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Analysis of a pressure-stabilized finite element

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equations

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Summary. The purpose of this paper is to analyze a finite element approximation of the stationary Navier-Stokes equations that allows the use of equal velocity-pressure interpolation. The idea is to introduce as unknown of the discrete problem the projection of the pressure gradient (multiplied by suitable algorithmic parameters) onto the space of continuous vector fields. The difference between these two vectors (pressure gradient and projection) is introduced in the continuity equation. The resulting formulation is shown to be stable and optimally convergent, both in a norm associated to the problem and in the L^2 norm for both velocities and pressure. This is proved first for the Stokes problem, and then it is extended to the nonlinear case. All the analysis relies on an inf-sup condition that is much weaker than for the standard Galerkin approximation, in spite of the fact that the present method is only a minor modification of this.

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1 Introduction

One of the most important subjects of research in the finite element approximation of the incompressible Navier-Stokes equations concerns the velocity-pressure interpolation. When the standard Galerkin formulation is used, the spaces chosen for the approximation of these variables have to satisfy the classical inf-sup or Babuška-Brezzi condition (see, e.g., [6]). Much

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effort has been invested in the development of approximation methods able to avoid the need to satisfy such stability requirement. In particular, one of the lines of research has been directed towards the development of methods accommodating *equal* velocity-pressure interpolation, considering this as the simplest choice from the computational point of view. Examples of these type of methods are those of Brezzi & Douglas [4], Douglas & Wang [12], the Galerkin/least-squares (GLS) technique of Hughes, Franca *et al.* [14,15,17] and first-order system least-squares methods (see e.g. [1] and references therein).

In this paper we analyze a stabilization technique of the above type, that is, a finite element formulation for the incompressible Navier-Stokes equations allowing the use of equal interpolation for the velocity and the pressure. This technique was developed for the Stokes problem in [10]. Now we extend it to the nonlinear stationary Navier-Stokes equations. Also, the linear problem is generalized to include the case in which the numerical parameters that define the method are computed elementwise. This allows to weaken the quasi-uniformity of the finite element partitions assumed in [10]. Furthermore, optimal L^2 error estimates are obtained.

The motivation for the formulation to be analyzed herein is the stabilization effect found in some fractional step methods when the pressure is computed via a Poisson equation (see the discussion in [10]). It can be shown that this enhancement comes from a modification of the continuity equation, which introduces the difference between two discrete pressure Laplacians computed in a different manner. The method proposed here has been designed to inherit this term, although it is applied to the stationary problem. The main idea of it consists in introducing as a new unknown of the discrete problem the projection of the discrete pressure gradient onto the velocity space. The difference between the pressure gradient and its projection is then introduced in the continuity equation, after being multiplied by adequate algorithmic parameters.

Here we present a 'classical' numerical analysis of the method outlined above, this meaning that the classical results known for the Galerkin approach using elements satisfying the inf-sup condition are recovered using the stabilized formulation. These results include a stability estimate and a convergence result in what we call 'natural' norm of the problem, as well as error estimates in L^2 norms using duality arguments. These results are proved first for the Stokes problem, and then they are extended to the nonlinear case.

Although our results are classical, they are based on non-standard arguments. To analyze the stability of the finite element approximation, we introduce a technique based on the decomposition of the vector space that contains both velocities and pressure gradients (multiplied by numerical parameters) into three orthogonal subspaces. We analyze stability for each of the components of the pressure gradient separately. One of these components can be bounded by the original equations and another by the stabilization term introduced. However, the stability of the third component must be explicitly required. As shown in [10], this leads to a *weak* inf-sup condition that turns out to be verified using equal velocity-pressure interpolation. We assume throughout that this condition holds.

An aspect that deserves special attention of the method analyzed here is its numerical implementation. As it has been said, the method introduces a new vector unknown to the discrete problem, thus increasing substantially the number of nodal unknowns of the final algebraic system. However, this new vector can be eliminated iteratively, keeping the matrix of the system unchanged. To our knowledge, the present formulation is the *simplest modification of the Galerkin approach* allowing equal velocity-pressure interpolations for general finite element interpolations. Likewise, its extension to the transient case is straightforward and can be made even computationally simpler if the projection of the pressure gradient is treated explicitly [11].

It has to be remarked that the only purpose of the stabilization technique presented here is to stabilize the pressure. The instabilities due to the convective term when the viscosity is very small are not considered in our formulation and thus they have to be treated by using other stabilization mechanisms, such as those studied in [9,13,21,22] (which also allow to use equal velocity-pressure interpolation). From the theoretical point of view, this is reflected by the fact that our estimates may depend on the inverse of the viscosity, as for the standard Galerkin method.

Let us summarize now the results to be proved in the following. In Sect. 2 we describe the main assumptions on the finite element discretization, as well as the discrete problem to be solved. In Sect. 3 we analyze the linear problem. Stability and an optimal order error estimate are proved in a mesh dependent norm, consisting of the H^1 velocity norm plus the L^2 norm of the pressure gradient multiplied by mesh dependent parameters and its projection onto the velocity space. Next, L^2 error estimates for both the velocity and the pressure are proved using duality arguments. In Sect. 4 these results are extended to the Navier-Stokes equations using the classical theory of approximation of branches of non-singular solutions [7,16]. The nearness between the standard Galerkin method and the present stabilization technique is in particular reflected by the fact that standard results for the former carry over to the latter with minor modifications.

2 Statement of the problem

2.1 The stationary incompressible Navier-Stokes equations

Let us consider the classical stationary Navier-Stokes equations for an incompressible fluid. Let Ω be an open, bounded and polyhedral domain of $\mathbb{R}^d$, where $d = 2$ or 3 is the number of space dimensions, and $\Gamma = \partial\Omega$ its boundary. The Navier-Stokes problem consists in finding a velocity $\boldsymbol{u}$ and a scaled pressure p (the kinematic pressure divided by the kinematic viscosity) such that

$$\lambda \left[\frac{1}{2}(\nabla \cdot \boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right] - \Delta \boldsymbol{u} + \nabla p = \lambda \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3}$$

where λ is the inverse of the kinematic viscosity and $\boldsymbol{f}$ is the force vector. We have considered the homogeneous Dirichlet boundary condition (3) for simplicity. The expression adopted for the nonlinear term will simplify the analysis, although for the continuous problem it could be replaced by $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$.

To write the weak form of problem (1)-(3) we need to introduce some notation. As usual, we denote by $H^m(\omega)$ the Sobolev space of m-th order in a set ω, consisting of functions whose distributional derivatives of order up to m belong to $L^2(\omega)$, and by $H_0^1(\omega)$ the subspace of $H^1(\omega)$ of functions with zero trace on Γ. A bold character is used for the vector counterpart of these spaces. The L^2 scalar product is denoted by $(\cdot,\cdot)_\omega$, and the H^m norm by $\|\cdot\|_{m,\omega}$. The subscript m is omitted when $m = 0$ and so is ω when it is Ω.

Let us now consider the spaces

$$\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega), \qquad Q = \left\{ q \in L^2(\Omega) \mid \int_\Omega q \, \mathrm{d}\Omega = 0 \right\}, \tag{4}$$

and the forms

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \tag{5}$$

$$b(q, \boldsymbol{v}) := (q, \nabla \cdot \boldsymbol{v}), \tag{6}$$

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \left(\frac{1}{2}(\nabla \cdot \boldsymbol{u})\boldsymbol{v} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w} \right), \tag{7}$$

with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ and $q \in Q$. All these forms are continuous and c is skew-symmetric in its last two arguments, that is

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -c(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}). \tag{8}$$

The norms of a and c are denoted by N_a and N_c, respectively. Thus,

$$a(\boldsymbol{u}, \boldsymbol{v}) \leq N_a \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1, \tag{9}$$

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \leq N_c \|\boldsymbol{u}\|_1 \|\boldsymbol{v}\|_1 \|\boldsymbol{w}\|_1. \tag{10}$$

Moreover, a is coercive as a consequence of the Poincaré-Friedrics inequality, and b satisfies the inf-sup or Babuška-Brezzi condition for the spaces $\boldsymbol{V}$ and Q introduced in (4). These conditions can be written as follows: there exist positive constants K_a and K_b such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq K_a \|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{11}$$

$$\inf_{q \in Q_1} \sup_{\boldsymbol{v} \in \boldsymbol{V}_1} b(q, \boldsymbol{v}) \geq K_b, \tag{12}$$

where Q_1 and $\boldsymbol{V}_1$ are defined as

$$Q_1 = \{q \in Q \mid \|q\| = 1\}, \qquad \boldsymbol{V}_1 = \{\boldsymbol{v} \in \boldsymbol{V} \mid \|\boldsymbol{v}\|_1 = 1\}.$$

If $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\boldsymbol{V}$ and its topological dual space $\boldsymbol{V}' = \boldsymbol{H}^{-1}(\Omega)$ where $\boldsymbol{f}$ is assumed to belong, the weak form of problem (1)-(3) consists in finding $[\boldsymbol{u}, p] \in \boldsymbol{V} \times Q$ such that

$$\lambda\, c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) + b(q, \boldsymbol{u}) = \lambda \langle \boldsymbol{f}, \boldsymbol{v} \rangle \tag{13}$$

for all $[\boldsymbol{v}, q] \in \boldsymbol{V} \times Q$. Since we assume that Ω is Lipschitz continuous, problem (13) has at least one solution (see [16]), which satisfies (1)-(3) in the sense of distributions. Uniqueness only holds for sufficiently small data or sufficiently large viscosity. In particular, the solution is unique if

$$\chi := \lambda^2 N_c K_a^{-2} \|\boldsymbol{f}\|_{-1} < 1. \tag{14}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ and $q = p$ in (13) and using the coercivity of a and the skew-symmetry of c, it follows that solutions to (13) satisfy the stability estimate

$$\|\boldsymbol{u}\|_1 \leq \lambda K_a^{-1} \|\boldsymbol{f}\|_{-1}. \tag{15}$$

If instead of having $\boldsymbol{f} \in \boldsymbol{V}' = \boldsymbol{H}^{-1}(\Omega)$ we require $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and Γ is sufficiently smooth, it is known that solutions of problem (13) verifies $\boldsymbol{u} \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ and $p \in Q \cap H^1(\Omega)$ (see e.g. [16]). This regularity for $\boldsymbol{u}$ will be needed only in Sects. 3.3 and 4.3 when proving L^2 error estimates. Likewise, convergence will be proven for smooth enough solutions. We define

Definition 1 *(a) The stationary incompressible Navier-Stokes equations (1)-(3) are called* regular *if* $[\boldsymbol{u}, p] \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega) \times Q \cap H^1(\Omega)$ *whenever* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$*, and there exists a constant* $C_r > 0$ *such that*

$$(16) \qquad \|\boldsymbol{u}\|_2 + \|p\|_1 \leq C_r \|\boldsymbol{f}\|.$$

(b) Let $k \geq 1$ *be an integer. Solutions* $[\boldsymbol{u}, p]$ *to (13) are called* k-regular *if* $[\boldsymbol{u}, p] \in \boldsymbol{V} \cap \boldsymbol{H}^{k+1}(\Omega) \times Q \cap H^k(\Omega)$.

It is known for example that problem (1)-(3) is regular when Ω is a bounded and *convex* polygon when $d = 2$ or when Ω is of class $\mathcal{C}^2$ in any space dimension [20]. Note that 1-regular solutions are not necessarily regular since the shift (16) may not hold.

2.2 Finite element discretization

Let $\mathcal{T}_h$ denote a finite element partition of the domain Ω of diameter h. For simplicity, we assume that all the element domains $K \in \mathcal{T}_h$ are the image of a reference element $\hat{K}$ through a polynomial mapping $\boldsymbol{F}_K$, affine for simplicial elements, bilinear for quadrilaterals and trilinear for hexahedra. On $\hat{K}$ we define the polynomial spaces $R_k(\hat{K})$ where, as usual, $R_k = P_k$ for simplicial elements and $R_k = Q_k$ for quadrilaterals and hexahedra. The finite element spaces we need are

$$(17) \quad Q_h = \left\{ q_h \in \mathcal{C}^0(\Omega) \mid \ q_h|_K = \hat{q} \circ \boldsymbol{F}_K^{-1}, \ \hat{q} \in R_{k_\mathrm{q}}(\hat{K}), \ K \in \mathcal{T}_h \right\},$$

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in [\mathcal{C}^0(\Omega)]^d \mid \ \boldsymbol{v}_h|_K = \hat{\boldsymbol{v}} \circ \boldsymbol{F}_K^{-1}, \right.$$

$$(18) \qquad \left. \hat{\boldsymbol{v}} \in [R_{k_\mathrm{v}}(\hat{K})]^d, \ K \in \mathcal{T}_h \right\},$$

$$(19) \ \boldsymbol{V}_{h,0} = \{ \boldsymbol{v}_h \in \boldsymbol{V}_h \mid \ \boldsymbol{v}_h|_\Gamma = \boldsymbol{0} \}.$$

Notice that all these finite element spaces are referred to the same partition and are made up with continuous functions.

The standard Galerkin finite element counterpart of problem (13) can now be written as follows: find $[\boldsymbol{u}_h, p_h] \in \boldsymbol{V}_{h,0} \times Q_h$ such that

$$(20) \ \lambda \, c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) + b(q_h, \boldsymbol{u}_h) = \lambda \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle,$$

for all $[\boldsymbol{v}_h, q_h] \in \boldsymbol{V}_{h,0} \times Q_h$. It is well known that if this formulation is used the velocity and pressure finite element spaces must satisfy the discrete counterpart of the inf-sup condition (12). For the finite element spaces (17)-(19) this happens if $k_\mathrm{v} = k_\mathrm{q} + 1$, i.e., for Taylor-Hood type elements [2,5, 18] This condition is not necessary using the method described next.

Let $\alpha_K,\ K \in \mathcal{T}_h$, be a family of mesh parameters depending on the element sizes h_K. The modification of problem (20) that we propose is: find $[\boldsymbol{u}_h, p_h, \tilde{\boldsymbol{u}}_h] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$ such that

$$\lambda\, c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) = \lambda \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle, \tag{21}$$

$$\sum_K \alpha_K^2 (\nabla p_h, \nabla q_h)_K - \sum_K \alpha_K (\tilde{\boldsymbol{u}}_h, \nabla q_h)_K + b(q_h, \boldsymbol{u}_h) = 0, \tag{22}$$

$$-\sum_K \alpha_K (\nabla p_h, \tilde{\boldsymbol{v}}_h)_K + (\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) = 0, \tag{23}$$

for all $[\boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$.

For a function $q \in H^1(\Omega)$ let us define $\nabla^h q$ by

$$\nabla^h q|_K = \alpha_K \nabla q|_K, \quad K \in \mathcal{T}_h. \tag{24}$$

From (23) it is seen that $\tilde{\boldsymbol{u}}_h$ is the projection of $\nabla^h p_h$ onto $\boldsymbol{V}_h$.

Let us also introduce the form

$$\begin{aligned} &\mathcal{B}(\lambda; \boldsymbol{u}_h, p_h, \tilde{\boldsymbol{u}}_h; \boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h) \\ &\quad := \lambda\, c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(p_h, \boldsymbol{v}_h) + b(q_h, \boldsymbol{u}_h) \\ &\qquad + (\nabla^h p_h - \tilde{\boldsymbol{u}}_h, \nabla^h q_h - \tilde{\boldsymbol{v}}_h). \end{aligned} \tag{25}$$

Problem (21)-(23) can be written now as: find $[\boldsymbol{u}_h, p_h, \tilde{\boldsymbol{u}}_h] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$ such that

$$\mathcal{B}(\lambda; \boldsymbol{u}_h, p_h, \tilde{\boldsymbol{u}}_h; \boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h) = \lambda \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \tag{26}$$

for all $[\boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$.

Let us describe now the assumptions we need on the family of finite element partitions $\{\mathcal{T}_h\}_{h>0}$. First of all, we assume that it is non-degenerate, that is, there exists a constant $\sigma > 0$ such that for all $h > 0$

$$\operatorname{diam}(B_K) \geq \sigma \operatorname{diam}(K)\ \forall K \in \mathcal{T}_h, \tag{27}$$

where B_K is the largest ball contained in $K \in \mathcal{T}_h$. Condition (27) is needed in order to have the following inverse estimates (see, e.g., [3]): there exist positive constants C_1 and C_2 such that

$$\|v_h\|_{1,K} \leq C_1 h_K^{-1} \|v_h\|_{0,K}, \tag{28}$$

$$\|v_h\|_{L^\infty(K)} \leq C_2 h_K^{-d/2} \|v_h\|_{0,K}, \tag{29}$$

where v_h is a function of any of the finite element spaces (17)-(19).

From now onwards we use C, possibly with subscripts, to denote a positive constant independent of the mesh size and λ, not necessarily the same at different occurrences.

Since we always have that

$$\|v_h\|_{0,K} \leq C h_K^{d/2} \|v_h\|_{L^\infty(K)},$$

from (29) it follows that

$$C_1 h_K^{d/2} \|v_h\|_{L^\infty(K)} \leq \|v_h\|_{0,K} \leq C_2 h_K^{d/2} \|v_h\|_{L^\infty(K)}. \tag{30}$$

Let n_{nod} be the number of nodes per element and N_i the shape function associated to the i-th node. If v_i is the value of a function v_h at this node and we are given a set of nodal parameters $\beta = \{\beta_1, ..., \beta_{n_{\text{nod}}}\}$, we define

$$\Pi_K(\beta v_h) := \sum_{i=1}^{n_{\text{nod}}} N_i \beta_i v_i. \tag{31}$$

Let $\max_K |\beta|$ be the maximum of the absolute value of the nodal parameters β in an element K. It is easy to see that

$$\|\Pi_K(\beta v_h)\|_{L^\infty(K)} \leq C \max_K |\beta| \|v_h\|_{L^\infty(K)},$$

which, by virtue of (30), implies that

$$\|\Pi_K(\beta v_h)\|_{0,K} \leq C \max_K |\beta| \|v_h\|_{0,K}. \tag{32}$$

We shall need a rather technical condition on the set of element parameters α_K, which is reflected by a condition on $\{\mathcal{T}_h\}_{h>0}$, since α_K depends on h_K, as we shall see. Let M_i be a macroelement obtained from the union of the elements to which a node i belongs. Let us define

$$\bar{\alpha}_i := \frac{1}{\text{meas} M_i} \sum_{K \subset M_i} \text{meas} K \alpha_K, \tag{33}$$

which is nothing but a weighted average of the parameters α_K. We need this average to converge to these parameters as the mesh is refined. More precisely, we need a continuously graded family of meshes, a concept introduced in the following:

Definition 2 *The family of finite element meshes $\{\mathcal{T}_h\}_{h>0}$ is* continuously graded *if there exists a function $\delta = \delta(h)$, tending to 0 as $h \to 0$, such that*

$$\max_{K \in \mathcal{T}_h} \left(\max_K \left| 1 - \frac{\bar{\alpha}}{\alpha_K} \right| \right) \leq \delta(h). \tag{34}$$

This condition is less restrictive than the quasi-uniformity of the family $\{\mathcal{T}_h\}_{h>0}$. It allows for instance local mesh refinement. What it does not permit, for example, is a constant ratio between the sizes of the elements sharing a fixed nodal point as the mesh is refined. From the practical point of view this is not a restriction, since the element sizes given to mesh generators are usually continuous functions obtained from the interpolation of element sizes at certain given points of the computational domain.

Condition (34) implies in particular that

$$\max_K |\bar{\alpha}| \leq (1 + \delta(h))\, \alpha_K. \tag{35}$$

Our stability and convergence analysis of the following sections will be strongly based on a decomposition of the pressure gradient that we describe next. Let $\nabla^h Q_h$ denote the space of vector functions in $\boldsymbol{L}^2(\Omega)$ which are of the form $\nabla^h q_h$, with $q_h \in Q_h$, and consider the vector space

$$\boldsymbol{E}_h := \boldsymbol{V}_h + \nabla^h Q_h = \boldsymbol{E}_{h,1} \oplus \boldsymbol{E}_{h,2} \oplus \boldsymbol{E}_{h,3}, \tag{36}$$

where $\boldsymbol{E}_{h,i},\ i = 1, 2, 3$, are three mutually L^2 orthogonal subspaces defined as

$$\boldsymbol{E}_{h,1} := \boldsymbol{V}_{h,0}, \tag{37}$$

$$\boldsymbol{E}_{h,2} := \boldsymbol{V}_{h,0}^{\perp} \cap \boldsymbol{V}_h, \tag{38}$$

$$\boldsymbol{E}_{h,3} := \boldsymbol{V}_h^{\perp} \cap \boldsymbol{E}_h. \tag{39}$$

Let us denote by $P_{h,i}$ the orthogonal projection from $\boldsymbol{E}_h$ to $\boldsymbol{E}_{h,i}$, and $P_{h,ij} := P_{h,i} + P_{h,j}$, $i, j = 1, 2, 3$. Also, we denote $\boldsymbol{E}_{h,ij} := \boldsymbol{E}_{h,i} \oplus \boldsymbol{E}_{h,j}$. In order to prove that the pressure gradient in problem (26) is stable, we shall bound independently the three terms in the decomposition

$$\nabla^h p_h = P_{h,1}(\nabla^h p_h) + P_{h,2}(\nabla^h p_h) + P_{h,3}(\nabla^h p_h). \tag{40}$$

To obtain error estimates for solutions of problem (26) we shall make use of the approximation properties of the spaces $\boldsymbol{V}_{h,0},\ Q_h$ and $\boldsymbol{V}_h$. These can be written as follows. If $\boldsymbol{v} \in \boldsymbol{H}^r(\Omega) \cap \boldsymbol{V}$, $r \geq 1$, and $q \in H^s(\Omega) \cap Q$, $s \geq 1$, there exist $I_{h,1}(\boldsymbol{v}) \in \boldsymbol{V}_{h,0},\ I_{h,2}(q) \in Q_h$ and $I_{h,3}(\nabla q) \in \boldsymbol{V}_h$ such that

$$\|\boldsymbol{v} - I_{h,1}(\boldsymbol{v})\|_m \leq C_1 h^{k_1} \|\boldsymbol{v}\|_{k_1},\ k_1 = \min\{r, k_{\mathrm{v}} + 1\} - m, \tag{41}$$

$$\|q - I_{h,2}(q)\|_m \leq C_2 h^{k_2} \|q\|_{k_2},\ k_2 = \min\{s, k_{\mathrm{q}} + 1\} - m, \tag{42}$$

$$\begin{aligned} &\|\nabla^h q - I_{h,3}(\nabla^h q)\|_{m,K} \leq C_3 h^{k_3} \|\nabla^h q\|_{k_3,K}, \\ &\qquad k_3 = \min\{s - 1, k_{\mathrm{v}} + 1\} - m. \end{aligned} \tag{43}$$

Notice that the last estimate is local, since $\nabla^h q$ is in general discontinuous.

The next ingredient we need is an inf-sup condition. The bound for the first component of the pressure gradient decomposition in (40) can be obtained from the momentum equation (21), whereas the third component can be bounded only assuming $\alpha_K > 0$. Thus, the stability provided by the method in comparison with the standard problem (20) is precisely in the control over the term $P_{h,3}(\nabla^h p_h)$, that is, the component of the pressure gradient orthogonal to the space of continuous vector fields $\boldsymbol{V}_h$.

The second component in (40) deserves special attention. It depends on the properties of the finite element spaces, and not on the problem actually solved. We assume that there is a positive constant K_2' such that

$$\|\nabla^h q_h\| \leq K_2' \|P_{h,13}(\nabla^h q_h)\| \qquad \forall q_h \in Q_h, \tag{44}$$

which means that $\|P_{h,2}(\nabla^h q_h)\|$ can be bounded by $\|P_{h,13}(\nabla^h q_h)\|$. This condition is equivalent to the existence of a constant $K_2 = 1/K_2' > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{E}_{h,13}} \frac{(\nabla q_h, \boldsymbol{v}_h)}{\|\nabla q_h\| \|\boldsymbol{v}_h\|} \geq K_2, \tag{45}$$

which is similar to the inf-sup condition of the standard problem, although much weaker and, in particular, verified when equal interpolation is used. This was proved in [10] for P_k interpolations (and a very similar proof can also be applied to the Q_1 case) under a mild condition on the family of finite element meshes, similar to that encountered in [2] for Taylor-Hood elements. The analysis in [10] is based on a generalization of the macroelement technique presented in [19].

Finally, we need to introduce a further assumption, now on the behavior of α_K: there exists constants α_0 and α_1, independent of h_K, such that

$$\alpha_0 h_K \leq \alpha_K \leq \alpha_1 h_K \qquad \forall K \in \mathcal{T}_h. \tag{46}$$

This completes the set of assumptions on the discrete finite element problem (26).

3 Stability and convergence I: Stokes problem

In this section we shall consider the Stokes problem, that is, the problem obtained with $\lambda = 0$ in (25).

Theorem 1 *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ of finite element partitions is non-degenerate, continuously graded and such that (45) and (46) hold. Then, for h small enough, there exists a unique solution to problem (26) that verifies the stability estimate*

$$\|\boldsymbol{u}_h\|_1 + \|\nabla^h p_h\| + \|\tilde{\boldsymbol{u}}_h\| \leq C\lambda \|\boldsymbol{f}\|_{-1}, \tag{47}$$

for a constant C independent of h.

Proof. Since the problem is linear and finite-dimensional, it is enough to prove that (47) holds. From the definition of the bilinear form $\mathcal{B}$ for $\lambda = 0$ it is easy to see that

$$\begin{aligned}\mathcal{B}(0;\boldsymbol{u}_h,p_h,\tilde{\boldsymbol{u}}_h;\boldsymbol{u}_h,p_h,\tilde{\boldsymbol{u}}_h) &= a(\boldsymbol{u}_h,\boldsymbol{u}_h) + \|\nabla^h p_h - \tilde{\boldsymbol{u}}_h\|^2 \\ &\le \lambda\|\boldsymbol{f}\|_{-1}\|\boldsymbol{u}_h\|_1.\end{aligned} \tag{48}$$

From the coercivity of the bilinear form a (see (11)) it follows that

$$\|\boldsymbol{u}_h\|_1 \le \lambda K_a^{-1}\|\boldsymbol{f}\|_{-1}. \tag{49}$$

On the other hand, (23) can now be written as $\tilde{\boldsymbol{u}}_h = P_{h,12}(\nabla^h p_h)$, and therefore from (48) it follows that

$$\left\|P_{h,3}(\nabla^h p_h)\right\|^2 = \|\nabla^h p_h - \tilde{\boldsymbol{u}}_h\|^2 \le \lambda\|\boldsymbol{f}\|_{-1}\|\boldsymbol{u}_h\|_1,$$

and, from estimate (49),

$$\left\|P_{h,3}(\nabla^h p_h)\right\| \le K_a^{-1/2}\lambda\|\boldsymbol{f}\|_{-1}. \tag{50}$$

If $\boldsymbol{v}_h \in \boldsymbol{E}_h$, let us define $\Pi(\bar{\alpha}\boldsymbol{v}_h)$ by $\Pi(\bar{\alpha}\boldsymbol{v}_h)|_K = \Pi_K(\bar{\alpha}\boldsymbol{v}_h)$, with Π_K defined in (31). We have that

$$\begin{aligned}\left\|P_{h,1}(\nabla^h p_h)\right\|^2 &= \sum_K \left(\alpha_K \nabla p_h, P_{h,1}(\nabla^h p_h)\right)_K \\ &= \left(\nabla p_h, \Pi\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right) \\ &\quad + \sum_K \left(\nabla p_h, \alpha_K P_{h,1}(\nabla^h p_h) - \Pi_K\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right)_K \\ &=: T_1 + T_2.\end{aligned} \tag{51}$$

Let us bound the terms T_1 and T_2. From the momentum equation (21) (without the nonlinear term) we have that

$$\begin{aligned}T_1 &= \lambda\left\langle \boldsymbol{f}, \Pi\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right\rangle - a\left(\boldsymbol{u}_h, \Pi\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right) \\ &\le \left[\lambda\|\boldsymbol{f}\|_{-1} + N_a\|\boldsymbol{u}_h\|_1\right]\left\|\Pi\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right\|_1 .\end{aligned} \tag{52}$$

Using the inverse estimate (28) and inequalities (32) and (35), together with assumption (46) on α_K, we obtain

$$\left\|\Pi\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right\|_1^2 = \sum_K \left\|\Pi_K\left(\bar{\alpha}P_{h,1}(\nabla^h p_h)\right)\right\|_{1,K}^2$$

$$\leq \sum_K \frac{C}{h_K^2} \left\| \Pi_K \left(\bar{\alpha} P_{h,1}(\nabla^h p_h) \right) \right\|_{0,K}^2$$

$$\leq \sum_K \frac{C}{h_K^2} \max_K |\bar{\alpha}|^2 \left\| P_{h,1}(\nabla^h p_h) \right\|_{0,K}^2$$

$$\leq C \left\| P_{h,1}(\nabla^h p_h) \right\|^2 . \tag{53}$$

Using this in (52), together with inequalities (32) and (49), it follows that

$$T_1 \leq C\lambda \|\boldsymbol{f}\|_{-1} \left\| P_{h,1}(\nabla^h p_h) \right\| . \tag{54}$$

To bound T_2 observe first that

$$\frac{1}{\alpha_K} \left(\alpha_K P_{h,1}(\nabla^h p_h) - \Pi_K \left(\bar{\alpha} P_{h,1}(\nabla^h p_h) \right) \right)$$

$$= \Pi_K \left(\beta^K P_{h,1}(\nabla^h p_h) \right) , \tag{55}$$

with $\beta_i^K = 1 - \bar{\alpha}_i / \alpha_K$, and therefore we obtain, using inequality (32), assumption (34), the weak inf-sup condition (44) and the bound (50),

$$T_2 = \sum_K \left(\alpha_K \nabla p_h, \Pi_K \left(\beta^K P_{h,1}(\nabla^h p_h) \right) \right)_K$$

$$\leq \|\nabla^h p_h\| \left(\sum_K \left\| \Pi_K \left(\beta^K P_{h,1}(\nabla^h p_h) \right) \right\|_{0,K}^2 \right)^{1/2}$$

$$\leq C\delta(h) \|\nabla^h p_h\| \left\| P_{h,1}(\nabla^h p_h) \right\|$$

$$\leq C\delta(h) \left(\left\| P_{h,1}(\nabla^h p_h) \right\|^2 + \left\| P_{h,1}(\nabla^h p_h) \right\| \left\| P_{h,3}(\nabla^h p_h) \right\| \right)$$

$$\leq C\delta(h) \left\| P_{h,1}(\nabla^h p_h) \right\|^2 + C\lambda \|\boldsymbol{f}\|_{-1} \left\| P_{h,1}(\nabla^h p_h) \right\| . \tag{56}$$

Using inequalities (54) and (56) in (51) we have that

$$(1 - C\delta(h)) \left\| P_{h,1}(\nabla^h p_h) \right\|^2 \leq C\lambda \|\boldsymbol{f}\|_{-1} \left\| P_{h,1}(\nabla^h p_h) \right\| ,$$

which for h small enough implies that

$$\left\| P_{h,1}(\nabla^h p_h) \right\| \leq C\lambda \|\boldsymbol{f}\|_{-1} . \tag{57}$$

This, together with (50) and the weak stability condition (44) implies the bound for $\nabla^h p_h$ in (47). Finally, the bound for $\tilde{\boldsymbol{u}}_h$ follows from the fact that $\tilde{\boldsymbol{u}}_h = P_{h,12}(\nabla^h p_h)$. □

A similar stability estimate to that of (47) can be obtained replacing α_K by h_K and using only the condition $\alpha_K \geq \alpha_0 h_K$ instead of (46) (see [10]). Observe that we have only used the fact that $\alpha_K \leq \alpha_1 h_K$ in (53). However, this part of assumption (46) is needed also for the error estimate presented next, which establishes convergence of the solution of problem (26) to the solution of the continuous Stokes problem in the norm in which stability has been proven, that is, in the H^1 norm for the velocity and the L^2 norm of the mesh dependent pressure gradient defined in (24) and its projection onto $\boldsymbol{V}_h$. This is what can be called the 'natural' norm of the method.

Theorem 2 *Under the same conditions as in Theorem 1, suppose also that the Stokes problem is k-regular, with $k \geq 1$. Then, for h small enough, the solution of problem (26) satisfies the error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\nabla^h p - \nabla^h p_h\| + \|\nabla^h p - \tilde{\boldsymbol{u}}_h\| \leq C h^r, \tag{58}$$

where $r = \min\{k, k_{\rm v}, k_{\rm q}+1\}$ and $\boldsymbol{u}$ and p are the solution of the continuous problem (13) (without the nonlinear term).

Proof. The proof is essentially the same as in Theorem 3 in [10]. The main difference is due to the fact that the parameters α_K are now allowed to change from element to element. This only affects the following bound:

$$\begin{aligned}\|P_{h,1}(\nabla^h q_h) \;-\; & P_{h,1}(\nabla^h p_h)\|^2 \\ &= (\nabla^h p - \nabla^h p_h, P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h)) \\ &+ (\nabla^h q_h - \nabla^h p, P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h)) \\ &=: T_1 + T_2, \end{aligned}\tag{59}$$

for all functions $q_h \in Q_h$. The first term T_1 can be written as

$$\begin{aligned} T_1 &= \sum_K \alpha_K \left(\nabla p - \nabla p_h, P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h) \right) \\ &= \sum_K \left(\nabla p - \nabla p_h, \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right)\right)_K \\ &+ \sum_K \Big(\nabla p - \nabla p_h, \alpha_K P_{h,1}(\nabla^h q_h) - \alpha_K P_{h,1}(\nabla^h p_h) \\ &\qquad - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) + \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right)\Big)_K \\ &=: T_{11} + T_{12}. \end{aligned}\tag{60}$$

From (21) (without the nonlinear term) we have that

$$\begin{aligned} T_{11} &= -a\left(\boldsymbol{u} - \boldsymbol{u}_h, \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right)\right) \\ &\leq N_a \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \left\| \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right)\right\|_1 . \end{aligned}$$

Using the same steps as those to arrive to (53) we get

$$\left\| \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right) \right\|_1 \leq C \left\| P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h) \right\|,$$

and therefore

$$T_{11} \leq C \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \left\| P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h) \right\|. \tag{61}$$

For T_{12} in (60) we have that

$$T_{12} = \sum_K \left(\nabla^h p - \nabla^h p_h, P_{h,1}(\nabla^h q_h) - \frac{1}{\alpha_K} \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h q_h)\right) - P_{h,1}(\nabla^h p_h) + \frac{1}{\alpha_K} \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h)\right) \right)_K,$$

and using the same steps as in (56) we obtain

$$T_{12} \leq \delta(h) \|\nabla^h p - \nabla^h p_h\| \left\| P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h) \right\|. \tag{62}$$

Using (60)-(62) in (59) we get

$$\|P_{h,1}(\nabla^h q_h) - P_{h,1}(\nabla^h p_h)\| \leq C \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \delta(h) \|\nabla^h p - \nabla^h p_h\| + \|\nabla^h p - \nabla^h q_h\|.$$

The proof concludes as in Theorem 3 of [10], assuming h to be sufficiently small (note that there the projection of the pressure gradient does not include the parameters α_K). □

We use now a duality argument to obtain pressure stability in $L^2(\Omega)$ and improved error estimates for the velocity and pressure, also in the space $L^2(\Omega)$, in a similar way to [4] for the GLS method. The shift used in these duality arguments needed for the velocity error estimates (not for the pressure) requires of more regularity of the problem than was needed up to now.

Theorem 3 *Under the same assumptions as in Theorem 1, the approximate pressure p_h satisfies:*

$$\|p_h\| \leq C \lambda \|\boldsymbol{f}\|_{-1}. \tag{63}$$

If the solution $[\boldsymbol{u}, p]$ of the continuous problem is k-regular, with $k \geq 1$, then

$$\|p - p_h\| \leq C h^r, \tag{64}$$

where $r = \min\{k, k_{\rm v}, k_{\rm q}+1\}$. *Moreover, if the Stokes problem is regular then the approximate velocity* $\boldsymbol{u}_h$ *satisfies*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \le C h^{r+1}. \tag{65}$$

Proof. We begin by the stability and error estimates for the pressure. Let $\gamma = 0$ or 1 and consider $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ and $\eta \in L^2(\Omega)$ as the solution of the Stokes problem:

$$-\Delta \boldsymbol{z} + \nabla \eta = 0 \quad \text{in} \quad \Omega, \tag{66}$$
$$\nabla \cdot \boldsymbol{z} = (\gamma p - p_h) \quad \text{in} \quad \Omega, \tag{67}$$
$$\boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \tag{68}$$

Standard results for this problem yield (see [16]):

$$\|\boldsymbol{z}\|_1 \le C\|\gamma p - p_h\|, \quad \|\eta\| \le C\|\gamma p - p_h\|. \tag{69}$$

If $\boldsymbol{z}_h \in V_{h,0}$ is an approximation to $\boldsymbol{z}$ satisfying:

$$\|\boldsymbol{z} - \boldsymbol{z}_h\|_{m,K} \le C h_K^{1-m} \|\boldsymbol{z}\|_{1,K}, \quad K \in \mathcal{T}_h, \tag{70}$$

for $m = 0, 1$, we have:

$$\begin{aligned}
\|\gamma p - p_h\|^2 &= (\gamma p - p_h, \gamma p - p_h) \\
&= (\nabla \cdot \boldsymbol{z}, \gamma p - p_h) \\
&= (\nabla \cdot (\boldsymbol{z} - \boldsymbol{z}_h), \gamma p - p_h) - (\boldsymbol{z}_h, \nabla(\gamma p - p_h)) \\
&= -(\boldsymbol{z} - \boldsymbol{z}_h, \nabla(\gamma p - p_h)) + a(\gamma \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}_h) + \gamma\lambda\langle \boldsymbol{f}, \boldsymbol{z}_h\rangle \\
&= -\sum_K \frac{1}{\alpha_K}(\boldsymbol{z} - \boldsymbol{z}_h, \gamma\nabla^h p - \nabla^h p_h))_K \\
&\quad + a(\gamma \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}_h - \boldsymbol{z}) + a(\gamma \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{z}) \\
&\quad + \gamma\lambda\langle \boldsymbol{f}, \boldsymbol{z}_h - \boldsymbol{z}\rangle + \gamma\lambda\langle \boldsymbol{f}, \boldsymbol{z}\rangle \\
&\le \sum_K \frac{1}{\alpha_K}\|\boldsymbol{z} - \boldsymbol{z}_h\|_{0,K}\|\gamma\nabla^h p - \nabla^h p_h\|_{0,K} \\
&\quad + C\left(\|\gamma \boldsymbol{u} - \boldsymbol{u}_h\|_1 + \gamma\lambda\|\boldsymbol{f}\|_{-1}\right)\left(\|\boldsymbol{z} - \boldsymbol{z}_h\|_1 + \|\boldsymbol{z}\|_1\right) \\
&\le C\sum_K \|\boldsymbol{z}\|_{1,K}\|\gamma\nabla^h p - \nabla^h p_h\|_{0,K} \\
&\quad + C\|\boldsymbol{z}\|_1\left(\|\gamma \boldsymbol{u} - \boldsymbol{u}_h\|_1 + \gamma\lambda\|\boldsymbol{f}\|_{-1}\right) \\
&\le C\left(\|\gamma\nabla^h p - \nabla^h p_h\| + \|\gamma \boldsymbol{u} - \boldsymbol{u}_h\|_1 + \gamma\lambda\|\boldsymbol{f}\|_{-1}\right)\|\gamma p - p_h\|. \qquad (71)
\end{aligned}$$

The stability estimate (63) for the pressure follows for $\gamma = 0$ and Theorem 1, whereas the error estimate (64) is obtained for $\gamma = 1$ using (58).

Let us prove now the estimate for the velocity. Let $\boldsymbol{y} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$ and $\xi \in H^1(\Omega) \cap Q$ be the solution of the regular problem:

$$-\Delta \boldsymbol{y} + \nabla \xi = \boldsymbol{u} - \boldsymbol{u}_h \quad \text{in} \quad \Omega, \tag{72}$$

$$\nabla \cdot \boldsymbol{y} = 0 \quad \text{in} \quad \Omega, \tag{73}$$

$$\boldsymbol{y} = \boldsymbol{0} \quad \text{on} \quad \Gamma, \tag{74}$$

which satisfies

$$\|\boldsymbol{y}\|_2 \leq C\|\boldsymbol{u} - \boldsymbol{u}_h\|, \quad \|\xi\|_1 \leq C\|\boldsymbol{u} - \boldsymbol{u}_h\|. \tag{75}$$

Let $\boldsymbol{y}_h \in V_{h,0}$ and $\xi_h \in Q_h$ be optimal order approximations to $\boldsymbol{y}$ and ξ, respectively, satisfying:

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{m,K} \leq C\, h_K^{2-m} \|\boldsymbol{y}\|_{2,K}, \tag{76}$$

$$\|\xi - \xi_h\|_{m,K} \leq C h_K^{1-m} \|\xi\|_{1,K}. \tag{77}$$

for $m = 0, 1$. We then have:

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|^2 &= (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= a(\boldsymbol{y}, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= [a(\boldsymbol{y} - \boldsymbol{y}_h, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi - \xi_h, \boldsymbol{u} - \boldsymbol{u}_h)] \\
&\quad + a(\boldsymbol{y}_h, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi_h, \boldsymbol{u} - \boldsymbol{u}_h) =: T_1 + T_2 + T_3.
\end{aligned}$$

We bound each term separately:

$$\begin{aligned}
T_1 &= a(\boldsymbol{y} - \boldsymbol{y}_h, \boldsymbol{u} - \boldsymbol{u}_h) - b(\xi - \xi_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&\leq N_a \|\boldsymbol{y} - \boldsymbol{y}_h\|_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + C\|\xi - \xi_h\| \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \\
&\leq C\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 (h\|\boldsymbol{y}\|_2 + h\|\xi\|_1) \\
&\leq Ch\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \|\boldsymbol{u} - \boldsymbol{u}_h\|
\end{aligned}$$

by (76), (77) and the shift (75). From the original momentum equation (with $c = 0$) we have:

$$\begin{aligned}
T_2 &= a(\boldsymbol{y}_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= -(\nabla p - \nabla p_h, \boldsymbol{y}_h) \\
&= \sum_K \frac{1}{\alpha_K} (\nabla^h p - \nabla^h p_h, \boldsymbol{y} - \boldsymbol{y}_h)_K \\
&\leq C \sum_K \frac{1}{h_K} \|\nabla^h p - \nabla^h p_h\|_{0,K} \|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,K} \\
&\leq C \sum_K h_K \|\boldsymbol{y}\|_{2,K} \|\nabla^h p - \nabla^h p_h\|_{0,K} \\
&\leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\| \|\nabla^h p - \nabla^h p_h\|,
\end{aligned} \tag{78}$$

by (76) and (75). Finally:

$$\begin{aligned}
T_3 &= -b(\xi_h, \boldsymbol{u} - \boldsymbol{u}_h) \\
&= (\nabla^h p - \nabla^h p_h, \nabla^h \xi_h) - (\nabla^h p - \tilde{\boldsymbol{u}}_h, \nabla^h \xi_h) \\
&= (\nabla^h p - \nabla^h p_h, \nabla^h \xi_h - \nabla^h \xi) + (\nabla^h p - \nabla^h p_h, \nabla^h \xi) \\
&- (\nabla^h p - \tilde{\boldsymbol{u}}_h, \nabla^h \xi_h - \nabla^h \xi) - (\nabla^h p - \tilde{\boldsymbol{u}}_h, \nabla^h \xi) \\
&\leq \sum_K \left(\|\nabla^h p - \nabla^h p_h\|_{0,K} + \|\nabla^h p - \tilde{\boldsymbol{u}}_h\|_{0,K} \right) \\
&\quad \times \left(\|\nabla^h \xi_h - \nabla^h \xi\|_{0,K} + \|\nabla^h \xi\|_{0,K} \right) \\
&\leq C \sum_K \left(\|\nabla^h p - \nabla^h p_h\|_{0,K} + \|\nabla^h p - \tilde{\boldsymbol{u}}_h\|_{0,K} \right) h_K \|\xi\|_{1,K} \\
&\leq C \left(\|\nabla^h p - \nabla^h p_h\| + \|\nabla^h p - \tilde{\boldsymbol{u}}_h\| \right) h \|\boldsymbol{u} - \boldsymbol{u}_h\|,
\end{aligned}$$

by (77) and (75). We obtain the error estimate for the velocity combining the above inequalities for T_1, T_2 and T_3. □

4 Stability and convergence II: Navier-Stokes equations

In this section we extend the results of the previous section to the nonlinear Navier-Stokes equations using the theory of approximation of branches of nonsingular solutions of [7, 16]. However, our first result concerns the case in which the uniqueness condition (14) holds. We show that this same condition ensures stability and uniqueness of solution of the discrete problem. To prove this we shall use as auxiliary problem a linearized form of it, namely, Picard's linearization. Denoting by a superscript the iteration counter, this problem is: given $\boldsymbol{u}_h^0$ arbitrary, for $i = 1, 2, ...,$ find $[\boldsymbol{u}^i, p_h^i, \tilde{\boldsymbol{u}}_h^i] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$ such that

$$\mathcal{B}_{(i)}(\lambda; \boldsymbol{u}_h^i, p_h^i, \tilde{\boldsymbol{u}}_h^i; \boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h) = \lambda \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \tag{79}$$

for all $[\boldsymbol{v}_h, q_h, \tilde{\boldsymbol{v}}_h] \in \boldsymbol{V}_{h,0} \times Q_h \times \boldsymbol{V}_h$. Here, $\mathcal{B}_{(i)}$ is the *bilinear* form obtained from $\mathcal{B}$ using $c(\boldsymbol{u}_h^{i-1}, \boldsymbol{u}_h^i, \boldsymbol{v}_h)$ as linearization of the nonlinear term.

As a by-product of the following theorem we shall have convergence of Picard's iterates, a property that does not hold when the solution to the nonlinear problem (13) is not unique (an alternative proof of the following result is to proceed the other way around, showing that when (14) holds the nonlinear operator associated to the variational problem (21)-(23) is contractive, and thus Banach's fixed point theorem implies that there is a unique solution to which Picard's linearization scheme converges).

Theorem 4 *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ of finite element partitions is non-degenerate, continuously graded and such that (45) and (46) hold. Suppose also that the uniqueness condition (14) holds. Then, for h small enough, there exists a unique solution to problem (26) that verifies the stability estimate*

$$\|\boldsymbol{u}_h\|_1 + \|\nabla^h p_h\| + \|\tilde{\boldsymbol{u}}_h\| \leq C\lambda \|\boldsymbol{f}\|_{-1}, \tag{80}$$

for a constant C independent of h.

Proof. We split the proof in three steps:
Step 1: The solution of the linear problem (79) exists, is unique and each iterate satisfies the stability estimate (80).

Since problem (79) is linear and finite dimensional, it is enough to prove the stability estimate (80). Due to the skew-symmetry of c (cf. (8)), this can be done exactly as for the Stokes problem, replacing $\mathcal{B}_{\text{lin}}$ by $\mathcal{B}_{(i)}$. The proof of Theorem 1 can be repeated here, the only difference being the bound for T_1 in (52). This term is now

$$\begin{aligned} T_1 = \lambda \left\langle \boldsymbol{f}, \varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right\rangle - a\left(\boldsymbol{u}_h^i, \varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right) \\ - \lambda\, c\left(\boldsymbol{u}_h^{i-1}, \boldsymbol{u}_h^i, \varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right), \end{aligned}$$

and, using the uniqueness condition (14) and the bounds for the velocity iterates (i.e., bound (49) for $\boldsymbol{u}_h^i$ and $\boldsymbol{u}_h^{i-1}$), the last term can be bounded by

$$\begin{aligned} &\lambda\, c\left(\boldsymbol{u}_h^{i-1}, \boldsymbol{u}_h^i, \varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right) \\ &\quad \leq \lambda N_c \|\boldsymbol{u}_h^{i-1}\|_1 \|\boldsymbol{u}_h^i\|_1 \left\|\varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right\|_1 \\ &\quad \leq \lambda \|\boldsymbol{f}\|_{-1} \left\|\varPi\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right)\right\|. \end{aligned}$$

and now we can proceed as in Theorem 1.
Step 2: Picard's iterates converge to a solution of the nonlinear problem (26).

Subtracting Eqs. (79) for i and $i-1$ and taking as test functions $\boldsymbol{v}_h = \boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}$, $q_h = p_h^i - p_h^{i-1}$ and $\tilde{\boldsymbol{v}}_h = \tilde{\boldsymbol{u}}_h^i - \tilde{\boldsymbol{u}}_h^{i-1}$, we get

$$\begin{aligned} &a(\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}, \boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}) + \|\nabla^h(p_h^i - p_h^{i-1}) - (\tilde{\boldsymbol{u}}_h^i - \tilde{\boldsymbol{u}}_h^{i-1})\|^2 \\ &+\lambda\, c(\boldsymbol{u}_h^{i-1} - \boldsymbol{u}_h^{i-2}, \boldsymbol{u}_h^i, \boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}) = 0. \end{aligned} \tag{81}$$

From the coercivity of a and the bound for $\boldsymbol{u}_h^i$ we have that

$$\begin{aligned} K_a \|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_1^2 &\leq \lambda N_c \|\boldsymbol{u}_h^{i-1} - \boldsymbol{u}_h^{i-2}\|_1 \|\boldsymbol{u}_h^i\|_1 \|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_1 \\ &\leq \lambda^2 N_c K_a^{-1} \|\boldsymbol{f}\|_{-1} \|\boldsymbol{u}_h^{i-1} - \boldsymbol{u}_h^{i-2}\|_1 \|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_1. \end{aligned}$$

Using inductively this inequality and the definition of χ in (14) we have that

$$\|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_1 \le C\chi^i. \tag{82}$$

Since $\chi < 1$ this proves convergence of the velocities. Using this in (81) we find that

$$\begin{aligned}\left\| P_{h,3}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\|^2 &= \|\nabla^h(p_h^i - p_h^{i-1}) - (\tilde{\boldsymbol{u}}_h^i - \tilde{\boldsymbol{u}}_h^{i-1})\|^2 \\ &\le \lambda N_c \|\boldsymbol{u}_h^{i-1} - \boldsymbol{u}_h^{i-2}\|_1 \|\boldsymbol{u}_h^i\|_1 \|\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}\|_1 \\ &\le C\chi^{2i},\end{aligned}$$

from where

$$\left\| P_{h,3}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\| \le C\chi^i. \tag{83}$$

To obtain a bound for $\left\| P_{h,1}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\|$ we proceed as in Theorem 1 to obtain the bound for $\|P_{h,1}(\nabla^h p_h)\|$. Equation (51) is also valid replacing p_h by $p_h^i - p_h^{i-1}$. The new term T_1 is now

$$\begin{aligned}T_1 = &-a\left(\boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}, \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^{i-1})\right)\right) \\ &-\lambda\, c\left(\boldsymbol{u}_h^{i-1} - \boldsymbol{u}_h^{i-2}, \boldsymbol{u}^{i-1}, \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^{i-1})\right)\right) \\ &-\lambda\, c\left(\boldsymbol{u}^{i-1}, \boldsymbol{u}_h^i - \boldsymbol{u}_h^{i-1}, \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^i)\right) - \mathit{\Pi}\left(\bar{\alpha} P_{h,1}(\nabla^h p_h^{i-1})\right)\right).\end{aligned}$$

Using the continuity of a and c, the bound for $\boldsymbol{u}_h^i$ and (82), as well as (53) with p_h replaced by $p_h^i - p_h^{i-1}$, we obtain that

$$T_1 \le C\chi^i \left\| P_{h,1}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\|. \tag{84}$$

For the new term T_2 we have, using the same steps as in (56),

$$\begin{aligned}T_2 &\le C\delta(h)\left\| P_{h,1}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\|^2 \\ &\quad + C\left\| P_{h,1}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\| \left\| P_{h,3}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\|.\end{aligned}$$

Using the bound (83) for $P_{h,3}\left(\nabla^h(p_h^i - p_h^{i-1})\right)$ and (84), for h small enough we have that,

$$\left\| P_{h,1}\left(\nabla^h(p_h^i - p_h^{i-1})\right)\right\| \le C\chi^i. \tag{85}$$

Convergence of the pressure follows now from (83), (85) and the weak inf-sup condition, and convergence of $\tilde{\boldsymbol{u}}^i$ follows from the fact that $\tilde{\boldsymbol{u}}_h^i = P_{h,12}(\nabla^h p_h^i)$.

Step 3: Problem (26) admits a unique solution.

Let $(\boldsymbol{u}_{h,1}, p_{h,1}, \tilde{\boldsymbol{u}}_{h,1})$ and $(\boldsymbol{u}_{h,2}, p_{h,2}, \tilde{\boldsymbol{u}}_{h,2})$ be two solutions of problem (26) and $(\bar{\boldsymbol{u}}_h, \bar{p}_h, \bar{\tilde{\boldsymbol{u}}}_h)$ their difference. It is easy to see that $\boldsymbol{u}_{h,1}$ and $\boldsymbol{u}_{h,2}$ satisfy the bound (80) and that

$$\begin{aligned}\mathcal{B}(\bar{\boldsymbol{u}}_h, \bar{p}_h, \bar{\tilde{\boldsymbol{u}}}_h; \bar{\boldsymbol{u}}_h, \bar{p}_h, \bar{\tilde{\boldsymbol{u}}}_h) &= a(\bar{\boldsymbol{u}}_h, \bar{\boldsymbol{u}}_h) + \|\nabla^h \bar{p}_h - \bar{\tilde{\boldsymbol{u}}}_h\| \\ &= -\lambda\, c(\bar{\boldsymbol{u}}_h, \boldsymbol{u}_{h,1}, \bar{\boldsymbol{u}}_h).\end{aligned}$$

The coercivity of a, the continuity of c and the bound (80) for $\boldsymbol{u}_{h,1}$ imply that

$$\|\bar{\boldsymbol{u}}_h\|_1^2 \leq \chi \|\bar{\boldsymbol{u}}_h\|_1^2,$$

and, since $\chi < 1$, $\bar{\boldsymbol{u}}_h = \boldsymbol{0}$. Also, $\bar{p}_h = 0$ and $\bar{\tilde{\boldsymbol{u}}}_h = \boldsymbol{0}$ follows from the fact that these variables are solution of a linear homogeneous problem and satisfy the stability estimates obtained in the linear case. □

Let us consider know the general case in which the uniqueness condition (14) does not hold. Regardless of the behavior of the continuous problem, it can be shown using exactly the same arguments as in [7] (also used in [22]) that problem (21)-(23) has solution. However, we consider directly the situation in which these solutions exist and approximate those of the continuous problem. For that, we need to recast it in the following abstract form. Let Λ be a compact subset of $\mathbb{R}^+$ and for each $\lambda \in \Lambda$ consider the mappings

$$\boldsymbol{V} \times Q \xrightarrow{G} \boldsymbol{V}' \xrightarrow{T} \boldsymbol{V} \times Q,$$

where

$$G(\lambda; \boldsymbol{v}, q) := \lambda \left[(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{2}(\nabla \cdot \boldsymbol{v})\boldsymbol{v} - \boldsymbol{f} \right],$$

and $[\boldsymbol{u}, p] := T\boldsymbol{g}$ is the solution of the continuous Stokes problem

$$a(\boldsymbol{u}, p) - b(p, \boldsymbol{v}) + b(q, \boldsymbol{u}) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle, \quad \forall [\boldsymbol{v}, q] \in \boldsymbol{V} \times Q.$$

Clearly, the solution of problem (13) is $[\boldsymbol{u}, p] = -TG(\lambda; \boldsymbol{u}, p)$, that is, the solution of

$$F(\lambda; \boldsymbol{u}, p) := [\boldsymbol{u}, p] + TG(\lambda; \boldsymbol{u}, p) = 0. \tag{86}$$

Likewise, the discrete Navier-Stokes problem can be written in a form similar to (86) with the help of the operators

$$\boldsymbol{V}_h \times Q_h \xrightarrow{G_h} \boldsymbol{V}' \xrightarrow{T_h} \boldsymbol{V}_h \times Q_h,$$

where $G_h(\lambda; \boldsymbol{v}_h, q_h) := G(\lambda; \boldsymbol{v}_h, q_h)$ and $[\boldsymbol{u}_h, p_h] := T_h\, \boldsymbol{g}$ is the solution of the discrete Stokes problem 'condensing' the pressure gradient projection:

$$a(\boldsymbol{u}_h, p_h) - b(p_h, \boldsymbol{v}_h) + b(q_h, \boldsymbol{u}_h) + \left(P_{h,3}(\nabla^h p_h), \nabla^h q_h\right) = \langle \boldsymbol{g}, \boldsymbol{v}_h \rangle,$$

for all $[\boldsymbol{v}_h, q_h] \in \boldsymbol{V}_h \times Q_h$. The solution $[\boldsymbol{u}_h, p_h]$ of problem (21)-(23) is the same as the solution of

$$F_h(\lambda; \boldsymbol{u}_h, p_h) := [\boldsymbol{u}_h, p_h] + T_h G_h(\lambda; \boldsymbol{u}_h, p_h) = 0. \tag{87}$$

Let $DF(\lambda; \boldsymbol{u}, p)$ denote the Fréchet derivative of F with respect to $[\boldsymbol{u}, p]$. A curve $\{(\lambda, [\boldsymbol{u}(\lambda), p(\lambda)]); \lambda \in \Lambda\}$ is called a *branch of nonsingular solutions* of (86) if $[\boldsymbol{u}(\lambda), p(\lambda)]$ is solution of this problem for all $\lambda \in \Lambda$, the map $\lambda \mapsto [\boldsymbol{u}(\lambda), p(\lambda)]$ is continuous and $DF(\lambda; \boldsymbol{u}(\lambda), p(\lambda))$ is a homeomorphism of $\boldsymbol{V} \times Q$.

The following results shows that if a branch of nonsingular solutions of (86) is regular enough, then problem (87) has also a unique branch of nonsingular solutions which gives an approximation of optimal order to it:

Theorem 5 *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ of finite element partitions is non-degenerate, continuously graded and such that (45) and (46) hold. Assume also that $\{(\lambda, [\boldsymbol{u}(\lambda), p(\lambda)]); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (86) such that $\lambda \mapsto [\boldsymbol{u}(\lambda), p(\lambda)] \in \boldsymbol{V} \cap \boldsymbol{H}^{k+1}(\Omega) \times Q \cap H^k(\Omega)$ is continuous for a certain integer $k \geq 1$. Then, for h small enough, there exists a unique branch of nonsingular solutions $\{(\lambda, [\boldsymbol{u}_h(\lambda), p_h(\lambda)]); \lambda \in \Lambda\}$ of problem (87) which satisfies*

$$\|\boldsymbol{u}(\lambda) - \boldsymbol{u}_h(\lambda)\|_1 + \|p(\lambda) - p_h(\lambda)\| \leq C(\lambda) h^r, \tag{88}$$

for all $\lambda \in \Lambda$, where $r = \min\{k, k_{\rm v}, k_{\rm q} + 1\}$ and $C(\lambda)$ depends on λ.

Proof. As in Theorem IV.4.1 of [16], the proof simply consists in checking that the assumptions of the abstract approximation result IV.3.3 of this reference are satisfied. First, we know that T is a bounded linear operator from $\boldsymbol{V}'$ to $\boldsymbol{V} \times Q$. By virtue of Theorem 1 and the pressure stability estimate (63) in Theorem 3, T_h is also a bounded linear operator from $\boldsymbol{V}'$ to $\boldsymbol{V}_h \times Q_h$ (endowing this space with the same norm as $\boldsymbol{V} \times Q$). Thus,

$$T \in \mathcal{L}(\boldsymbol{V}', \boldsymbol{V} \times Q), \quad T_h \in \mathcal{L}(\boldsymbol{V}', \boldsymbol{V}_h \times Q_h). \tag{89}$$

On the other hand, G is a $\mathcal{C}^\infty$ map whose Fréchet derivative with respect to $[\boldsymbol{u}, p]$, $DG(\lambda; \boldsymbol{u}, p)$, maps $\boldsymbol{V} \times Q$ to $\boldsymbol{V}'$ for each $[\boldsymbol{u}, p] \in \boldsymbol{V} \times Q$ and is given by

$$\begin{aligned} &DG(\lambda; \boldsymbol{u}, p) \cdot [\boldsymbol{v}, q] \\ &\qquad = \lambda \left[(\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + \frac{1}{2}(\nabla \cdot \boldsymbol{v})\boldsymbol{u} + \frac{1}{2}(\nabla \cdot \boldsymbol{u})\boldsymbol{v} \right]. \end{aligned}$$

Furthermore, Sobolev's embedding theorem and Hölder's inequality imply that in fact $DG(\lambda;\boldsymbol{u},p)\cdot[\boldsymbol{v},q]\in Z:=L^{3/2}(\Omega)$ whenever $\boldsymbol{u},\ \boldsymbol{v}\in\boldsymbol{V}$. Therefore, for G we have

(90) $$G\in\mathcal{C}^\infty(\boldsymbol{V}\times Q,\boldsymbol{V}'),\quad DG(\lambda;\boldsymbol{u},p)\in\mathcal{L}(\boldsymbol{V}\times Q,Z).$$

On the other hand, Theorem 2 and (64) in Theorem 3 imply

(91) $$\|(T-T_h)\boldsymbol{g}\|_{\boldsymbol{V}\times Q}\leq Ch^r\quad\forall\boldsymbol{g}\in\boldsymbol{V}'.$$

Since $Z=L^{3/2}(\Omega)$ is compactly embedded in $\boldsymbol{V}'=\boldsymbol{H}^{-1}(\Omega)$, this last estimate implies

(92) $$\lim_{h\to 0}\|T-T_h\|_{\mathcal{L}(Z,\boldsymbol{V}\times Q)}=0.$$

Properties (89)-(92) are precisely the assumptions needed to apply Theorem IV.3.3 in [16], from where estimate (88) follows. The dependence of $C(\lambda)$ with λ appears through the inverse of the homeomorphism $DF(\lambda;\boldsymbol{u}(\lambda),p(\lambda))$. □

Finally, optimal L^2 estimates for the velocity can be obtained if the Stokes problem is regular. The following result can be proved adapting the proof of Theorem IV.4.2 in [16] as done above in Theorem 5:

Theorem 6 *Under the same assumptions as in Theorem 5, if, in addition, the Stokes problem is regular, then the error estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|\leq C(\lambda)h^{r+1}$$

holds for all $\lambda\in\Lambda$.

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