + c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big).
\end{aligned}
$$

Due to the continuity properties of the trilinear form c, its skew symmetry in its last two arguments, the results of Theorem 1, the regularity assumed for the continuous solution $\boldsymbol{u}$ and Young's inequality, we have

$$
\begin{aligned}
c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_c^{n+1}, \boldsymbol{e}_d^{n+1}\big) &\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1\big\|\boldsymbol{e}_c^{n+1}\big\|_1\big\|\boldsymbol{e}_d^{n+1}\big\|_1 \\
&\leqslant C\delta t^{1/2}\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1\big\|\boldsymbol{e}_d^{n+1}\big\|_1 \\
&\leqslant \frac{\nu}{10}\big\|\boldsymbol{e}_d^{n+1}\big\|_1^2 + C\delta t\nu\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1^2, \\
-c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_c^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) &\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1\big\|\boldsymbol{e}_c^{n+1}\big\|_1 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant C\delta t^{1/2}\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant \delta t\nu\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1^2 + C I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big)^2, \\
-c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}(t_{n+1}), \boldsymbol{e}_d^{n+1}\big) &\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2\big\|\boldsymbol{e}_d^{n+1}\big\|_1 \\
&\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_0^2 + \frac{\nu}{10}\big\|\boldsymbol{e}_d^{n+1}\big\|_1^2, \\
c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{u}(t_{n+1}), \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) &\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_0\big\|\boldsymbol{u}(t_{n+1})\big\|_2 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_0^2 + I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big)^2, \\
c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) &= 0, \\
-c\big(\boldsymbol{e}_d^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) &\leqslant C\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1\big\|\boldsymbol{e}_d^{n+1}\big\|_1 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant \frac{\nu}{10}\big\|\boldsymbol{e}_d^{n+1}\big\|_1^2 + C\nu\big\|\boldsymbol{e}_d^{n+\varepsilon}\big\|_1^2 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big)^2, \\
c\big(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) &= 0, \\
-c\big(\boldsymbol{e}_c^{n+\varepsilon}, \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) &\leqslant C\big\|\boldsymbol{e}_c^{n+\varepsilon}\big\|_1\big\|\boldsymbol{e}_d^{n+1}\big\|_1 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant \frac{\nu}{10}\big\|\boldsymbol{e}_d^{n+1}\big\|_1^2 + C\nu\big\|\boldsymbol{e}_c^{n+\varepsilon}\big\|_1^2 I_1\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big)^2, \\
-c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, \boldsymbol{e}_d^{n+1}\big) &= 0, \\
c\big(\boldsymbol{u}(t_{n+\varepsilon}), \boldsymbol{e}_d^{n+1}, \boldsymbol{u}^{n+1} - \boldsymbol{v}_h\big) &\leqslant C\big\|\boldsymbol{u}(t_{n+\varepsilon})\big\|_2\big\|\boldsymbol{e}_d^{n+1}\big\|_1 I_0\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big) \\
&\leqslant \frac{\nu}{10}\big\|\boldsymbol{e}_d^{n+1}\big\|_1^2 + C I_0\big(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h\big)^2.
\end{aligned}
$$

Taking all the previous inequalities into account, and using (20), we find

$$\begin{aligned}
&\|\boldsymbol{e}_d^{n+1}\|_0^2 - \|\boldsymbol{e}_d^n\|_0^2 + \|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \nu\delta t\|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2 G_{n+1}^2 \\
&\quad \leqslant C\delta t\bigg(I_1(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2 + h^2 I_1(p^{n+1}, q_h)^2 + h^2 I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h)^2 \\
&\qquad\qquad + \frac{1}{h^2} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2 + I_0(p^{n+1}, q_h)^2\bigg) + C I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2 \\
&\qquad + \delta t\nu(\|\boldsymbol{e}_d^{n+1}\|_1^2 + \|\boldsymbol{e}_d^n\|_1^2)\frac{C}{h} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) + \delta t\nu(\|\boldsymbol{e}_c^{n+1}\|_1^2 + \|\boldsymbol{e}_c^n\|_1^2)\frac{C}{h} I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
&\qquad + C\delta t^2\nu\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + C\delta t\|\boldsymbol{e}_d^{n+\varepsilon}\|_0^2 + C\frac{\delta t\nu^{1/2}}{h}\|\boldsymbol{e}_d^n\|_1 I_0(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h) \\
&\qquad + C\delta t\nu\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 I_1(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2 + C\delta t\nu\|\boldsymbol{e}_c^{n+\varepsilon}\|_1^2 I_1(\boldsymbol{u}^{n+1}, \boldsymbol{v}_h)^2.
\end{aligned}$$

Taking the infimum with respect to $(\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h) \in V_{h,0} \times Q_h \times V_h$, we get

$$\begin{aligned}
&\|\boldsymbol{e}_d^{n+1}\|_0^2 - \|\boldsymbol{e}_d^n\|_0^2 + \|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \nu\delta t\|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2 G_{n+1}^2 \\
&\quad \leqslant C\delta t\big(E_n(h)\big)^2 + Ch^2\big(E_n(h)\big)^2 \\
&\qquad + C\delta t\nu(\|\boldsymbol{e}_d^{n+1}\|_1^2 + \|\boldsymbol{e}_d^n\|_1^2 + \|\boldsymbol{e}_c^{n+1}\|_1^2 + \|\boldsymbol{e}_c^n\|_1^2) E_n(h) \\
&\qquad + C\delta t^2\nu\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + C\delta t\|\boldsymbol{e}_d^{n+\varepsilon}\|_0^2 + C\delta t\nu^{1/2}\|\boldsymbol{e}_d^n\|_1 E_n(h) \\
&\qquad + C\delta t\nu(\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + \|\boldsymbol{e}_c^{n+\varepsilon}\|_1^2)\big(E_n(h)\big)^2.
\end{aligned} \tag{34}$$

Adding up (34) from $n = 0$ to N, using assumption H4, the definition of $E(h)$ and the estimates of Theorem 1, we get

$$\begin{aligned}
&\|\boldsymbol{e}_d^{N+1}\|_0^2 + \sum_{n=0}^N \|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2 \sum_{n=0}^N G_{n+1}^2 \\
&\quad \leqslant C\big(E(h)\big)^2 + C\left(\nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+1}\|_1^2\right)E(h) + C\left(\nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_c^{n+1}\|_1^2\right)E(h) \\
&\qquad + C\nu\delta t^2\sum_{n=0}^N \|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + C\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+\varepsilon}\|_0^2 + C\left(\delta t\nu^{1/2}\sum_{n=0}^N \|\boldsymbol{e}_d^n\|_1\right)E(h) \\
&\qquad + C\left(\nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + \nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_c^{n+\varepsilon}\|_1^2\right)\big(E(h)\big)^2 \\
&\quad \leqslant C\big(E(h)\big)^2 + C\left(\nu\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+1}\|_1^2\right)E(h) + C\delta t^2 E(h) \\
&\qquad + C\nu\delta t^2\sum_{n=0}^N \|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + C\delta t\sum_{n=0}^N \|\boldsymbol{e}_d^{n+\varepsilon}\|_0^2 + C\left(\delta t\nu\sum_{n=0}^N \|\boldsymbol{e}_d^n\|_1^2\right)^{1/2} E(h)
\end{aligned}$$

$$\leqslant C\big(E(h)\big)^2 + C\left(\nu\delta t\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2\right)E(h) + C\delta t^2 E(h)$$
$$+ C\nu\delta t^2\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+\varepsilon}\|_1^2 + C\delta t\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+\varepsilon}\|_0^2 + \frac{1}{2}\left(\delta t\nu\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2\right)$$

since $E(h) \leqslant Ch$, $(E(h))^2 \leqslant E(h)$ for h small enough and $|Z|_{l^1(X)} \leqslant C|Z|_{l^2(X)}$ for any Z and X (see Remark 1). For sufficiently small h, the second term in the right hand side can be passed over to the left hand side, since $E(h)$ tends to 0 as h tends to 0. By the discrete Gronwall inequality, this implies, for sufficiently small δt in the case $\varepsilon = 1$,

$$\|\boldsymbol{e}_d^{N+1}\|_0^2 + \sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^{n}\|_0^2 + \nu\delta t\sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2\sum_{n=0}^{N}G_{n+1}^2 \leqslant C\big(E(h)\big)^2 + C\delta t^2 E(h) \tag{35}$$

and (33) follows. □

Remark 3. For equal order interpolations of degree k, the spatial error function $E(h)$ behaves like h^k, the worst case being that of linear (P_1) and multilinear (Q_1) elements. In general, one always has $E(h) \leqslant Ch$; due to assumption (32), this result proves in particular that the discrete velocities are bounded in $l^\infty(\boldsymbol{H}_0^1(\Omega))$ by a constant independent of δt and h, since

$$\|\boldsymbol{u}_h^{n+1}\|_1 \leqslant \|\boldsymbol{u}(t_{n+1})\|_1 + \|\boldsymbol{e}_c^{n+1}\|_1 + \|\boldsymbol{e}_d^{n+1}\|_1$$
$$\leqslant C\left(1 + \delta t^{1/2} + \left(\frac{(E(h))^2}{\delta t}\right)^{1/2}\right) \leqslant C\left(1 + \left(\frac{h^2}{\delta t}\right)^{1/2}\right) \leqslant C.$$

This is the key point to obtain improved stability estimates in the next section.

Remark 4. The last term in the estimate (33) for the discrete velocity is due to the presence of the convective term in the equations (it is not present in an analysis of the linear Stokes case) and arises from the estimates of the semidiscrete problem. Again, since $E(h) \leqslant Ch$, this extra term is always smaller than δt^2, and the method remains first order accurate in time for the velocity.

3.4. *Improved stability estimate*

As a consequence of the convergence analysis of the previous section, the stability results of Section 3.2 can be improved as follows:

Proposition 2. *Assume* (R1), (R2), (R4) *and* H1–H4 *hold*; *then, there exists a constant* $C > 0$ *independent of* δt *and* h *such that, for small enough* h *and, if* $\varepsilon = 1$, *small enough* δt,

$$\delta t h^2\sum_{n=0}^{N}\|\nabla p_h^{n+1}\|_0^2 \leqslant C. \tag{36}$$

Proof. In a similar way to [7], taking $\boldsymbol{v} = \boldsymbol{u}_h^{n+1}$ in (8) (with $\theta = 1$), $q_h = p_h^{n+1}$ in (9) and $\boldsymbol{\eta}_h = \alpha\boldsymbol{\xi}_h^{n+1}$ in (10), and adding them up, we get

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n}}{\delta t}, \boldsymbol{u}_h^{n+1}\right) + \nu\|\nabla\boldsymbol{u}_h^{n+1}\|_0^2 + \alpha\|\nabla p_h^{n+1} - \boldsymbol{\xi}_h^{n+1}\|_0^2 = \left(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}\right). \tag{37}$$

From (36), it is found that

$$\|\boldsymbol{u}_h^{N+1}\|_0^2 + \sum_{n=0}^{N}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_0^2 + \nu\delta t\sum_{n=0}^{N}\|\boldsymbol{u}_h^{n+1}\|_1^2 + \alpha\delta t\sum_{n=0}^{N}\|\nabla p_h^{n+1} - \boldsymbol{\xi}_h^{n+1}\|_0^2$$
$$\leqslant C\left(\delta t\sum_{n=0}^{N}\|\boldsymbol{f}^{n+1}\|_0^2 + 1\right) \leqslant C\left(\int_0^T\|\boldsymbol{f}(t)\|_0^2\,\mathrm{d}t + 1\right).$$

Thus, the third component $P_{h,3}(\nabla p_h^{n+1}) = \nabla p_h^{n+1} - \boldsymbol{\xi}_h^{n+1}$ in the decomposition of ∇p_h^{n+1} in E_h is bounded; due to assumption (23), it only remains to bound $P_{h,1}(\nabla p_h^{n+1})$, which belongs to $V_{h,0}$. Using the continuity of the forms a and c, the inverse estimate (22) and the result of Remark 2, we have

$$\begin{aligned}\|P_{h,1}(\nabla p_h^{n+1})\|_0^2 &= (\nabla p_h^{n+1}, P_{h,1}(\nabla p_h^{n+1}))\\ &= -\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\delta t}, P_{h,1}(\nabla p_h^{n+1})\right) - a(\boldsymbol{u}_h^{n+1}, P_{h,1}(\nabla p_h^{n+1}))\\ &\quad - c(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, P_{h,1}(\nabla p_h^{n+1})) + (\boldsymbol{f}^{n+1}, P_{h,1}(\nabla p_h^{n+1}))\\ &\leqslant \|P_{h,1}(\nabla p_h^{n+1})\|_0\left(\frac{1}{\delta t}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_0 + \|\boldsymbol{f}^{n+1}\|_0\right.\\ &\qquad\left. + \frac{C\nu}{h}\|\boldsymbol{u}_h^{n+1}\|_1 + \frac{C}{h}\|\boldsymbol{u}_h^{n+\varepsilon}\|_1\|\boldsymbol{u}_h^{n+1}\|_1\right)\\ &\leqslant \|P_{h,1}(\nabla p_h^{n+1})\|_0\left(\frac{1}{\delta t}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_0 + \|\boldsymbol{f}^{n+1}\|_0 + \frac{C}{h}\right).\end{aligned}$$

Dividing this estimate by $\|P_{h,1}(\nabla p_h^{n+1})\|_0$, squaring the result, multiplying by $\delta t h^2$ and adding up for $n = 0, \ldots, N$, we find

$$\delta t h^2\sum_{n=0}^{N}\|\nabla p_h^{n+1}\|_0^2 \leqslant C\left(\frac{h^2}{\delta t}\sum_{n=0}^{N}\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_0 + 1\right) \leqslant C$$

due to the assumed behaviour (32) on the time step size. □

3.5. *Error estimates for the pressure*

We begin this section with an estimate for the discrete pressure gradient:

Proposition 3. *Assume* (R1), (R2), (R3), (R4) *and* H1–H4 *hold*; *then, there exists a constant* $C > 0$ *independent of* δt *and* h *such that, for small enough* h *and, if* $\varepsilon = 1$, *small enough* δt,

$$\delta t h^2\sum_{n=0}^{N}\|\nabla r_d^{n+1}\|_0^2 \leqslant C\left((E(h))^2 + E(h)\delta t^2\right). \tag{38}$$

Proof. From Lemma 1, we have

$$\|\nabla r_d^{n+1}\|_0^2$$
$$\leqslant C\left\{\left(I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h)\right)^2 + \left(I_1(p^{n+1}, q_h)\right)^2 + \frac{1}{\delta t^2}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \frac{\nu}{h^2}\|\boldsymbol{e}_d^{n+1}\|_1^2 + (G_{n+1})^2\right\}.$$

Thus,

$$\delta t h^2 \sum_{n=0}^{N} \|\nabla r_d^{n+1}\|_0^2 \leqslant C\Biggl\{\delta t h^2 \sum_{n=0}^{N}\bigl(I_0(\nabla p^{n+1}, \boldsymbol{\eta}_h)\bigr)^2 + \delta t h^2 \sum_{n=0}^{N}\bigl(I_1(p^{n+1}, q_h)\bigr)^2 + \frac{h^2}{\delta t}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \nu\delta t \sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2 \sum_{n=0}^{N}(G_{n+1})^2\Biggr\}.$$

Taking the infimum with respect to $\boldsymbol{\eta}_h$ and q_h and using (32), this implies

$$\delta t h^2 \sum_{n=0}^{N} \|\nabla r_d^{n+1}\|_0^2 \leqslant C\Biggl\{\delta t \sum_{n=0}^{N}\bigl(E_n(h)\bigr)^2 + \sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n\|_0^2 + \nu\delta t \sum_{n=0}^{N}\|\boldsymbol{e}_d^{n+1}\|_1^2 + \delta t h^2 \sum_{n=0}^{N}(G_{n+1})^2\Biggr\},$$

and (38) follows from (35) and the definition of $E(h)$, (29). □

Since we have obtained error estimates for the fully discrete pressure gradient and the semidiscrete pressure itself, we now present some estimates for the fully discrete pressure solution, which are based on a classical duality argument:

Proposition 4. *Assume* (R1)–(R4) *and* H1–H4 *hold*; *then, there exists a constant $C > 0$ independent of δt and h such that, for small enough h and, if $\varepsilon = 1$, small enough δt,*

$$\delta t^2 \sum_{n=0}^{N} \|r_d^{n+1}\|_0^2 \leqslant C\bigl(\bigl(E(h)\bigr)^2 + \delta t^2\bigr). \tag{39}$$

Proof. Let $z \in \boldsymbol{H}_0^1(\Omega)$ and $\xi \in L_0^2(\Omega)$ be the solution of the following Stokes problem:

$$\begin{aligned} -\Delta z + \nabla\xi &= 0 \quad \text{in } \Omega,\\ \nabla\cdot z &= r_d^{n+1} \quad \text{in } \Omega,\\ z &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{40}$$

Standard results for this problem yield

$$\|z\|_1 \leqslant C\|r_d^{n+1}\|_0, \qquad \|\xi\|_0 \leqslant C\|r_d^{n+1}\|_0. \tag{41}$$

If $z_h \in V_{h,0}$ now satisfies

$$\|z - z_h\|_m \leqslant C h^{1-m}\|z\|_1 \tag{42}$$

for $m = 0, 1$, we have

$$\begin{aligned} \|r_d^{n+1}\|_0^2 &= (r_d^{n+1}, r_d^{n+1}) = (\nabla\cdot z, r_d^{n+1}) = -(z, \nabla r_d^{n+1})\\ &= -(z - z_h, \nabla r_d^{n+1}) - (z_h, \nabla r_d^{n+1})\\ &= -(z - z_h, \nabla r_d^{n+1}) + \left(\frac{\boldsymbol{e}_d^{n+1} - \boldsymbol{e}_d^n}{\delta t}, z_h\right) + \nu(\nabla\boldsymbol{e}_d^{n+1}, \nabla z_h)\\ &\quad - c(\boldsymbol{u}_h^{n+\varepsilon}, \boldsymbol{u}_h^{n+1}, z_h) + c(\boldsymbol{u}^{n+\varepsilon}, \boldsymbol{u}^{n+1}, z_h). \end{aligned}$$

Furthermore,

$$\begin{aligned}
&-(z-z_h,\nabla r_d^{n+1})\leqslant \|z-z_h\|_0\|\nabla r_d^{n+1}\|_0\leqslant Ch\|\nabla r_d^{n+1}\|_0\|z\|_1\leqslant Ch\|\nabla r_d^{n+1}\|_0\|r_d^{n+1}\|_0,\\
&\left(\frac{\boldsymbol{e}_d^{n+1}-\boldsymbol{e}_d^n}{\delta t},z_h\right)\leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1}-\boldsymbol{e}_d^n\|_0\|z_h\|_0\leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1}-\boldsymbol{e}_d^n\|_0(\|z-z_h\|_0+\|z\|_0)\\
&\qquad\leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1}-\boldsymbol{e}_d^n\|_0(Ch\|z\|_1+C\|z\|_1)\leqslant \frac{1}{\delta t}\|\boldsymbol{e}_d^{n+1}-\boldsymbol{e}_d^n\|_0\|r_d^{n+1}\|_0,\\
&\nu(\nabla\boldsymbol{e}_d^{n+1},\nabla z_h)\leqslant C\nu^{1/2}\|\boldsymbol{e}_d^{n+1}\|_1\|z_h\|_1\leqslant C\nu^{1/2}\|\boldsymbol{e}_d^{n+1}\|_1\|r_d^{n+1}\|_0,\\
&-c(\boldsymbol{u}_h^{n+\varepsilon},\boldsymbol{u}_h^{n+1},z_h)+c(\boldsymbol{u}^{n+\varepsilon},\boldsymbol{u}^{n+1},z_h)=c(\boldsymbol{u}^{n+\varepsilon},\boldsymbol{e}_d^{n+1},z_h)+c(\boldsymbol{e}_d^{n+\varepsilon},\boldsymbol{u}_h^{n+1},z_h)\\
&\qquad\leqslant C(\|\boldsymbol{u}^{n+\varepsilon}\|_1\|\boldsymbol{e}_d^{n+1}\|_1+\|\boldsymbol{e}_d^{n+\varepsilon}\|_1\|\boldsymbol{u}_h^{n+1}\|_1)\|z_h\|_1\\
&\qquad\leqslant C(\|\boldsymbol{e}_d^{n+1}\|_1+\|\boldsymbol{e}_d^{n+\varepsilon}\|_1)(\|z-z_h\|_1+\|z\|_1)\\
&\qquad\leqslant C\nu^{1/2}(\|\boldsymbol{e}_d^{n+1}\|_1+\|\boldsymbol{e}_d^{n+\varepsilon}\|_1)\|r_d^{n+1}\|_0.
\end{aligned}$$

Estimate (39) is obtained dividing by $\|r_d^{n+1}\|_0$ throughout, squaring the result, multiplying by δt^2 and adding up from $n=0$ to N, due to (35) and (38).

3.6. Global error behaviour

As a consequence of the previous results, we have:

Corolary 1. *Assume* (R1)–(R4) *and* H1–H4 *hold*; *assume also that, for* $n=0,\dots,N$, $\boldsymbol{u}^{n+1}\in \boldsymbol{H}^r(\Omega)$, $r\geqslant 2$ *and* $p^{n+1}\in H^s(\Omega)$, $s\geqslant 1$, *and that they are uniformly bounded in these spaces. Then, there exists a constant* $C>0$ *independent of* δt *and* h *such that, for small enough* h *and, if* $\varepsilon=1$, *small enough* δt,

$$\|\boldsymbol{e}^{N+1}\|_0^2+\nu\delta t\sum_{n=0}^{N}\|\boldsymbol{e}^{n+1}\|_1^2+\delta t^2\sum_{n=0}^{N}\|r^{n+1}\|_0^2\leqslant C(\delta t^2+h^{2k}), \tag{43}$$

where $k=\min(r-1,s,k_v,k_q+1)$.

Proof. This estimate follows from Theorems 1 and 3, Propositions 1 and 4, assumption (32), the regularity assumed of the semidiscrete solution $(\boldsymbol{u}^{n+1},p^{n+1})$ and the approximating properties of the finite element spaces considered. □

Remark 5. The condition $\delta t\geqslant Ch^2$ arises due to the proof technique employed, which deals with the temporal error first and then the spatial error. However, according to the results of Corollary 1, accuracy considerations indicate that, when equal order interpolation of degree k is used, δt should be of order h^k; for linear (P_1) and bilinear (Q_1) elements, one has $k=1$, so that assumption H4 is fulfilled. Even for quadratic (P_2) and biquadratic (Q_2) elements, one still has $k=2$, making H4 acceptable.

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