

# PETROV–GALERKIN METHODS FOR THE TRANSIENT ADVECTIVE–DIFFUSIVE EQUATION WITH SHARP GRADIENTS

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## SUMMARY

A Petrov–Galerkin formulation based on two different perturbations to the weighting functions is presented. These perturbations stabilize the oscillations that are normally exhibited by the numerical solution of the transient advective–diffusive equation in the vicinity of sharp gradients produced by transient loads and boundary layers. The formulation may be written as a generalization of the Galerkin Least-Square method.

**KEY WORDS:** transient loads; advective–diffusive equations; Petrov–Galerkin; Galerkin Least-Square; boundary layers

## 1. INTRODUCTION

We focus on the numerical solution of the transient advective–diffusive equation

$$\frac{\partial \phi}{\partial t} - \nabla \cdot k \nabla \phi + \mathbf{u} \nabla \phi = f \quad (1)$$

using the finite element method. We will assume for simplicity that the convective velocity  $\mathbf{u}$  and the diffusion coefficient  $k$  are constants and that the equation is valid over a finite domain  $\Omega$ , together with appropriate initial and boundary conditions.

It is well known that this kind of equation represents a simplified model of several industrial processes in which the unknown variable  $\phi$  may represent temperature, concentration of a species, or other scalar variables. It is also accepted that this scalar equation is representative of more complicated advective–diffusive systems such as the Navier–Stokes equations and constitutes a good simplified model to study the numerical behaviour of convective–diffusive systems in general. For this reason, the analysis of the solutions of equation (1), even in the most simplified cases, is the first step towards a more detailed analysis. Unacceptable numerical solutions detected in this equation act as the warning light to examine other more complicated linear and non-linear system of equations whose behaviour is not well understood.

The numerical solution of equation (1) using Galerkin formulations normally exhibits global spurious oscillations in advection-dominated problems, especially in the vicinity of sharp gradients. In recent years, a variety of finite element algorithms have been proposed to deal with such

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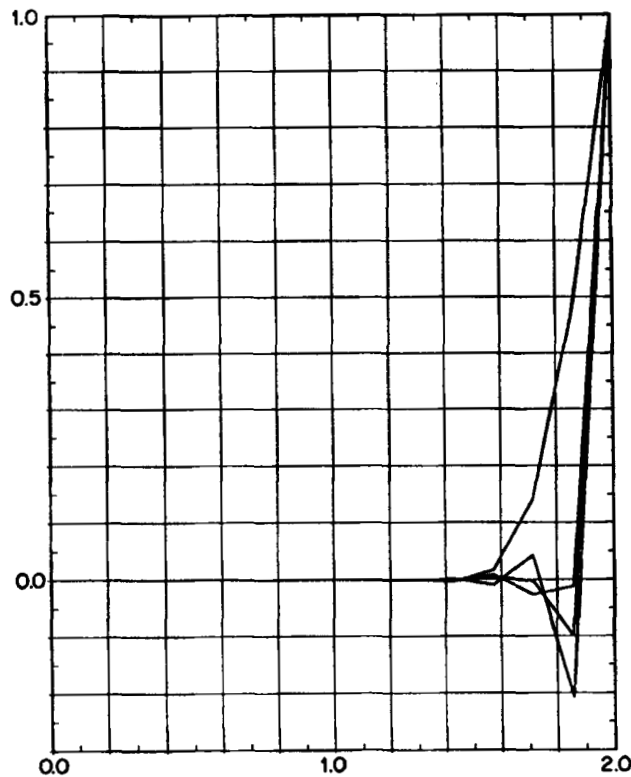


Figure 1. First 3 time steps and 20th time step for the diffusive dominant problem.  $u = 10^{-3}$ ;  $k = 1$ ;  $\Delta t = 10^{-3}$

situations, these methods stabilize the numerical scheme by adding a perturbation to the weighting functions and thus, producing an oscillation-free solution.<sup>1-3</sup> These perturbation is proportional to the gradient of the standard interpolation functions. The dimensionless Peclet number gives an accurate measure of the magnitude of the perturbation to be incorporated. Most of these perturbed methods have been developed for the time-independent advection-diffusion equation, and may have resolved the problem successfully in this case.<sup>1-4</sup>

Some of these techniques have been extended successfully to the time-dependent problem in the case when the initial and boundary conditions are smooth.<sup>4-7</sup> Nevertheless, in transient problems, additional difficulties arise, associated with the occurrence of local oscillations normally associated with sharp transient loads.<sup>8</sup> These spurious oscillations are of a different type, firstly because they are not global, and secondly because they are not directly related to the advective term, they appear even in the absence of convection. To illustrate this, Figure 1 shows the numerical solution of the one-dimensional equation:

$$\frac{\partial \phi}{\partial t} - k \frac{\partial^2 \phi}{\partial x^2} + u \frac{\partial \phi}{\partial x} = 0 \quad (2)$$

with initial condition

$$\phi(x, t_0) = 0$$

and boundary conditions

$$\phi(0, t) = 0$$

$$\phi(2, t) = 1$$

solved using a  $\theta$ -scheme for the time integration with  $\theta = 1/2$  and 14 equal size linear finite elements. With  $k = 1$  and  $\nu = 10^{-3}$  the solution is dominated by diffusion and shows strong oscillations at the early stages of the calculation.

These local oscillations are not as pervasive as the advective ones when steady-state solutions are sought, because of their local character and because they normally disappear as the solution approaches steady state. However, they become very dangerous in non-linear problems in which the oscillating behaviour may considerably slow down or prevent convergence altogether. The local oscillations can be eliminated using particular integration techniques, non-consistent mass matrices or varying the time step in order to eliminate transient sharp gradients.<sup>8</sup> However, to the authors' knowledge, there is no general method available that can eliminate spurious oscillations over the range from non-advective to advection-dominated problems; and that can be applied independently of the specific time-integration technique and time step to be used. Particularly if a consistent mass matrix is used in the finite element formulation.

This paper is an attempt to give an answer to the above problem. We present a Petrov–Galerkin formulation based on two different perturbations to the weighting functions. One of them is similar to that of the now standard Petrov–Galerkin method for advective problems. The second one is a symmetric perturbation similar to that proposed in Reference 9 for the convection–reaction–diffusion equation. The proportionality constant for each perturbation depends on two-dimensionless numbers: the Peclet number ( $Pe$ ) and the Transient number ( $r$ ). The latter depends on the time-integration technique used, the time step and some coefficient taking into account the stationarity of the problem. The method may be understood as a generalization of the Galerkin Least-Square method<sup>14</sup> with two different stabilizing parameters  $\tau_1$  and  $\tau_2$ .

In Section 2 the stabilized numerical method is developed for the one-dimensional advection–reaction–diffusion equation. The formulation of this scheme in the form of a Petrov–Galerkin method, and the new Galerkin Least-Squares formulation are presented in Section 3. The extension to transient problems is described in Section 4 and Section 5 shows some numerical results obtained for a range of  $Pe$  and  $r$  numbers.

## 2. 'BALANCING DIFFUSION' AND 'BALANCING ADVECTION'

Let us consider the time-dependent advection–diffusion equation over a domain  $\Omega$  with boundary  $\Gamma = \Gamma_1 + \Gamma_2$ :

$$\frac{\partial \phi}{\partial t} - \nabla \cdot k \nabla \phi + \mathbf{u} \nabla \phi = f$$

together with initial and boundary conditions,

$$\phi(x, t_0) = \phi^0(x)$$

$$\phi(x, t) = \phi_i(t) \quad \text{for } x \in \Gamma_1$$

$$k \nabla \phi(x, t) \cdot \mathbf{n} = -\phi_q(t) \quad \text{for } x \in \Gamma_2 \tag{3}$$

Equation (3) is first integrated in time using any finite differences or finite element discretization in time. In all cases, the problem will be to find the unknown function  $\phi^{n+1}$  at time  $t_{n+1} = t_n + \Delta t$

as a function of the values  $\phi^n = \phi(t_n)$  at the previous time step. For instance using a  $\theta$ -method (3) may be written as

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = (\nabla \cdot k \nabla \phi - \mathbf{u} \nabla \phi)^{n+1} \theta + (\nabla \cdot k \nabla \phi - \mathbf{u} \nabla \phi)^n (1 - \theta) + \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(t) dt \quad (4)$$

or

$$c \phi^{n+1} - \nabla \cdot k \nabla \phi^{n+1} + \mathbf{u} \nabla \phi^{n+1} = f^n \quad (5)$$

with

$$f^n = \frac{1}{\theta \Delta t} \int_{t_n}^{t_{n+1}} f(t) dt + \frac{1 - \theta}{\theta} (\nabla \cdot k \nabla \phi^n - \mathbf{u} \nabla \phi^n) + \frac{1}{\theta \Delta t} \phi^n \quad (6)$$

and

$$c = \frac{1}{\theta \Delta t} \quad (7)$$

It must be noted that any other choice of the integration scheme may be reduced to a similar equation as (5) in which the unknown function  $\phi^{n+1}$  must be evaluated as a function of values at the previous steps  $\phi^n, \phi^{n-1}$  and so on, with the appropriate changes in the definition of  $f^n$  and  $c$ .

Equation (5) represents a problem of reaction–diffusion–advection for which, as it is well known, the numerical solution has problems associated with the existence of local and global oscillations near regions of sharp gradients.

Recently, Idelsohn *et al.*<sup>9</sup> presented a method that eliminates these spurious oscillations for the case of a constant forcing term  $f^n$ . The method consists in adding two different perturbations to the weighting functions, one of them is an antisymmetric perturbation upwinded in the flow direction as is usual in streamline upwind Petrov–Galerkin methods, the other one is symmetric. The idea of using two different stabilization parameters is very similar to the idea used for incompressible flows by Tezduyar.<sup>10</sup> Alternatives to this solution have been presented by Tezduyar and Park<sup>11</sup> and also by Franca *et al.*<sup>12</sup> and both are based on the introduction of a switch to determine if the problem is reactive or advective. However they cannot properly handle the problem when both terms are important. Codina<sup>13</sup> introduced a shock capturing term to stabilize the reactive effects. However, this approach introduces a non-linearity even in linear one-dimensional problems.

To the authors knowledge, none of the above-mentioned ideas on reactive–diffusion–advective problems have been used to solve the transient advective–diffusion equation. In this paper we will use the approach reported in Reference 9, which may also be seen as a generalization of the Galerkin Least-Square method (GLS), to approximate the solution of the time-dependent equation (1).

It is well known that in the Petrov–Galerkin approach, a ‘balancing diffusion’  $k^*$  is added in order to have the exact nodal solution of the homogeneous one-dimensional linear problem. In the present formulation, both a ‘balancing diffusion’  $k^*$  and a ‘balancing advection’  $u^*$  will be added as shown below.

Let

$$\bar{k} = k + k^*$$

$$\bar{u} = u + u^*$$

be the total diffusion and advective velocity coefficients. For a uniform mesh of size  $h$  and linear finite elements the Galerkin formulation applied to equation (5) for one-dimensional problems gives the following difference equation at each node  $i$ :

$$\frac{(k + k^*)}{h^2} \begin{pmatrix} -\phi_{i-1} \\ 2\phi_i \\ -\phi_{i+1} \end{pmatrix} + \frac{(u + u^*)}{2h} \begin{pmatrix} -\phi_{i-1} \\ 0\phi_i \\ \phi_{i+1} \end{pmatrix} + \frac{c}{6} \begin{pmatrix} \phi_{i-1} \\ 4\phi_i \\ \phi_{i+1} \end{pmatrix} = f_i \tag{8}$$

The exact solution to equation (5) with  $f_i = 0$  is of the form

$$\phi(x) = a e^{\lambda_1 x} + b e^{\lambda_2 x} \tag{9}$$

where

$$\lambda_{1,2} = \frac{u}{2k} \pm \sqrt{\left(\frac{u}{2k}\right)^2 + \frac{c}{k}} \tag{10}$$

Replacing (9) in (8) we find:

$$k + k^* = -ch^2 \frac{2 + e^{\lambda_1 h} + e^{\lambda_2 h} + 2e^{(\lambda_1 + \lambda_2)h}}{6(1 - e^{\lambda_1 h})(1 - e^{\lambda_2 h})} \tag{11}$$

$$u + u^* = ch \frac{1 - e^{(\lambda_1 + \lambda_2)h}}{(1 - e^{\lambda_1 h})(1 - e^{\lambda_2 h})} \tag{12}$$

When the reactive term  $c$  is small, the  $k^*$  tends to the known balancing diffusivity:

$$k^* = -k + \frac{uh}{2} \coth\left(\frac{uh}{2k}\right) \tag{13}$$

and the numerical advection  $u^*$  goes to zero.

On the other hand, when the advective term is small, the numerical diffusion behaves like:

$$k^* = -k + \frac{ch^2}{6} + \frac{ch^2}{4 \sin h^2 \left(\sqrt{\frac{ch^2}{4k}}\right)} \tag{14}$$

and  $u^*$  goes to zero, which is the result obtained by Tezduyar *et al.*<sup>11</sup> for reactive dominant flows.

### 3. THE GENERALIZED GALERKIN LEAST-SQUARE METHOD (GLS<sub>2</sub>)

A consistent alternative to introducing the numerical coefficients  $k^*$  and  $u^*$ , as shown in the previous section, is to find weighting functions that yield the same results as equations (8)–(12). In this way the physical equation is not changed, and the weighting functions are perturbed in order to obtain the desired effect. These methods are called, in general, Petrov–Galerkin methods.<sup>1–3</sup> The best-known Petrov–Galerkin methods are the streamline-diffusion algorithms in which the weighting functions are perturbed in an unsymmetrical way in the upwind direction and the perturbation function is proportional to the gradient of the weighting function. The SUPG (streamline upwind Petrov–Galerkin) method is one of them, and it has been shown to be effective for the finite element solutions of linear advective–diffusive systems.<sup>3,7</sup> More recently, the Galerkin Least-Square method<sup>14</sup> has been introduced as a general methodology to obtain

consistent finite element schemes that can accommodate a wider class of interpolation functions. In the GLS approach, the perturbation functions are not only proportional to the gradient of the shape functions, but to the whole operator including the Laplacian of the function. We will generalize this concept in order to include the stabilization of the reactive–diffusive–advective problem.

Let equation (1) be written as

$$\sum_{i=0}^2 \mathcal{L}_i(\phi) = f \tag{15}$$

where

$$\mathcal{L}_2(\phi) = -\nabla \cdot k \nabla \phi \tag{16}$$

$$\mathcal{L}_1(\phi) = \mathbf{u} \nabla \phi \tag{17}$$

$$\mathcal{L}_0(\phi) = c \phi \tag{18}$$

A weighted residuals method applied to equation (15) consists in finding  $\hat{\phi}$  such that

$$\phi \simeq \hat{\phi} = \sum_{i=1}^n w_i(x) \phi_i \tag{19}$$

by imposing that

$$\int_{\Omega} \tilde{w}_i \left( \sum_{i=0}^2 \mathcal{L}_i(\hat{\phi}) - f \right) d\Omega = 0 \tag{20}$$

where  $\tilde{w}_i$  are weighting functions.

The following approaches are recovered by an appropriate selection of the weighting functions:

(a) The Galerkin approach

$$\tilde{w} = w \tag{21}$$

(b) The SUPG technique

$$\tilde{w} = w + \tau \mathcal{L}_1(\hat{\phi}) \tag{22}$$

where  $\tau$  is the upwind coefficient necessary to achieve stability in the proposed scheme

(c) The GLS method

$$\tilde{w} = w + \tau \left( \sum_{i=0}^2 \mathcal{L}_i(\hat{\phi}) \right) \tag{23}$$

The name of Least-Square method was used because the perturbation to the weighting functions are the same as the operator itself.

In the proposed Generalized Galerkin Least-Square method (GLS2)  $\tilde{w}$  is given by

$$\tilde{w} = w + \sum_{i=0}^2 \tau_i \mathcal{L}_i(\hat{\phi}) \tag{24}$$

which requires the use of different stabilizing parameters for each of the operators involved.

In fact, we can normalize one of the coefficients  $\tau_i$  in order to have 2 independent parameters as

$$\tilde{w} = w + \tau_1 \mathbf{u} \nabla w + \tau_2 (-\nabla \cdot k \nabla w) \tag{25}$$

This formulation, includes all the previous ones as particular cases i.e.,

$$\begin{aligned} \tau_1 = \tau_2 = 0 &\rightarrow \text{Galerkin} \\ \tau_1 \neq 0; \tau_2 = 0 &\rightarrow \text{SUPG} \\ \tau_1 = \tau_2 \neq 0 &\rightarrow \text{GLS} \\ \tau_1 \neq \tau_2 \neq 0 &\rightarrow \text{GLS}_2 \end{aligned}$$

In order to evaluate the stabilizing parameters  $\tau_1$  and  $\tau_2$  we will write the weighting functions (25) as in Reference 9

$$\tilde{w} = w + \alpha h \mathbf{e}_n \nabla w + \gamma P_2(x) \tag{26}$$

where

$$\alpha = \frac{\tau_1 |\mathbf{u}|}{h}; \quad \gamma = -k\tau_2; \quad P_2(x) = \nabla \cdot \nabla w \tag{27}$$

$h$  is the size of the element and  $\mathbf{e}_n$  is unit vector in the direction of  $\mathbf{u}$ .

The weak form of equation (20) is

$$\int_{\Omega} (\nabla \tilde{w} k \nabla \phi + \tilde{w} \mathbf{u} \nabla \phi + \tilde{w} c \phi - \tilde{w} f) \, d\Omega - \int_{\Gamma} \tilde{w} k \nabla \phi \cdot \mathbf{n} \, d\Gamma = 0 \tag{28}$$

and using (26)

$$\begin{aligned} &\int_{\Omega} (\nabla w k \nabla \phi + w \mathbf{u} \nabla \phi + w c \phi - w f) \, d\Omega \\ &+ \int_{\Omega} (P_2 k \alpha h \mathbf{e}_n \nabla \phi + \nabla w \alpha h \mathbf{e}_n \nabla \phi + \nabla w \alpha h c \mathbf{e}_n \phi - \nabla w \alpha h \mathbf{e}_n f) \, d\Omega \\ &+ \int_{\Omega} (\nabla P_2 \gamma k \nabla \phi + P_2 \gamma \mathbf{u} \nabla \phi + P_2 c \gamma \phi - P_2 \gamma f) \, d\Omega \\ &- \int_{\Gamma} (w + \alpha \mathbf{e}_n \nabla w + \gamma P_2) k \nabla \phi \cdot \mathbf{n} \, d\Gamma = 0 \end{aligned} \tag{29}$$

For linear finite elements ( $\nabla \phi = \text{constant}$ ) and constant  $f$ , equation (29) shows that the results involve specific averages of  $P_2(x)$  and  $\nabla P_2(x)$ . For simplicity, we denote such averages as

$$a = \frac{1}{\Omega_e} \int_{\Omega_e} P_2(x) \, d\Omega; \quad m_i = \frac{1}{\Omega_e h} \int_{\Omega_e} x_i P_2(x); \quad P_0 = \int_{\Omega_e} \nabla P_2(x) \, d\Omega \tag{30}$$

where  $\Omega_e$  is the volume of each finite element.

For linear finite elements, the definition of  $P_2(x)$  as (27) is rather arbitrary because  $\nabla \cdot \nabla w$  vanishes within each element and it is a Dirac  $\delta$ -function at the interfaces. On the other hand, equation (29) shows that the results are independent of the precise definition of  $P_2(x)$ , depending only on some average values over each element. Thus, any function giving the same  $a$ ,  $m_i$  and  $P_0$  values yields the same results. In Reference 9, the authors analysed the effect of varying the parameters  $a$ ,  $m_i$  and  $P_0$ . A basic requirement is that the proportionality constants  $\alpha$  and  $\gamma$  must be bounded for all combinations of the coefficients  $k$  and  $c$ . In that reference, the use of

$$a = \frac{1}{6}; \quad m_i = \frac{1}{12} \quad \text{and} \quad P_0 = 0 \tag{31}$$

was proposed, but different values may be used with similar results.

The stabilizing parameters  $\alpha$  and  $\gamma$  (and then,  $\tau_1$  and  $\tau_2$ ) are computed so as to obtain the exact nodal values in the one-dimensional homogeneous problem. This situation is equivalent to the use of balancing diffusion  $k^*$  and balancing advection  $u^*$  defined in (24) and (25), respectively.

The following Peclet and reaction dimensionless numbers are defined

$$Pe = \frac{|\mathbf{u}|h}{2k} \quad \text{and} \quad r = \frac{ch^2}{k} \quad (32)$$

in which  $r$ , for transient problems, is a function of the time step and the time-integration technique used according to equation (7).

The values of  $\alpha$  and  $\gamma$  are obtained by solving the following  $2 \times 2$  system:

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad (33)$$

where

$$\begin{aligned} g_{j1} &= 4Pe(1 - \cosh(\lambda_j)) - r \sinh(\lambda_j) \\ g_{j2} &= 2 \cosh(\lambda_j)(rm_i - P_0) + 4Pe a \sinh(\lambda_j) + 2(P_0 - m_i r + ar) \\ h_j &= 2 \cosh(\lambda_j)(\frac{1}{6}r - 1) + 2Pe \sinh(\lambda_j) + (2 + \frac{2}{3}r) \\ \lambda_j &= Pe + (-1)^{j-1}(Pe^2 + r)^{1/2} \end{aligned} \quad (34)$$

Figure 2 shows the curves of  $\alpha$  and  $\gamma$  for different values of  $Pe$  and  $r$  when  $a = \frac{1}{6}$ ;  $m_i = \frac{1}{12}$  and  $P_0 = 0$ .

It must be noted that both parameters  $\alpha$  and  $\gamma$  (and then  $\tau_1$  and  $\tau_2$ ) depend on both dimensionless numbers  $Pe$  and  $r$ , i.e.,

$$\tau_1 = \tau_1(Pe, r)$$

$$\tau_2 = \tau_2(Pe, r)$$

In the limit case in which one of these dimensionless numbers becomes zero, (e.g.:  $r = 0$  for the stationary case, or  $Pe = 0$  for a non-advective problem), one of the parameters becomes zero, and the other one becomes a function of the remaining dimensionless number,

$$\begin{aligned} r = 0 &\rightarrow \begin{cases} \tau_1 = \tau_1(Pe) \\ \tau_2 = 0 \end{cases} \\ Pe = 0 &\rightarrow \begin{cases} \tau_1 = 0 \\ \tau_2 = \tau_2(r) \end{cases} \end{aligned}$$

#### 4. THE TRANSIENT SOLUTION

Using the GLS<sub>2</sub> method with the stabilizing parameters proposed in previous section, the reactive-advective-diffusive problem with constant source terms can be solved giving exact nodal values in the one-dimensional case. However, the transient advective-diffuse problem proposed in equation (5) has a variable generalized source term  $f^n$  which is not constant even in the case when the source term  $f$  is. In particular, in the stationary limit, the generalized source terms  $f^n$



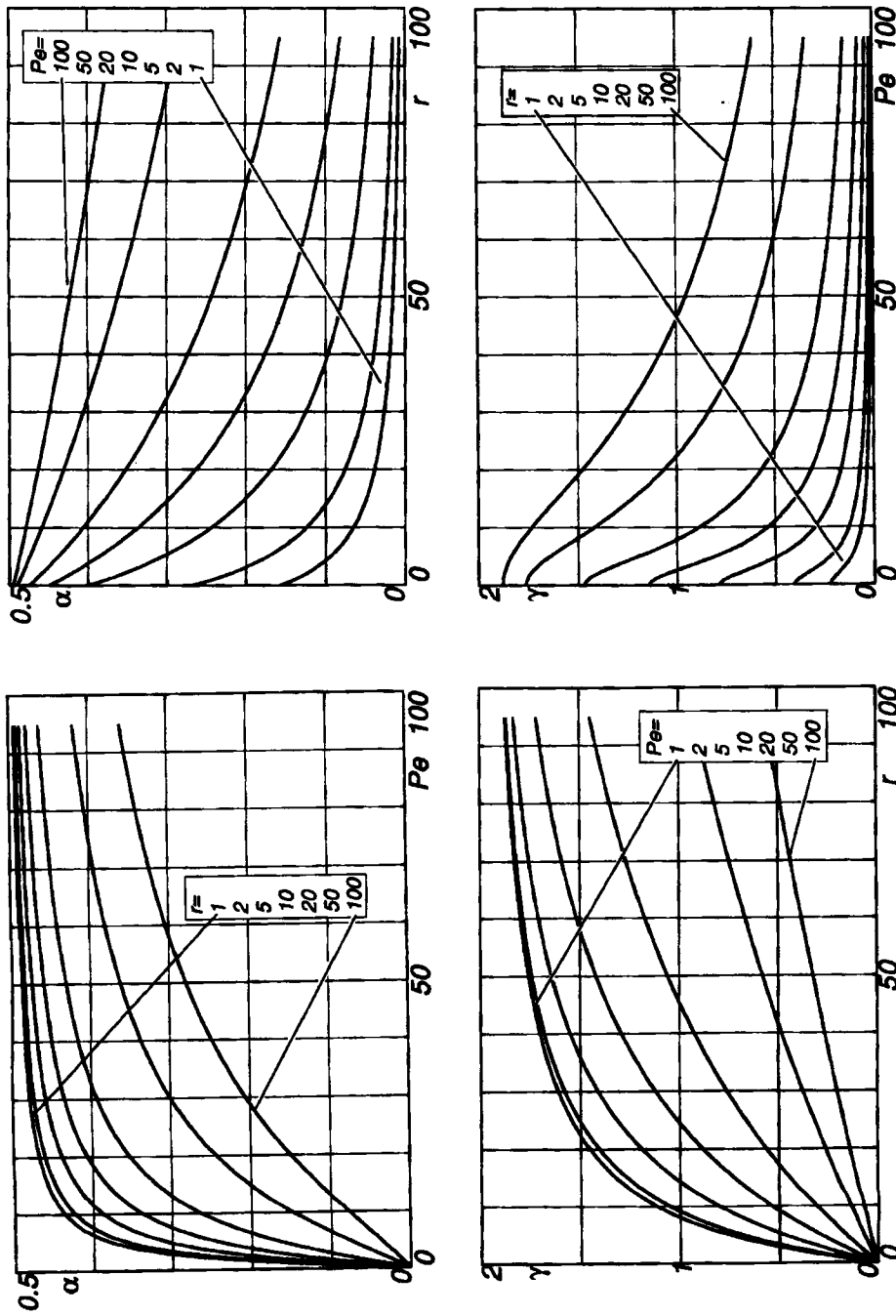


Figure 2. Stabilization parameters as a function of  $Pe$  and  $r$

becomes equal to the reactive term  $c\phi^{n+1}$  (see equations (5) and (6)). In this limit, the equation must be solved as a non-reactive equation and the stabilizing parameter becomes the optimal parameter for the advective–diffuse case.

To overcome this difficulty a modification on the definition of the coefficient  $c$  is introduced. Equation (5) is now written:

$$c^*(x, t)\phi^{n+1} - \nabla \cdot k\nabla\phi^{n+1} + \mathbf{u}\nabla\phi^{n+1} = 0 \quad (35)$$

with

$$c^*(x, t) = \frac{c\phi^{n+1} - f^n}{\phi^{n+1}} \quad (36)$$

The problem is transformed into a homogeneous one but with a non-linear reactive coefficient that varies both in space and time.

This coefficient may be approximated in order to retain the linearity of the problem using:

$$c^*(x, t) \simeq \frac{c\phi^{n+1} - c\phi^n}{\phi^{n+1}} \simeq c \frac{\phi^n - \phi^{n-1}}{\phi^n} \quad (37)$$

furthermore, to obtain a constant average  $c^*(t)$  on all the domain:

$$c^*(t) \simeq c \frac{\max(|\phi^n - \phi^{n-1}|)}{\max(|\phi^n|)} \quad (38)$$

This last approximation has been used in the examples presented below. It must be noted that the value of  $c^*$ , given by equation (38) should be used in the evaluation of the stabilizing parameters  $\alpha$  and  $\gamma$  (or  $\tau_1$  and  $\tau_2$ ) only. That is, the approximation (38) is introduced only for the evaluation of the perturbations to the weighting functions  $\tilde{w}$  but not in the equation to be solved. This is important in order to retain the consistency of the solution.

Using the GLS<sub>2</sub> method for the transient advective–diffusive equation, with the approach (35)–(38) in the evaluation of the time-dependent parameters, no spurious oscillations during the transient part and an optimally stabilized solution near the stationary state are ensured as it will be shown in the next examples.

## 5. NUMERICAL RESULTS AND ACCURACY ANALYSIS

The problem of finding numerical approximations to the equation:

$$\frac{\partial\phi}{\partial t} - k \frac{\partial^2\phi}{\partial x^2} + \mathbf{u} \frac{\partial\phi}{\partial x} = 0 \quad (39)$$

with initial and boundary conditions:

$$\begin{aligned} \phi(x, t_0) &= 0 \\ \phi(0, t) &= 0 \\ \phi(2, t) &= 1 \end{aligned} \quad (40)$$

is presented for various combinations of parameters and boundary conditions.

This simple equation was chosen because it has the two types of sharp gradients under consideration. For high Peclet numbers, a boundary layer develops in the right end due to the

elliptic-hyperbolic character of the equation. On the other hand, for all Peclet numbers, a sharp gradient appears at the right end during the first few time steps due to the transient solution. This sharp gradient, which is similar to a shock in a fluid mechanics problem, disappears after a few time steps if the problem is dominated by diffusion, and it will remain as the solution approaches the stationary state if the problem is dominated by advection.

It is important to note, that the way to eliminate the spurious oscillations is different when the sharp gradients are induced by the transient evolution, than when they are produced by the advective terms.

Figure 3 shows the first three time steps and the 20th time step for  $k = 1$  and  $u = 10$ . This is a case dominated by diffusion. The time step used was  $\Delta t = 10^{-3}$ , we use 14 equal size linear elements in space and  $\theta = \frac{1}{2}$  for the time-integration scheme.

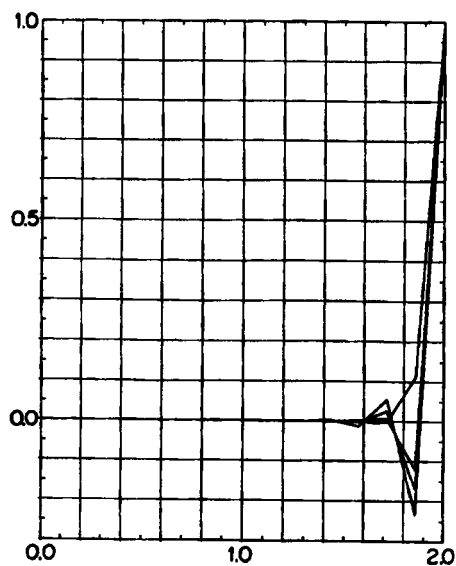
We can see that the Galerkin approach as well as the upwinded approach using the standard SUPG with optimal upwinding parameter both give very similar results, with spurious oscillations during the first time steps. These local oscillations disappear before the 20th time step. The solutions using the new Galerkin Least-Square method (GLS<sub>2</sub>) do not present any significant oscillations.

Figure 4 shows the same problem for  $u = 20$ . This case represents a more interesting situation because the advective terms are important enough to induce oscillations in the stationary state. The Galerkin approach (Figure 4(a)) produces spurious oscillations at all time steps, including the stationary state. The optimal upwinding approach stabilizes the stationary solution but not the initial steps where large negative values of the function are present. Figure 4(c) shows the perfectly stabilized GLS<sub>2</sub> solution from  $t = 0$  until the last time step.

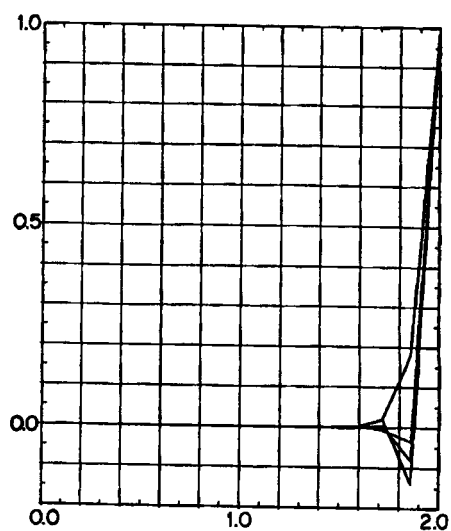
Finally in Figure 5 the advection-dominated flow with  $u = 100$  is tested, for which the boundary layer is smaller than the first element, even in the stationary state. The exact solution will be  $\phi = 0$  in all the interior nodes from the first time step. Figure 5(a) displays the oscillating behaviour obtained with the standard Galerkin approach. In Figure 5(b), the optimal upwinding solution is shown. Note that no negative oscillations are obtained although the first three steps are overdiffusive. The solution approaches the correct steady state but from above, which is not in agreement with the physics of the problem. Figure 5(c) shows the GLS<sub>2</sub> method in which the stationary solution is obtained from the first time step.

In order to analyse the accuracy of the method, we have compared the results with the analytical solution of this problem proposed in Reference 15. The results are plotted in Figures 3(d), 4(d) and 5(d). We can see that in the 3 cases, the first time step is perfectly captured with 0 per cent error while the other methods (Galerkin and SUPG) show large negative values (23 per cent and 14 per cent error, respectively). The second and third time step, show some overdiffuse results which are more important in the diffusive-dominant case, with errors which are never larger than 7 per cent (step 3,  $u = 10$ ). Finally, when the time approaches the stationary value, the proposed method is coincident with SUPG and, as it is well known, this method gives exact nodal values for this particular case.

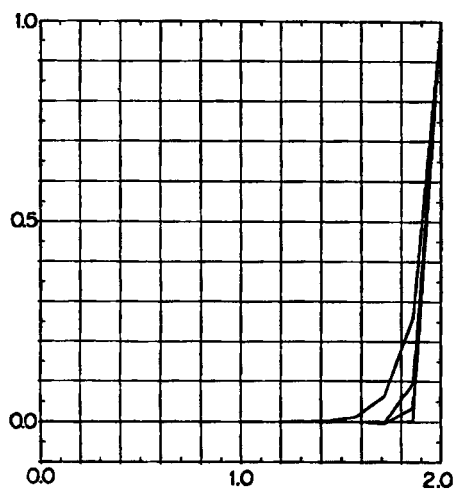
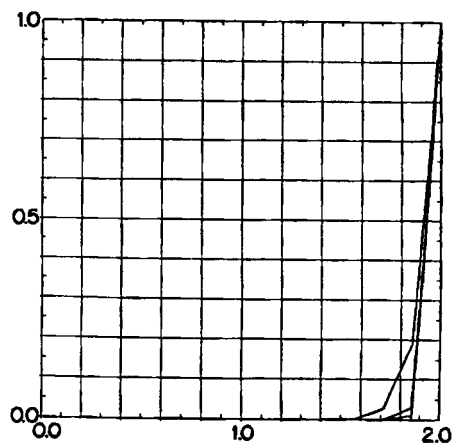
The accuracy of the method may be analysed in a more general context. First of all, it must be noted that oscillations may be the source of larger errors in non-linear problems (for instance in phase-change problems). On the other hand, the accuracy of the method is closely related to the integration technique used. A detailed study of this kind is outside the scope of this paper. Nevertheless, limiting the analysis to the  $\theta$ -methods, the proposed scheme eliminates the oscillations for any value of  $\theta$  used. It is well known that  $\theta = 0.5$  introduces the largest oscillations in standard  $\theta$ -methods although it is the more accurate value regarding the numerical diffusion. The present method allows to use this very accurate value of  $\theta$  while eliminating the spurious oscillations.



a) Galerkin



b) SUPG

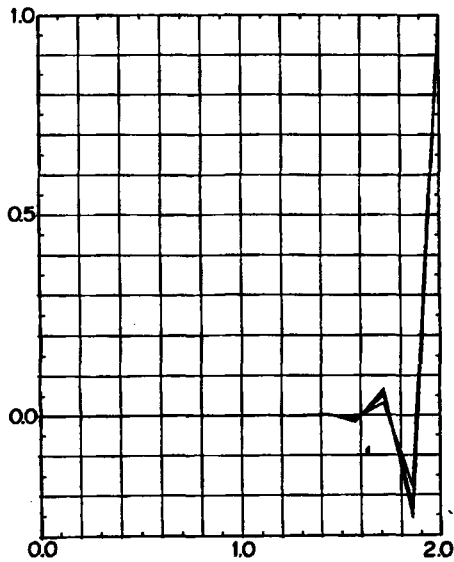
c) GLS<sub>2</sub>

d) Analytical

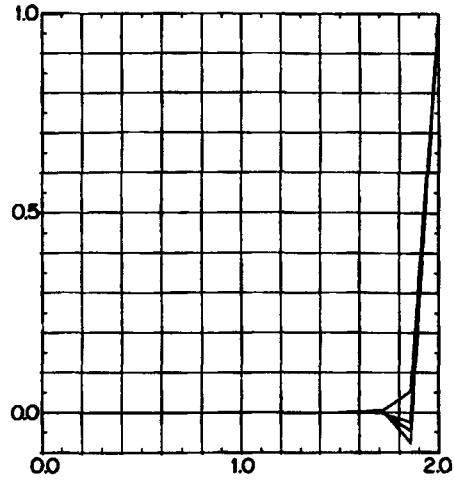
Figure 3. Diffusive dominant problem I ( $\mu = 10$ )

The second example tested is the same as the previous one with the following boundary conditions:

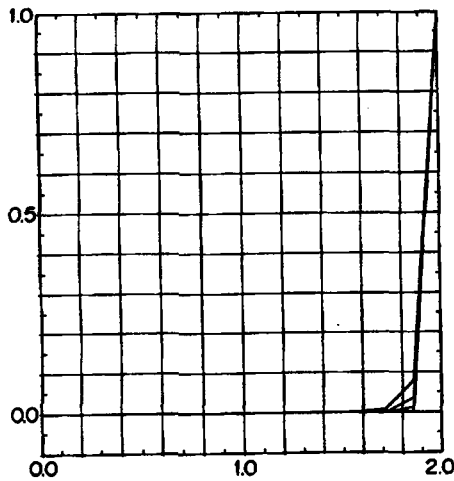
$$\begin{aligned}\phi(0, t) &= 1 \\ \phi(2, t) &= 0\end{aligned}\tag{41}$$



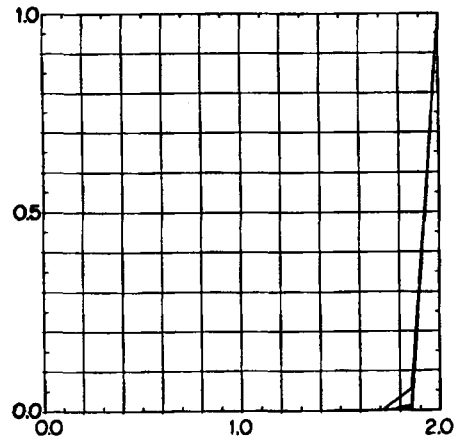
a) Galerkin



b) SUPG



c) GLS<sub>2</sub>



d) Analytical

Figure 4. Advective-diffusive problem I ( $u = 20$ )

and the same initial condition  $\phi(x, t_0) = 0$ .

In this case, the sharp gradients developed during the transient solution occur at the left end of the domain, while those due to the convective term start on the right. It is interesting to see the behaviour of the Galerkin and GLS methods for this case, and to observe the capability of the

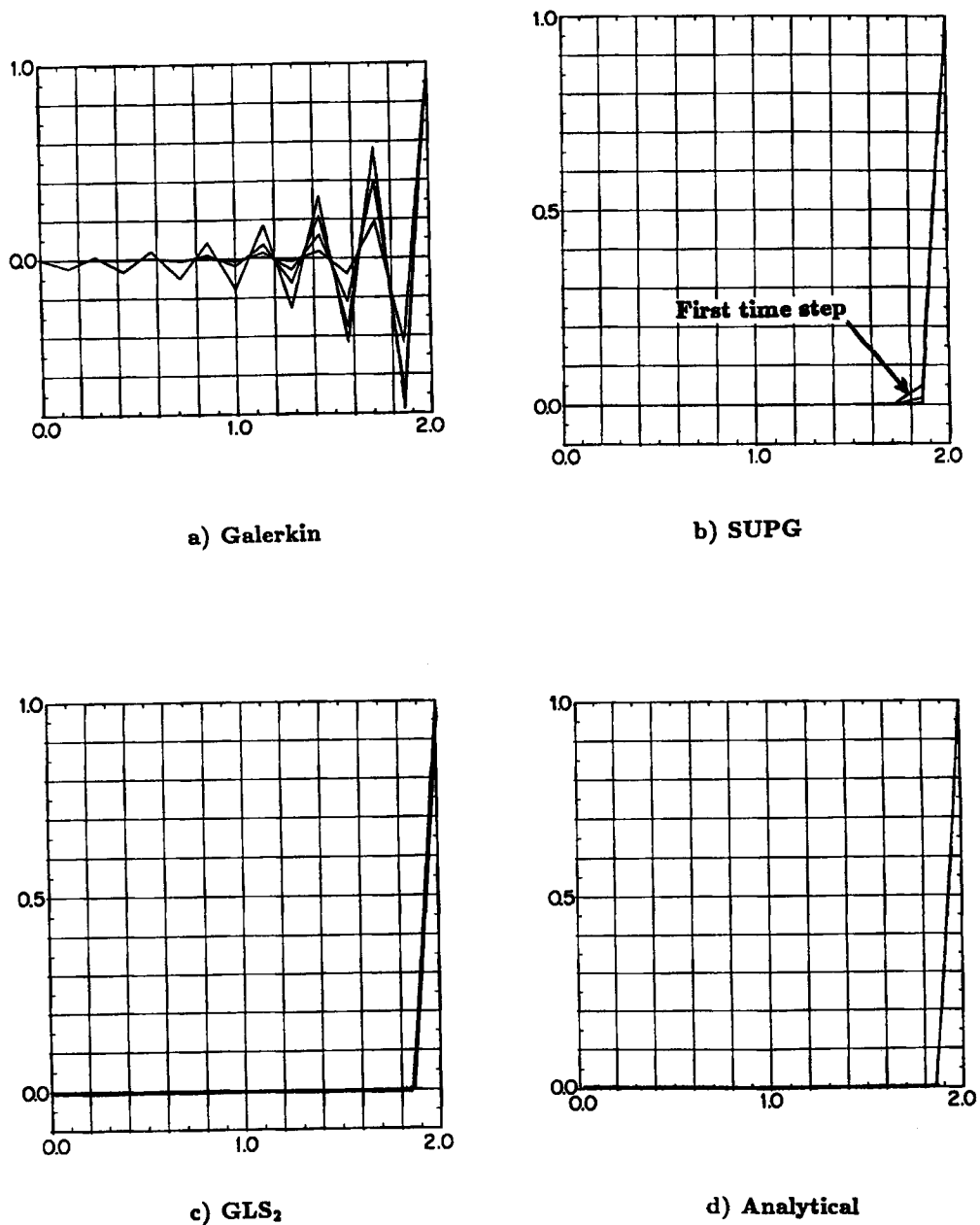


Figure 5. Advective-dominant problem I ( $u = 100$ )

GLS<sub>2</sub> method to recognize automatically the two different types of gradients, and to introduce the correct Petrov–Galerkin correction to eliminate oscillations at different locations and different times.

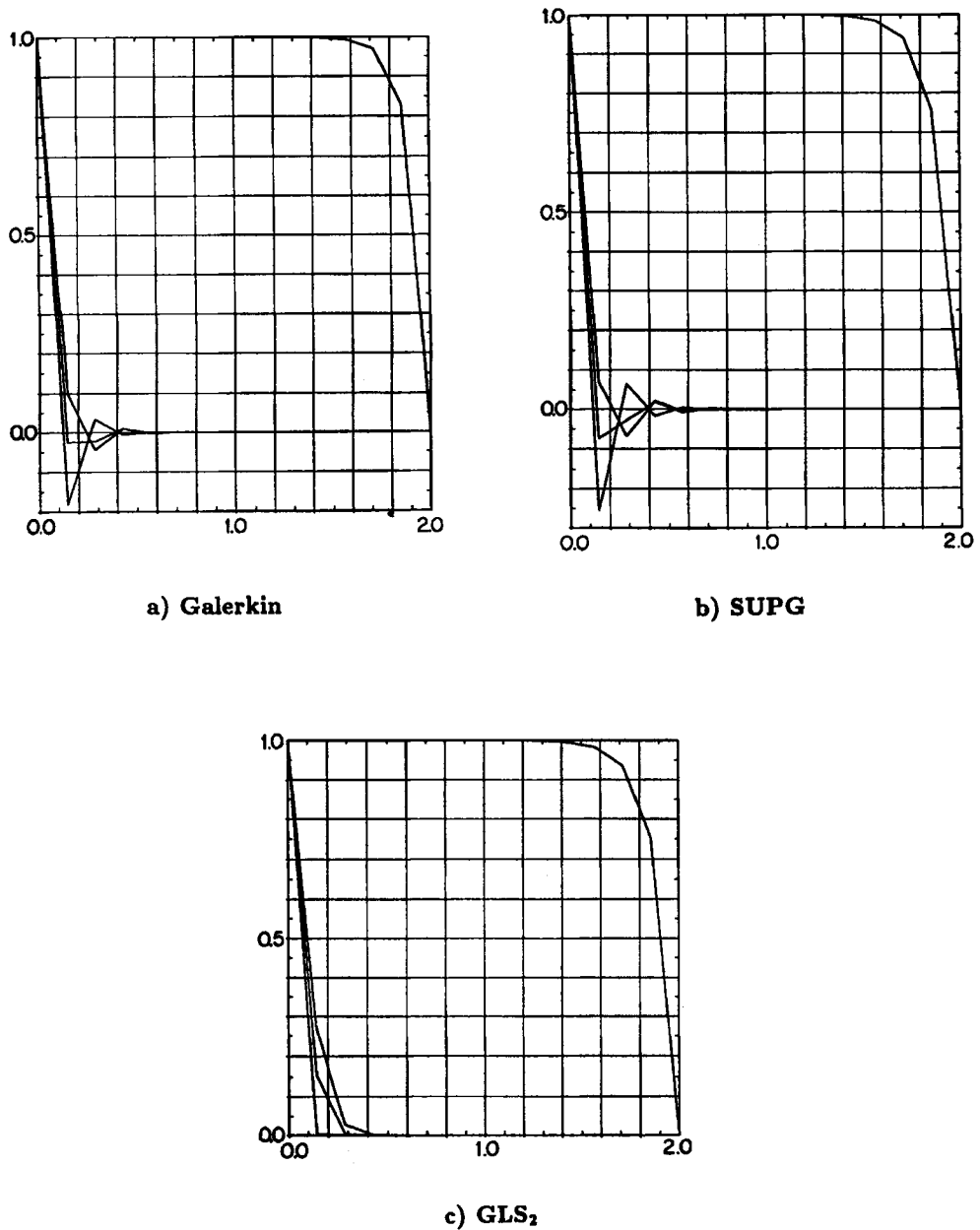
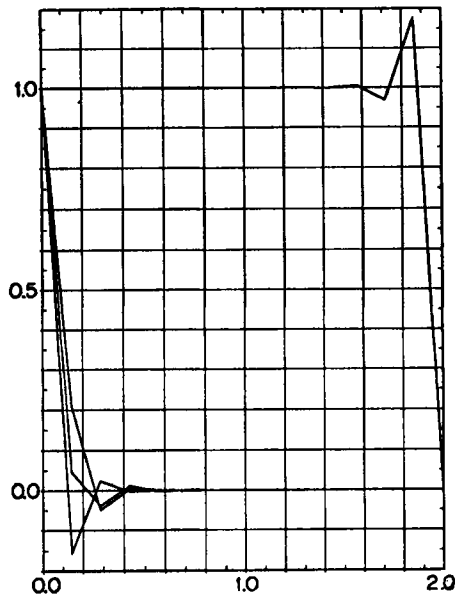
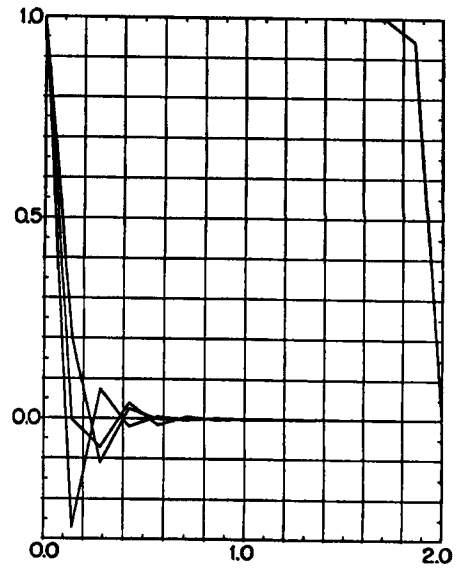


Figure 6. Diffusive-dominant problem II ( $u = 10$ )

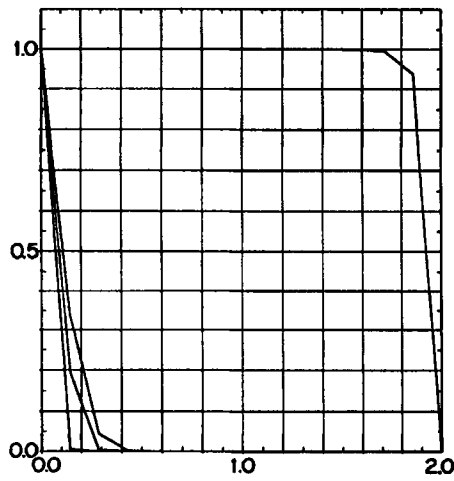
Figure 6 shows the diffusion-dominated problem. The same time step and  $\theta$  method as in the previous problem was used. The three initial time steps and the stationary solution are shown. For  $u = 10$ , all three methods give stable solutions on the stationary state but just the GLS<sub>2</sub> stabilizes the initial steps. For  $u = 20$  (Figure 7), the Galerkin approach gives oscillatory



a) Galerkin



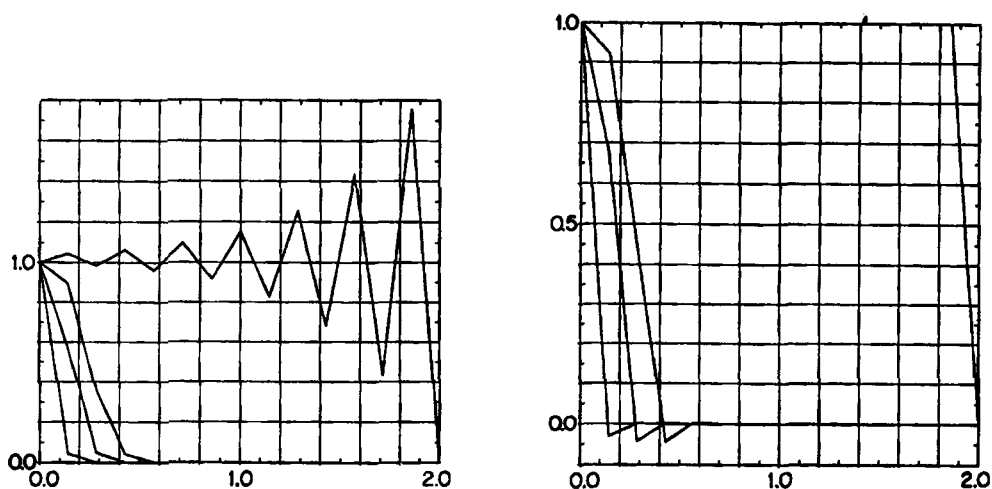
b) SUPG



c) GLS<sub>2</sub>

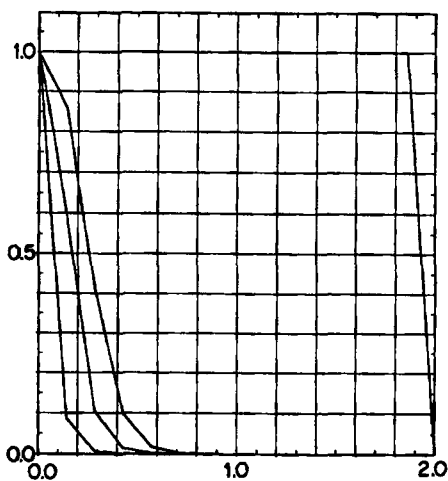
Figure 7. Advective-diffusive problem II ( $u = 20$ )





a) Galerkin

b) SUPG



c) GLS<sub>2</sub>

Figure 8. Advective-dominant problem II ( $\nu = 100$ )

behaviour in both ends while the optimal upwinding solution oscillates only during the initial steps. Finally, Figure 8 shows the advection-dominated case.

It must be noted, that the oscillations that occur during the transient on the Galerkin and GLS schemes, are related to the time-integration technique, the  $\theta$  parameter and the time step.

No oscillations were found for such schemes for some very particular combination of the  $Pe$ , and  $\theta$  parameters. Nevertheless, the  $GLS_2$  strategy stabilizes the results independently of the Peclet number and the time-integration scheme chosen and for any possible time step used. This is the most important advantage which will be explored further in future works.

## 6. CONCLUSIONS

A new Petrov–Galerkin formulation for the solution of the transient advective–diffusive equation has been presented. The proposed method stabilizes the oscillations which appear in the numerical solution of these equations when sharp gradients are present. The paper shows that the way to stabilize the oscillations is different depending on the nature of the gradients. Those sharp gradients produced by transient sharp loads must be stabilized in a different way than the ones existing in the vicinity of boundary layers or shocks when the advective terms are dominant.

Petrov–Galerkin techniques with weighting functions based on unsymmetric perturbation to the shape functions have been used to stabilize convection-dominated problems. In this paper we have shown that Petrov–Galerkin formulations that use a symmetric perturbation to the weighting functions can also play an important role to stabilize time-dependent equations with sharp gradients when these are produced by transient loads.

Finally, it is interesting to note that this new Petrov–Galerkin approach can be seen as a generalization of the Galerkin Least-Square method, introducing two independent parameters for stabilization. This idea can also be generalized to systems of equations such as the compressible and incompressible Navier–Stokes equations.

## REFERENCES

1. J. C. Heinrich, P. S. Huyakorn, O. C. Zienkiewicz and A. R. Mitchell, 'An upwind finite element scheme for two-dimensional convective transport equation', *Int. j. numer. methods eng.*, **11**, 131–143 (1977).
2. D. W. Kelly, S. Nakazawa, O. C. Zienkiewicz and J. C. Heinrich, 'A note on upwinding and anisotropic balancing dissipation in finite element approximations to convective diffusion problems', *Int. j. numer. methods eng.*, **15**, 1705–1711 (1980).
3. A. Brooks and T. J. R. Hughes, 'Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations', *Comput. Methods Appl. Mech. Eng.*, **32**, 199–259 (1982).
4. C. C. Yu and J. C. Heinrich, 'Petrov–Galerkin method for multidimensional, time-dependent, convective–diffusion equations', *Int. j. numer. methods eng.*, **24**, 2201–2215 (1987).
5. C. Johnson, U. Navert and J. Pitkaranta, 'Finite element methods for linear hyperbolic problems', *Comput. Methods Appl. Mech. Eng.*, **45**, 285–321 (1984).
6. J. J. Westerink and D. Shea, 'Consistent higher degree Petrov–Galerkin methods for the solution of the transient convection–diffusion equation', *Int. j. numer. methods eng.*, **28**, 1077–1102 (1989).
7. T. E. Tezduyar and T. J. R. Hughes, 'Finite element formulations for convection dominated flows with particular emphasis on the compressible Euler equations', *AIAA Paper 83-0125, Proc. AIAA 21st Aerospace Sciences Meetings*, Reno, Nevada, 1983.
8. W. L. Wood and R. W. Lewis, 'A comparison of time marching schemes for the transient heat conduction equation', *Int. j. numer. methods eng.*, **9**, 679–689 (1975).
9. S. R. Idelsohn, M. A. Storti, N. Nigro and G. Buscaglia, 'A Petrov–Galerkin formulation for advection–reaction–diffusion problem', *Comput. Methods Appl. Mech. Eng.* (1996), to appear.
10. T. E. Tezduyar, 'Stabilized finite element formulations for incompressible flows computations', *Adv. Appl. Mech.*, **28**, 1–44 (1991).
11. T. Tezduyar and Y. Park, 'Discontinuity capturing finite element formulation for nonlinear convection–diffusion–reaction equations', *Comput. Methods Appl. Mech. Eng.*, **59**, 307–325 (1986).

12. L. P. Franca and E. G. Dutra do Carmo, 'The Galerkin gradient least-squares method', *Comput. Methods Appl. Mech. Eng.*, **74**, 41–54 (1989).
13. R. Codina, 'A shock capturing anisotropic diffusion for the finite element solution of the diffusion-convection-reaction equation', in E. Oñate (ed.), *Numerical Methods in Engineering and Applied Sciences*, CIMNE, Barcelona, Spain, 1993.
14. T. J. R. Hughes, L. P. Franca and G. Hulbert, 'A new element formulation for computational fluid dynamics: VIII. The Galerkin-least-squares method for advective-diffusive equations', *Comput. Methods Appl. Mech. Eng.*, **59**, 307–325 (1986).
15. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd edn., Oxford University Press, Oxford, 1959.