INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING

Equivalent FE and BE Forms of a Substructure Oriented Boundary Solution Approach

J. Jirousek

Publication, CIMNE Nº 20, Mayo 1992

Equivalent FE and BE Forms of a Substructure Oriented Boundary Solution Approach

J. Jirousek

Publication CIMNE Nº-20, Mayo 1992

International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

1. INTRODUCTION

More than a decade ago, the so-called hybrid-Trefftz (HT) FE model[2] has been presented as an attempt to combine the high accuracy and fast convergence rate exhibited by the boundary solution (BS) approaches with the versatility of the FE approach. In this model a hybrid method has been applied, including the use of an auxiliary interelement displacement or traction frame, to link up the internal displacement fields of the elements. Such fields, chosen so as to a priori satisfy the governing differential equations (Lamé-Navier) have conveniently been represented as the sum of a particular integral of the non homogeneous equations and a suitably truncated T-complete set of regular homogeneous solutions multiplied by undetermined coefficients. The element formulation, during which the internal parameters are eliminated at the element level, in the end leads to the standard force-displacement relationship, with a symmetric positive definite stiffness matrix. Various versions of these elements, including the highly efficient HT p-elements[3,4] have later successfully been implemented, along with an afficient modul for adaptive reliability assurance[4] , into the FE program SAFE[7]. The HT approach has succeded in preserving the advantages of the BS and BE approaches and avoid their drawbacks (such as the fully populated non symmetric coefficient matrices of the resulting BEM equations, the need for elaborate integration schemes due to the singular Kelvin's type fundamental solutions, need for special technics to solve problems governed by non homogeneous equations...etc.). Zielinski and Zienkiewicz[1] have studied an alternative method whereby, the BS subdomains have been linked up by a least square procedure without an auxiliary frame. Unfortunately, in the form presented in their paper, this highly efficient approach (where again the internal fields are identical to those used in the HT elements) is not suited for implementation into FE programs. Indeed the resulting equations are formed directly, without the usual FE assembly process.

To by-pass this difficulty, the present paper outlines two equivalent formulations which take respectively form of frameless Trefftz-type finite elements and of nonconventional symmetric boundary elements. The theorical formulation is presented in the next section. These elements are assembled following the standard assembly procedure of the direct stiffness method and their implementation into FE codes is straightforward. Pending numerical assessment, the last section speculates upon the expected advantages of the presented approach over the, now well established, HT elements.

satisfied. This implies

$$\mathcal{L}\overset{\circ}{\mathrm{u}}_i = \bar{\mathrm{b}} \quad \text{and} \quad \mathcal{L}\mathrm{N}_i = 0 \quad \text{on} \quad \Omega_i\,. \tag{2a, b}$$

In practice (to insure convergence towards the exact solution when m is increased), N_i should be obtained by truncating an infinite series of functions which must be T-complete (i.e. be capable of representing, under very general conditions, any possible solution over the subdomain[1,4]). Furthermore, to prevent numerical problems, the trial functions have to be defined in terms of the local coordinates originating at the "center" i of Ω_i (Fig.1a) rather than in terms of the global coordinates x,y.

We designate now by

$$I = I(\mathrm{c}) = \quad \text{Min.}, \tag{3}$$

where

$$\mathrm{c} = \{\mathrm{c}_1, \mathrm{c}_2 \ldots \mathrm{c}_n\}, \tag{3a}$$

the least square statement used to enforce in an average integral way, the boundary conditions (1a,b) and the interdomain continuity. If Ω_1 and Ω_2 are two neighbouring subdomains, such conditions may be defined as

$$\mathrm{u}_1 - \mathrm{u}_2 = 0 \quad \text{and} \quad \mathrm{t}_1 + \mathrm{t}_2 = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2\,. \tag{4a, b}$$

The vanishing variation of I may be written as

$$\delta I = \delta\mathrm{c}^T \frac{\partial I}{\partial \mathrm{c}} = \delta\mathrm{c}^T(\overset{\circ}{\mathrm{R}} + \mathrm{Kc}) = 0 \tag{5}$$

which yields for the undetermined coefficients c the following symmetric system of linear equations:

$$\mathrm{Kc} = -\overset{\circ}{\mathrm{R}}\,. \tag{5a}$$

2.1.1 FE form of the approach

For implementation into FE codes, the equations (5a) have to be obtained by the standard assembly process (direct stiffness method) from the individual contributions of suitably defined finite elements. The domain, initially divided into subdomains Ω_i, will now be covered by a mesh of triangular and quadrilateral elements shown in Fig 1b. Here, two kinds of nodes appear:

- nodes o, numbered $i = 1, 2, \ldots n$ and associated each with corresponding vector of "nodal" parameters $\mathbf{c}_i$;
- geometrical nodes $\bullet$, not involved in the assembly process.

The FE mesh involves two types of elements displayed in Fig.2:

- the quadrilateral internal elements, such as the element of Fig. 2a with the displacement field defined in a piecewise manner as

  $\mathbf{u} = \mathbf{u}_1 = \overset{\circ}{\mathbf{u}}_1 + \mathbf{N}_1\mathbf{c}_1$ on the triangle 1AB

  $\mathbf{u} = \mathbf{u}_2 = \overset{\circ}{\mathbf{u}}_2 + \mathbf{N}_2\mathbf{c}_2$ on the triangle 2BA;

- the triangular elements adjacent to the portions Γ_u or Γ_t of $\partial\Omega$, such as the elements of Fig. 2b,c wich possess each a single displacement field

$$\mathbf{u} = \mathbf{u}_1 = \overset{\circ}{\mathbf{u}}_1 + \mathbf{N}_1\mathbf{c}_1 .$$

If applied to the whole FE mesh, this definition results in the initial subdomain pattern of Fig 1a. As a consequence, the required displacement and traction conditions at interelement bounderies are automatically

2.2.1. Internal element (Fig. 2a)

In order to link the two displacement fields, $\mathbf{u}_1$ and $\mathbf{u}_2$, we set simply

$$I^e = \alpha \frac{E}{d} \int_A^B (\mathbf{u}_1^T - \mathbf{u}_2^T)(\mathbf{u}_1 - \mathbf{u}_2) ds + \beta \frac{d}{E} \int_A^B (\mathbf{t}_1^T + \mathbf{t}_2^T)(\mathbf{t}_1 + \mathbf{t}_2) ds \quad (8)$$

The factors $\frac{E}{d}$ and $\frac{d}{E}$ ($E=$ Young modulus, $d=$ arbitrarily chosen length) serve the purpose of getting a physically meaningful functional. Furthemore, α and β are optional nondimensional weighting factors which may be used to optimally tune the satisfaction of the displacement and traction conditions over the FE mesh. Since $\mathbf{c}^e = \{\mathbf{c}_1, \mathbf{c}_2\}$, the equation (7b) yields the relationship (6) in the form

$$\begin{Bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{Bmatrix} = \begin{Bmatrix} \overset{\circ}{\mathbf{r}}_1 \\ \overset{\circ}{\mathbf{r}}_2 \end{Bmatrix} + \begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{Bmatrix} \quad (9)$$

where $\mathbf{k}_{21} = \mathbf{k}_{12}^T$. The subscript "e" has been dropped. Substituting into (8) for displacement and tractions from

$$\mathbf{u}_i = \overset{\circ}{\mathbf{u}}_i + \mathbf{N}_i \mathbf{c}_i \rightarrow \mathbf{t}_i = \overset{\circ}{\mathbf{t}}_i + \mathbf{T}_i \mathbf{c}_i \qquad \text{(i=1,2)} \quad (10)$$

and performing the variation, we get

$$\left.\begin{aligned} \overset{\circ}{\mathbf{r}}_1 &= \alpha \tfrac{E}{d} \textstyle\int_A^B \mathbf{N}_1^T (\overset{\circ}{\mathbf{u}}_1 - \overset{\circ}{\mathbf{u}}_2) ds + \beta \tfrac{d}{E} \textstyle\int_A^B \mathbf{T}_1^T (\overset{\circ}{\mathbf{t}}_1 + \overset{\circ}{\mathbf{t}}_2) ds \\ \overset{\circ}{\mathbf{r}}_2 &= \alpha \tfrac{E}{d} \textstyle\int_A^B \mathbf{N}_2^T (\overset{\circ}{\mathbf{u}}_2 - \overset{\circ}{\mathbf{u}}_1) ds + \beta \tfrac{d}{E} \textstyle\int_A^B \mathbf{T}_2^T (\overset{\circ}{\mathbf{t}}_2 + \overset{\circ}{\mathbf{t}}_1) ds \end{aligned}\right\} \quad (11a)$$

and

$$\left.\begin{aligned} \mathbf{k}_{11} &= \alpha \tfrac{E}{d} \textstyle\int_A^B \mathbf{N}_1^T \mathbf{N}_1 ds + \beta \tfrac{d}{E} \textstyle\int_A^B \mathbf{T}_1^T \mathbf{T}_1 ds \\ \mathbf{k}_{12} = \mathbf{k}_{21}^T &= -\alpha \tfrac{E}{d} \textstyle\int_A^B \mathbf{N}_1^T \mathbf{N}_2 ds + \beta \tfrac{d}{E} \textstyle\int_A^B \mathbf{T}_1^T \mathbf{T}_2 ds \\ \mathbf{k}_{22} &= \alpha \tfrac{E}{d} \textstyle\int_A^B \mathbf{N}_2^T \mathbf{N}_2 ds + \beta \tfrac{d}{E} \textstyle\int_A^B \mathbf{T}_2^T \mathbf{T}_2 ds \end{aligned}\right\} \quad (11b)$$

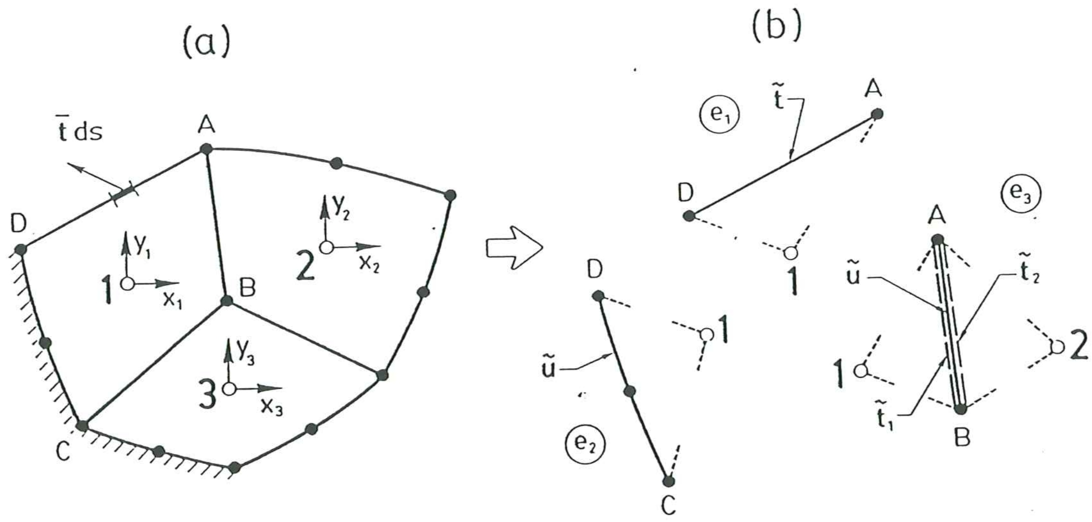

Fig. 3 BE version of approach: (a) Subdivision into subdomains $\Omega_1, \Omega_2 \ldots$ with piece-wise aproximations $\mathbf{u}_1, \mathbf{u}_2 \ldots$; (b) Boundary elements with BE fields $(\tilde{\mathrm{u}}, \tilde{\mathrm{t}})$ and associated external nodes

associated with one, and each internal element with two, external nodes (Fig. 3b) along with the following definition of the element displacement or/and traction fields:

$$e_1 : \tilde{\mathrm{u}} = \bar{\mathrm{u}} \tag{17a}$$

$$e_2 : \tilde{\mathrm{t}} = \bar{\mathrm{t}} \tag{17b}$$

$$\left.\begin{aligned} e_3 : \tilde{\mathrm{u}} &= \tfrac{1}{2}(\mathrm{u}_1 + \mathrm{u}_2) = \tfrac{1}{2}(\mathring{\mathrm{u}}_1 + \mathring{\mathrm{u}}_2) + \tfrac{1}{2}(\mathbf{N}_1\mathbf{c}_1 + \mathbf{N}_2\mathbf{c}_2) \\ \tilde{\mathrm{t}}_1 &= \tfrac{1}{2}(\mathrm{t}_1 - \mathrm{t}_2) = \tfrac{1}{2}(\mathring{\mathrm{t}}_1 - \mathring{\mathrm{t}}_2) + \tfrac{1}{2}(\mathbf{T}_1\mathbf{c}_1 - \mathbf{T}_2\mathbf{c}_2) \\ \tilde{\mathrm{t}}_2 &= \tfrac{1}{2}(\mathrm{t}_2 - \mathrm{t}_1) = \tfrac{1}{2}(\mathring{\mathrm{t}}_2 - \mathring{\mathrm{t}}_1) + \tfrac{1}{2}(\mathbf{T}_2\mathbf{c}_2 - \mathbf{T}_1\mathbf{c}_1) \end{aligned}\right\} \tag{17c}$$

Note that the definitions (17c) are such that

$$\mathrm{u}_1 - \tilde{\mathrm{u}} = 0 \quad \text{or} \quad \mathrm{u}_2 - \tilde{\mathrm{u}} = 0 \quad \text{implies} \quad \mathrm{u}_1 - \mathrm{u}_2 = 0$$

and

$$\mathrm{t}_1 - \tilde{\mathrm{t}}_1 = 0 \quad \text{or} \quad \mathrm{t}_2 - \tilde{\mathrm{t}}_2 = 0 \quad \text{implies} \quad \mathrm{t}_1 + \mathrm{t}_2 = 0$$

3. CONCLUDING REMARKS

An aproach has been presented which attempts to optimally combine the advantages of the conventional FE and BS procedures. The approach can indifferently be considered as a FE or a nonconventional substructure-oriented BE method. Since the resulting element stiffness matrices are of the symmetric positive definite type, and the rules of the standard assembly procedure (direct stiffness method) hold true, the implementation into existing FE codes is straightforward. The elements are of the p-type and the accuracy can be controlled within very large limits by simply adjusting (uniformly or selectively) the number of DOF associated with their nodes.

Compared to the well established HT elements (working towards a similar aim), the presented approach appears as substancially more efficient from the computational point of view. Indeed, the coefficients of the element stiffness matrix are now obtained straightforwardly and the time consuming generation and inversion of the auxiliary matrix **H**, associated with the hybrid method, is avoided.

Although the HT elements are well known to be exceptionally accurate[6], it is expected that the presented approach has the capacity to reach even higher degree of accuracy. Indeed the two basic HT versions, the displacement frame model (HT-D) and the dual traction frame model (HT-T[8]) enforce more strongly either the displacement conformity or the traction reciprocity (and

ACKNOWLEDGEMENT

This work was carried out while visiting the International Center for Numerical Methods in Engineering in Barcelona. The author wishes to thank Professor E. Oñate for making the visit possible and for the use of the Center's facilities.

8. J. Jirousek and A.P. Zielinski, '*Dual hybrid-Trefftz Formulation Based on Independent Boundary Traction Frame*', Internal Report LSC 91/21, Swiss Federal Institute of Technology, Lausanne, 1991.