Super (a, d)-Edge Antimagic Total Labeling of Connected Ferris Wheel Graph

Djoni Budi Sumarno[3,4], Dafik[1,2,*], Kiswara Angung Santoso[3]

[1]CGANT - University of Jember
[2]Department of Mathematics Education - FKIP University of Jember,
[3]Department of Mathematics - FMIPA University of Jember,
[4]SMP Negeri 1 Bondowoso,
[*]d.dafik@unej.ac.id

Abstract. Let G be a simple graph of order p and size q. Graph G is called an (a, d)*-edge-antimagic total* if there exist a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ such that the edge-weights, $w(uv) = f(u)+f(v)+f(uv); u, v \in V(G), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a graph G is called *super* if the smallest possible labels appear on the vertices. In this paper we study super (a, d)-edge antimagic total properties of connected of Ferris Wheel $FW_{m,n}$ by using deductive axiomatic method. The results of this research are a lemma or theorem. The new theorems show that a connected ferris wheel graphs admit a super (a, d)-edge antimagic total labeling for $d = 0, 1, 2$. It can be concluded that the result of this research has covered all feasible d.

Key Words : *(a, d)-edge antimagic vertex labeling, super (a, d)-edge antimagic total labeling, Ferris Wheel graph $FW_{m,n}$.*

Introduction

In daily life, many problems can be modeled by a graph. One of the interesting topics in graph theory is graph labeling. There are various types of graph labe-ling, one is a super (a, d)-edge antimagic total labeling (SEATL). This problem is quite difficult as assigning a label on each vertex is considered to be NP hard problem, in other word it can not be traced in a polynomial times. There is no guarantee that for a specific family of graph always admit a super (a, d)-edge antimagic total labeling for all feasible d, see Dafik (2007) for detail.

By a *labeling* we mean any mapping that carries a set of graph elements onto a set of numbers, called *labels*. In this paper, we deal with labelings in which the domain are the set of all vertices and edges. Such type of labeling belongs to the class of *total* labelings. We define the *edge-weight* of an edge $uv \in E(G)$ under a total labeling to be the sum of the vertex and edge labels which respectively corresponds to vertices u, v and edge uv.

An (a, d)*-edge-antimagic total labeling* on a graph G is a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v), uv \in E(G)$, form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)*-edge-antimagic total graph*. Such a graph G is called *super* if the smallest possible labels appear on the vertices.

Thus, a *super (a, d)-edge-antimagic total graph* is a graph that admits a super (a, d)-edge-antimagic total labeling.

These labelings, introduced by Simanjuntak et al. (2000), are natural extensions of the concept of magic valuation, studied by Hartsfield and Ringel (2002); see also (Bača et al., 2001), (Bodendiek and Walther, 1996), (Bača et al., 2008), (Ringel and Lladó, 1996), (Wallis et al., 2000). The concept of super edge-magic labeling, firstly defined by Enomoto et al. (1998) gave motivations to other researchers to investigate the different forms of antimagic graphs. For example, Bača et al. (2008), Bača et al. (2001), and Dafik et al. (2008) investigated the existence of the super (a, d)-edge-antimagic total graph.

Some constructions of super (a, d)-edge-antimagic total labelings for the disjoint union of stars and the disjoint union of s-Crowns have been shown by Dafik et al. (2008) and Bača et al. (2009) respectively, and super (a, d)-edge-antimagic total labelings for disjoint union of caterpillars have been described by Bača et al. (2008). Dafik et al. (2013) also found some families of well-defined Graph which admits super (a, d)-edge-antimagic total labelings, namely Triangular Book and Diamond Ladder. The existence of super (a, d)-edge-antimagic total labeling for connected Disc Brake graph had been found also by (Arianti et al., 2014).

In this paper we investigate the existence of super (a, d)-edge-antimagic total labeling of Ferris Wheel graph, denoted by $FW_{m,n}$.

Two Useful Lemmas

In this section, we recall two known lemmas that will be useful in the next section. The first lemma, see Sugeng et al. (2006), is a necessary condition for a graph to be super (a,d)-edge antimagic total, providing a least upper bound for feasible value of d.

Lemma 1 *(Sugeng et al., 2006). If a (p, q)-graph is super (a, d)-edge-antimagic total then $d \leq \frac{2p+q-5}{q-1}$.*

Proof. Assume that a (p, q)-graph has a super (a, d)-edge-antimagic total labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v), uv \in E(G)$, form an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. The minimum possible edge-weight in the labeling f is at least $1 + 2 + p + 1 = p + 4$. Thus, $a \geq p + 4$. On the other hand, the maximum possible edge-weight is at most $(p-1) + p + (p+q) = 3p + q - 1$. Thus, $a + (q-1)d \leq 3p + q - 1$. It gives the desired upper bound for the difference d. $\square$

The second lemma obtainded by Figueroa-Centeno et al. (2001), gives a necessary and sufficient condition for a graph to be super $(a, 0)$-edge-antimagic total.

Lemma 2 *(Figueroa-Centeno et al., 2001). A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $a = p + q + s$, where $s = min(S)$ and $S = \{a - (p+1), a - (p+2), \ldots, a - (p+q)\}$.*

Previously, the lemma states that a (p, q)-graph G is super $(a, 0)$-edge-antimagic total if and only if there exists an (a-p-q; 1)-edge-antimagic vertex labeling.

Research Methods

To find the existence of a super (a, d)-edge-antimagic total labeling of Ferris Wheel graph, we use a pattern recognition and axiomatic deductive approach. The approach was carried out through the following steps: (1) obtaining the number of vertex p and size q of graph $FW_{m,n}$, (2) determining the upper bound of feasible d, (3) By using a pattern recognition, we determine the label of the vertices of $FW_{m,n}$, such type of labeling belongs to *EAVL (edge antimagic vertex labeling)*, (4) if the label of EAVL is expandable, then we continue to determine its bijective function, (5) By using deductive approach, we search the label of the edges of $FW_{m,n}$, it extends to *SEATL (super-edge antimagic total labeling)* with feasible values of d, (6) Finally, determine the bijective function of super-edge antimagic total labeling of graph $FW_{m,n}$.

The Result

From now on, we will describe the result of the existence of a super (a, d)-edge-antimagic total labeling of Ferris Wheel graph, denoted by $FW_{m,n}$. Ferris Wheel Graph is a connected graph with the following cardinality: Vertex set $V(FW_{m,n}) = \{x_{i,j}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_{i,j,k}; 2 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2\}$ and edge set $E(FW_{m,n}) = \{x_{i,j}x_{i,j+1}; 1 \leq j \leq m, 1 \leq i \leq n-1\} \cup \{x_{i,j}x_{i,j,k}; 2 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2\} \cup \{x_{i,j}x_{i+1,j,k}; 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2\} \cup \{x_{i,j,k}x_{i,j+1,k+1}; 2 \leq i \leq m, 1 \leq j \leq n-1, k = 1\} \cup \{x_{i,j,k}x_{i,j-1,k-1}; 2 \leq i \leq m, j = n, k = 2\} \cup \{x_{i,j,k-1}x_{i,j-(n-1),k}; 2 \leq i \leq m, j = n, k = 2\}$.

Considering the vertex set and the edge set, we have obtained that the order and the size of $FW_{m,n}$ are respectively $p = 3mn - 2n$ and $q = 6mn - 5n$. If Ferris Wheel graph has a super (a, d)-edge-antimagic total labeling then, it follows from Lemma 1 that the upper bound of d is $d \leq 2$ or $d \in \{0, 1, 2\}$. We start the result by providing the following lemma. It describes an $(a, 1)$-edge-antimagic vertex labeling for Ferris Wheel.

$\diamond$ **Teorema 1** *If $m \geq 2$ and $n \geq 1$ then the Ferris Wheel graph $FW_{m,n}$ has an $(\frac{n+3}{2}, 1)$-edge-antimagic vertex labeling.*

Proof. Define the vertex labeling

$$\begin{aligned}
f_1(x_{ij}) &= 3n(i-1) + \frac{j+1}{2}, \text{ for } 1 \leq i \leq m, 1 \leq j \leq n \\
f_1(x_{ij}) &= \frac{n+1}{2} + 3ni - 3n + \frac{j}{2}, \text{ for } 1 \leq i \leq m, 1 \leq j \leq n-1 \\
f_1(x_{ijk}) &= \frac{n+1}{2} + (3 + (1 + (-1)^{j+1}) + 3ni + (k-7)n + \left\lfloor \frac{j+1}{2} \right\rfloor, \\
&\quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n-1, 1 \leq k \leq 2 \\
f_1(x_{ijk}) &= \frac{n+1}{2} + 3 + nk + 3ni - 7n, \text{ for } 2 \leq i \leq m, j \text{ otherwise}, 1 \leq k \leq 2
\end{aligned}$$

The vertex labeling f_1 is a bijective function. The edge-weights of $FW_{m,n}$, under the labeling f_1, constitute the following sets

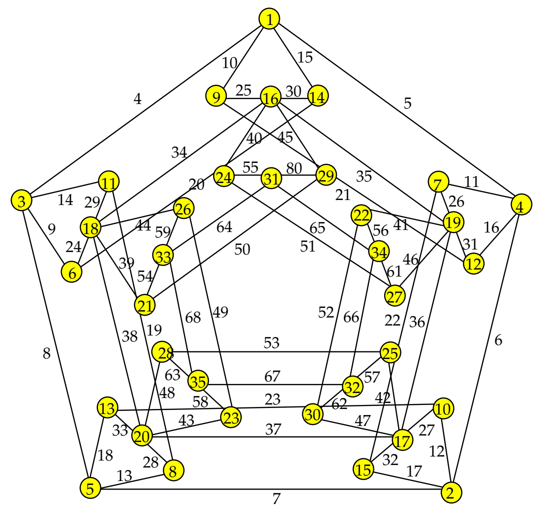

Figure 1: The example of $(\frac{n+3}{2}, 1)$-edge antimagic vertex labeling of $FW_{3,5}$

$$\begin{aligned}
w^1_{f_1}(x_{ij}x_{ij+1}) &= \frac{n+3}{2} + 6n(i-1) + j, \\
&\quad \text{for } 1 \le i \le m\,, 1 \le j \le n-1 \\
w^2_{f_1}(x_{ij}x_{i+1jk}) &= \frac{n+3}{2} + 6ni + (6-k)n + j, \\
&\quad \text{for } 1 \le i \le m\,, 1 \le j \le n-1\,, 1 \le k \le 2 \\
w^3_{f_1}(x_{ijk}x_{ij+1k+1}) &= \frac{n+3}{2} + 3n(2i-3), \\
&\quad \text{for } 2 \le i \le m\,, j = n-1\,, k = 1 \\
w^4_{f_1}(x_{ijk}x_{ij+1k+1}) &= \frac{n+3}{2} + 3n(2i-3) + j + 1, \\
&\quad \text{for } 2 \le i \le m\,, 1 \le j \le n-2\,, k = 1 \\
w^5_{f_1}(x_{ijk}x_{ijk}) &= \frac{n+3}{2} + 6ni + j + (k-9)n, \\
&\quad \text{for } 2 \le i \le m\,, 1 \le j \le n-1\,, 1 \le k \le 2 \\
w^6_{f_1}(x_{ij}x_{ij+1}) &= \frac{n+3}{2} + 6n(i-1) \\
&\quad \text{for } 1 \le i \le m\,, j \text{ otherwise}
\end{aligned}$$

$$
\begin{aligned}
w^7_{f_1}(x_{ij}x_{i+1jk}) &= \frac{n+3}{2} + 6n(i-1) + kn, \\
& \quad \text{for } 1 \le i \le m\,, j \,\text{otherwise}\,, 1 \le k \le 2 \\
w^8_{f_1}(x_{ijk-1}x_{ij-(n-1)k}) &= \frac{n+3}{2} + 3n(2i-3) + 1, \\
& \quad \text{for } 2 \le i \le m\,, j \,\text{otherwise}\,, k = 2 \\
w^9_{f_1}(x_{ijk}x_{ijk}) &= \frac{n+3}{2} + 3n(2i-3) + kn, \\
& \quad \text{for } 2 \le i \le m\,, j \,\text{otherwise}\,, k = 2
\end{aligned}
$$

It is easy to see that the set $\bigcup_{t=1}^{9} w^t_{f_1} = \{\frac{n+3}{2}, \frac{n+3}{2}+1, \dots, 6mn - \frac{9n+1}{2}\}$ consists of consecutive integers. Thus f_1 is a $(\frac{n+3}{2}, 1)$-edge antimagic vertex labeling. □

With the Theorem 1 in hand, and using Lemma 2, we obtain the following theorem.

◇ **Teorema 2** *The graph $FW_{m,n}$ has a super $(\frac{14mn+13n+3}{2}, 0)$-edge-antimagic total labeling for $n \ge 1$.*

◇ **Teorema 3** *The graph $FW_{m,n}$ has a super $(\frac{n+3}{2} + 3mn - 3m, 2)$-edge-antimagic total labeling for $n \ge 1$.*

Proof. Label the vertices of $FW_{m,n}$ by $f_2(x_{ij}) = f_1(x_{ij})$ and $f_2(x_{ijk}) = f_1(x_{ijk})$, and the edges by the following:

$$
\begin{aligned}
f_2(x_{ij}x_{i+1jk}) &= mn(1+2i) - m + j - 1, \\
& \quad \text{for } 1 \le i \le m\,, 1 \le j \le n-1,\ k = 1 \\
f_2(x_{ij}x_{i+1jk}) &= mn(1+2i) + j + 1, \\
& \quad \text{for } 1 \le i \le m\,, 1 \le j \le n-1,\ k = 2 \\
f_2(x_{ijk}x_{ij+1k+1}) &= mn(2i-1) + m + n + j - 1, \\
& \quad \text{for } 2 \le i \le m\,, 1 \le j \le n-2,\ k = 1 \\
f_2(x_{ij}x_{ijk}) &= mn(2i-1) + 2m + j + kn, \\
& \quad \text{for } 2 \le i \le m\,, 1 \le j \le n-1,\ 1 \le k \le 2 \\
f_2(x_{ij}x_{ij+1}) &= mn(1+2i) - m - n + j - 1, \\
& \quad \text{for } 1 \le i \le m\,, 1 \le j \le n-1 \\
f_2(x_{ij}x_{ij+1}) &= mn(1+2i) - 3m, \\
& \quad \text{for } 1 \le i \le m\,, j \,\text{otherwise}
\end{aligned}
$$

$$\begin{aligned}
f_2(x_{ij}x_{i+1jk}) &= mn(1+2i)+n(k-2)+1, \\
&\quad \text{for } 1 \le i \le m\,, j \text{ otherwise, } 1 \le k \le 2 \\
f_2(x_{ijk}x_{ij-1k-1}) &= mn(2i-1)+2m, \\
&\quad \text{for } 2 \le i \le m\,, j \text{ otherwise, } k = 2 \\
f_2(x_{ijk-1}x_{ij-(n-1)k}) &= mn(2i-1)+2m+1, \\
&\quad \text{for } 2 \le i \le m\,, j \text{ otherwise, } k = 2 \\
f_2(x_{ij}x_{ijk}) &= mn(2i-1)+2m+kn, \\
&\quad \text{for } 2 \le i \le m\,, j \text{ otherwise, } 1 \le k \le 2
\end{aligned}$$

It is clear to see that the edge label is a bijective function $f_2 : E(FW_{m,n}) \rightarrow \{3mn-2n+1, 3mn-2n+2\ldots, 4mn-4n+1\}$. Let W_{f_2} be a total edge weight of super $(a,2)$-edge antimagic total labeling of $FW_{m,n}$. The total edge weight is derived by adding the associated bijective function w_{f_1} and f_2, namely $W_{f_2} = w_{f_1} + f_2$. It is not difficult to see that $\bigcup_{t=1}^{10} W_{f_2}^t = \{\frac{n+3}{2} + 3mn - 3m, \frac{n+3}{2} + 3mn - 3m + 2, \ldots, \frac{n+3}{2} + 11mn - 1\}$ form an arithmetics sequence of difference $d = 2$. Thus, the graph $FW_{m,n}$ admits a super $(\frac{n+3}{2} + 3mn - 3m, 2)$-edge antimagic total labeling for $n \ge 2$. $\square$

To show the existence of super $(a,1)$-edge antimagic of $FW_{m,n}$, we will use the following lemma, found by Dafik, et. al in Dafik et al. (2012).

Lemma 3 *Dafik et al. (2012) Let Υ be a sequence of consecutive number $\Upsilon = \{c, c+1, c+2, \ldots c+k\}$, k even. Then there exists a permutation $\Pi(\Upsilon)$ of the elements of Υ such that $\Upsilon + \Pi(\Upsilon) = \{2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, 2c+\frac{k}{2}+3, \ldots, 2c+\frac{3k}{2}, 2c+\frac{3k}{2}+1\}$ is also a sequence of consecutive number.*

Proof. Let Υ be a sequence $\Upsilon = \{a_i |\, a_i = c+(i-1),\ 1 \le i \le k+1\}$ and k be even. Define a permutation $\Pi(\Upsilon) = \{b_i |\, 1 \le i \le k+1\}$ of the elements of Υ as follows:

$$b_i = \begin{cases} c+k+\frac{3-i}{2} & \text{if } i \text{ is odd}, 1 \le i \le k+1 \\ c+\frac{k}{2}+\frac{2-i}{2} & \text{if } i \text{ is even}, 2 \le i \le k. \end{cases}$$

By direct computation, we obtain that $\Upsilon + \Pi(\Upsilon) = \{a_i + b_i |\, 1 \le i \le k+1\} = \{2c+k+\frac{1+i}{2} |\, i \text{ odd}, 1 \le i \le k+1\} \cup \{2c+\frac{k}{2}+\frac{i}{2} |\, i \text{ even}, 2 \le i \le k\} = \{2c+\frac{k}{2}+1, 2c+\frac{k}{2}+2, 2c+\frac{k}{2}+3, \ldots, 2c+\frac{3k}{2}, 2c+\frac{3k}{2}+1\}$.
$\square$

Directly from Theorem 1 together with Lemma 3, it follows that the graph $FW_{m,n}$ has a super $(a,1)$-edge-antimagic total labeling.

$\diamond$ **Teorema 4** *If $n \ge 1$, then the graph Ferris Wheel has a super $(6mn-4n+2, 1)$-edge-antimagic total labeling.*

Proof. From Theorem 1, the graph $FW_{m,n}$ has a $(\frac{n+3}{2}, 1)$-edge-antimagic vertex labeling. Let $\Upsilon = \{c, c+1, c+2, \ldots c+k\}$, for k even, be a set of the edge weights of the vertex labeling f_3, for $c = \frac{n+3}{2}$ and $k = 6mn-5n-1$. In light of Lemma 3, there exists a permutation $\Pi(\Upsilon)$ of the elements of Υ such that $\Upsilon + [\Pi(\Upsilon) - c + p + 1] =$

$\{6mn-4n+2, 6mn-4n+3, \ldots, 12mn-9n+1\}$. If $[\Pi(\Upsilon) - c + p + 1]$ is an edge labeling of $FW_{m,n}$ then $\Upsilon + [\Pi(\Upsilon) - c + p + 1]$ gives the set of the total edge weights of $FW_{m,n}$, which implies that the graph $FW_{m,n}$ has super $(a, 1)$-edge-antimagic total, where $a = 6mn - 4n + 2$. This concludes that the graph $FW_{m,n}$ admits a super $(6mn-4n+2, 1)$-edge antimagic total labeling. □

Conclusion

The results shows that there are a super (a, d)-edge-antimagic total labeling of graph $FW_{m,n}$ for $n \geq 3$, odd, and $m \geq 2$. For the case n is odd the results cover all feasible $d \in \{0, 1, 2\}$. However for n is even, we propose the following open problem.

Open Problem 1 *Determine the existence of a super (a, d)-edge-antimagic total labeling of Ferris Wheel graph $FW_{m,n}$ for n even and feasible $d \in \{0, 1, 2\}$.*

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