Modeling Total Maintenance Cost of a Repairable System in Some Bounded Time Interval

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ABSTRACT

In this paper we model the total maintenance cost of a repairable system where the inter-maintenance times are modeled as discrete-time and continuous-time renewal processes. The maintenance cost is assumed to be a function of the lifetime of the system. We derive the probability distribution, including the mean and the variance, of the total maintenance cost. The results are presented in the form of generating functions and Laplace transforms that in general have to be inverted numerically. Some examples are presented in this paper.

Keywords : renewal process, Laplace transform, generating function, numerical inversion methods

INTRODUCTION

Maintenance of a system is needed to keep the system works properly. The maintenance action of the system is immediately carried out after the failure of the system. We assume that the maintenance actions bring the system back to a state as good as new. In general the maintenance times are random, so the number of maintenance actions in some time interval is a random variable. We will model the maintenance times as renewal processes. This means that the inter-maintenance times are assumed to be independent and identically distributed (i.i.d.), non-negative random variables.

Associate to each maintenance action the maintenance cost. This cost usually depends on the lifetime of the system, the shorter the lifetime the cheaper the cost. So the maintenance cost is also a random variable. Of course, as a special case, we can take a constant as the maintenance cost. Then a quantity that is interesting to be analyzed is the total maintenance cost over some time interval.

Some authors have discussed this total maintenance cost. Van Noortwijk (1988) discussed optimal replacement decisions for systems under stochastic deterioration in unbounded time interval. Asymptotic property of the expected value of the total maintenance cost was discussed by Van Noortwijk &d Frangopol (2004). Van Noortwijk (2003) discussed asymptotic property of the variance of the total maintenance cost. In this paper we discuss probability distributions, including the mean and the variance, of the total maintenance cost in some *bounded time interval*.

This paper is organized as follows. In Section 2 we model the total maintenance cost where the inter-maintenance times are modeled as a discrete renewal process. Then in Section 3 we discuss its probability distribution. The results are presented in the form of generating function. Section 4 deals with the total maintenance cost where the inter-maintenance times are modeled as a continuous renewal process. The probability distribution of the total maintenance cost is discussed in Section 5 where the results are presented in the form of Laplace transforms.

RESULTS AND DISCUSSION

Discrete-Time model for the total maintenance cost

Let $0 = S_0 < S_1 < S_2 < \ldots$ be the times at which maintenance of a system take places. Maintenance is carried out such that the system condition as good as new. So we may assume that the times between consecutive maintenance

$$T_j = S_j - S_{j-1}, \qquad j = 1, 2, 3, \ldots$$

as i.i.d., non-negative random variables. In this section the random variable T_j is assumed to be discrete non-negative integer valued random variables. The number of maintenance taken in the time interval $[0,n]$, $n = 0, 1, 2, \ldots$, is

$$N(n) = \max\{j \geq 0 : S_j \leq n\}.$$

The process $(N(n), n = 0, 1, 2, \ldots)$ is called a discrete-time renewal process. Denote by C_j the maintenance cost associated with the event $\{T_i$

= $j\}$. Then the total maintenance cost in the time interval $[0,n]$ can be represented by

$$K(n) = \sum_{i=1}^{N(n)} C_{T_i} \tag{1}$$

We assume that the maintenance cost depend only on the age of system. In the next section we discuss the probability distribution of $K(n)$.

Probability distribution of the discrete-time model for the total maintenance cost

Let for all j = 1, 2, ...,

$$P(T_j = n) = p_n, \qquad n \geq 1$$

where $p_n \geq 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} p_n = 1$. The probability distribution of the total maintenance cost $K(n)$ is given in the following theorem in the form of generating function.

Theorem 1

If the times between consecutive maintenance (T_j) are non-negative-integer valued, then

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j}{1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j}. \tag{2}$$

Proof:

Using conditional expectation, the generating function of $K(n)$ can be written as

$$E[z^{K(n)}] = \sum_{j=1}^{\infty} E[z^{K(n)} \mid T_1 = j]P(T_1 = j)$$

$$= \sum_{j=1}^{n} z^{C_j} E[z^{K(n-j)} \mid T_1 = j]p_j + \sum_{j=n+1}^{\infty} p_j$$

$$= \sum_{j=1}^{n} z^{C_j} E[z^{K(n-j)}]p_j + \sum_{j=n+1}^{\infty} p_j$$

Define

$$\varphi(u) = \sum_{n=1}^{\infty} E[z^{K(n)}]u^n .$$

Then

$$\varphi(u) = \sum_{n=1}^{\infty}\sum_{j=1}^{n} z^{C_j} E[z^{K(n-j)}]p_j u^n + \sum_{n=1}^{\infty}\sum_{j=n+1}^{n} p_j u^n$$

$$= \sum_{j=1}^{\infty} z^{C_j} p_j \sum_{n=j}^{\infty} E[z^{K(n-j)}]u^n + \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{n} p_j$$

$$= \sum_{j=1}^{\infty} z^{C_j} p_j u^j \sum_{m=0}^{\infty} E[z^{K(m)}]u^m + \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{n} p_j$$

$$= [1+\varphi(u)]\sum_{j=1}^{\infty} z^{C_j} p_j u^j + \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{n} p_j$$

It follows that

$$\varphi(u) = \frac{\sum_{j=1}^{\infty} z^{C_j} p_j u^j + \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j}{1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j}.$$

Adding 1 on both sides of this equation we get

$$1+\varphi(u) = 1+\frac{\sum_{j=1}^{\infty} z^{C_j} p_j u^j + \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j}{1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j}$$

or, equivalently,

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j}{1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j}.$$

The mean and the higher moments of $K(n)$ can be obtained by taking derivatives with respect to z on both sides of equation (2) and then setting z = 1.

Theorem 2

Generating functions of the mean and the second moment of $K(n)$ are as follows:

a. $$\sum_{n=0}^{\infty} E[K(n)]u^n = \frac{Q_1[1+R]}{(1-Q_0)^2}. \tag{3}$$

b. $$\sum_{n=0}^{\infty} E[K^2(n)]u^n = \frac{[Q_2(1-Q_0)+2Q_1^2](1+R)}{(1-Q_0)^3} \tag{4}$$

where $Q_i = \sum_{j=1}^{\infty} C_j^i p_j u^j, \quad i = 0,1,2,$ and $R = \sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j$.

Proof:

Recall the equation (2), that is,

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j}{1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j}.$$

If we take the derivative with respect to z on the left hand side of this equation and set z = 1, then we get $\sum_{n=0}^{\infty} E[K(n)]u^n$. The derivative with respect to z of right side of the equation is

$$\frac{\left[1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j\right]\sum_{j=1}^{\infty} C_j z^{(C_j-1)} p_j u^j}{\left[1-\sum_{j=1}^{\infty} z^{C_j} p_j u^j\right]^2}.$$

Replacing z by 1 in this equation we get

$$\frac{\left[1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p_j\right]\sum_{j=1}^{\infty} C_j p_j u^j}{\left[1-\sum_{j=1}^{\infty} p_j u^j\right]^2}$$

and part (a) of the theorem is proved. For part (b) take the second derivatives with respect to z on both sides of equation (2) and then set $z = 1$. ●

Example 1

Suppose that the inter-maintenance times T_i, i = 1, 2, 3, … are i.i.d. with probability distributions

$$P(T_i = j) = p_j = p(1-p)^{j-1}, \quad 0 < p < 1, \quad j = 1, 2, ...$$

and the maintenance cost $C_j \equiv c > 0$. Using (2) we get

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p(1-p)^{j-1}}{1-\sum_{j=1}^{\infty} z^c p(1-p)^j u^j}$$

Note that

$$1+\sum_{n=1}^{\infty} u^n \sum_{j=n+1}^{\infty} p(1-p)^{j-1} = 1+\sum_{n=1}^{\infty}[(1-p)u]^n$$

$$= \frac{1}{1-(1-p)u}$$

and

$$1-\sum_{j=1}^{\infty} z^c p(1-p)^{j-1} u^j = 1-\frac{z^c p(1-p)u}{(1-p)[1-(1-p)u]}$$

$$= \frac{1-(1-p+z^c p)u}{1-(1-p)u}$$

So

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1}{1-(1-p+z^c p)u}$$

$$= \sum_{n=0}^{\infty}(1-p+z^c p)u^n$$

It follows that

$$E[z^{K(n)}] = (1-p+z^c p)^n = E[z^{cY}]$$

where Y is binomially distributed with parameters n and p. So $K(n)$ and cY have the same distribution. Since $E(Y) = np$ and $\mathrm{Var}(Y) = np(1-p)$,

$E[K(n)] = npc$ dan $\mathrm{Var}[K(n)] = np(1-p)c^2$.

As an illustration, if $p = 0.25$ and $c = 1$ then the distribution of $K(10)$ is given in Table 1.

Table 1. The probability distribution of $K(10)$ when T_j is geometrically with parameter $p = 0.25$.

k	$p_{k,n}$	k	$p_{k,n}$
0	0.0563	5	0.0584
1	0.1877	6	0.0162
2	0.2816	7	0.0031
3	0.2503	8	0.0004
4	0.1460	9	0.0000

Example 2

Now suppose that the inter-maintenance times T_i, i = 1, 2, 3, … are i.i.d. with shifted Poisson distributions given by

$$p_j = \frac{\lambda^{j-1}e^{-\lambda}}{(j-1)!}, \quad j = 1, 2, ... \tag{5}$$

and the maintenance cost $C_j \equiv c > 0$. Using (2) we get

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \frac{1-ue^{-\lambda(1-u)}}{(1-u)[1-z^c ue^{-\lambda(1-u)}]}.$$

Note that

$$\sum_{n=0}^{\infty} E[z^{K(n)}]u^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P[K(n) = ck]z^{ck}u^n$$

Let

$z^c = v$ and $P[K(n) = ck] = p_{k,n}$.

Then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{k,n} v^k u^n = \frac{1-ue^{-\lambda(1-u)}}{(1-u)[1-z^c ue^{-\lambda(1-u)}]} \tag{6}$$

To obtain the probability distributions $p_{k,n}$ of the total maintenance cost $K(n)$ we have to invert the transform (5) numerically. An approximation formula for $p_{k,n}$ is

$$p_{k,n} \approx \frac{1}{4knr_1^k r_2^n}\sum_{j=0}^{2k-1}\sum_{l=0}^{2n-1}(-1)^{j+l} G(r_1 e^{i\pi j/k}, r_2 e^{i\pi l/n})$$

where $G(v,u)$ denotes the right-hand side of (6), see Choudhury *et al*. (1994). As an example, when n = 10, c = 1, and λ = 3 we get the approximation distribustion of $K(n)$ as shown in Table 2.

Table 2. The probability distribution of $K(10)$ when T_j has shifted Poisson distribution with parameter $\lambda = 3$.

k	$p_{k,n}$	k	$p_{k,n}$
0	0.0003	5	0.0027
1	0.1517	6	0.0001
2	0.5233	7	0.0000
3	0.2781	8	0.0000
4	0.0430	9	0.0000

From Table 2 we can calculate the approximation of the mean of $K(10)$:

$$E[K(10)] \cong \sum_{k=0}^{9} kp_{k,n} = 2.22$$

and

$$Var[K(10)] \cong 0.58 .$$

Of course we can also calculate the mean and the variance of $K(n)$ by using (3) and (4).

Continuous-Time model for the total maintenance cost

In this section we model the maintenance times $0 = S_0 < S_1 < S_2 < \dots$ of the system as continuous random variables. Similar to the discrete case, it is assumed that the maintenance actions bring the system back to a state as good as new. So the inter-maintenance times

$$T_j = S_j - S_{j-1}, \qquad j = 1, 2, 3, \dots$$

can be modeled as i.i.d., non-negative continuous random variables. We will denote the cumulative distribution function of T_j by F, that is,

$$F(t) = P(T_j < t).$$

The number of maintenance actions taken in the time interval $[0, t]$, $t > 0$, is

$$N(t) = \max\{j \geq 0 : S_j \leq t\}.$$

The process $(N(t), t > 0)$ is called a continuous-time renewal process.

In this setting we denote by $C(T_j)$ the maintenance cost associated with the inter-maintenance time T_j, where C is some non-negative measurable function. Then the total maintenance cost over the time interval $[0, t]$ can be represented by

$$K(t) = \sum_{j=1}^{N(t)} C(T_j) \tag{7}$$

The probability distribution of $K(t)$ is discussed in the next section.

Probability distribution of the continuous-time model for the total maintenance cost

Since the maintenance times are modeled as continuous random variables, we will present the probability distribution of the total maintenance cost $K(t)$ in in the form of Laplace transforms instead of generating functions.

The model (7) can be considered as a modification of the instantaneous renewal reward process considered by Suyono (2002). In Suyono (2002) the probability distribution of the instantaneous renewal reward process was derived by using the theory of point processes. In the instantaneous renewal reward process we include the reward (or the cost in this paper) at the time interval $[t - S_{N(t)}]$ (the time from the last maintenance action up to time t), that is $C(t - S_{N(t)})$. The model we are discussing does not include the cost over the time interval $[t - S_{N(t)}]$ because there is no maintenance action in that time interval.

The probability distribution of $K(t)$ is given in the following theorem in the form of double Laplace transform.

Theorem 3

For $\alpha, \beta > 0$,

$$\int_0^\infty E\left[e^{-\alpha K(t)}\right] e^{-\beta t} dt = \frac{1 - F^*(\beta)}{\beta\left[1 - \int_0^\infty e^{-\beta t - \alpha C(t)} dF(t)\right]} \tag{8}$$

where F^* is the Laplace-Stieltjes transform of F, that is, $F^*(\beta) = \int_0^\infty e^{-\beta t} dF(t)$.

Proof:

Conditioning on the event $\{T_1 = x\}$ we get

$$\begin{aligned} E[e^{-\alpha K(t)}] &= \int_0^\infty E[e^{-\alpha K(t)} \mid T_1 = x] dF(x) \\ &= \int_0^t E[e^{-\alpha K(t)} \mid T_1 = x] dF(x) \\ &\quad + \int_t^\infty E[e^{-\alpha K(t)} \mid T_1 = x] dF(x) \\ &= \int_0^t E[e^{-\alpha C(x) - \alpha K(t-x)}] dF(x) + \int_t^\infty dF(x) \\ &= \int_0^t e^{-\alpha C(x)} E[e^{-\alpha K(t-x)}] dF(x) + [1 - F(t)] \end{aligned}$$

where we have used the fact that $K(t) = 0$ if $x > t$. Taking Laplace transform of both sides of this equation we get

$$\int_0^\infty E[e^{-\alpha K(t)}]e^{-\beta t}dt = \int_{t=0}^\infty \int_{x=0}^t e^{-\alpha C(x)}E[e^{-\alpha K(t-x)}]dF(x) + [1-F(t)]e^{-\beta t}dt$$

$$= \int_{x=0}^\infty \int_{t=x}^\infty e^{-\alpha C(x)}E[e^{-\alpha K(t-x)}]e^{-\beta t}dtdF(x) + \frac{1}{\beta} + \frac{1}{\beta}\int_0^\infty F(t)e^{-\beta t}dt$$

$$= \int_{x=0}^\infty \int_{y=0}^\infty e^{-\alpha C(x)}E[e^{-\alpha K(y)}]e^{-\beta(x+y)}dydF(x) + \frac{1}{\beta} - \frac{1}{\beta}\int_0^\infty e^{-\beta t}dF(t)$$

$$= \int_{x=0}^\infty e^{-\alpha C(x)-\beta x}dF(x)\int_{y=0}^\infty E[e^{-\alpha K(y)}]e^{-\beta y}dy + \frac{1-F^*(\beta)}{\beta}$$

It follows that

$$\int_0^\infty E\left[e^{-\alpha K(t)}\right]e^{-\beta t}dt = \frac{1-F^*(\beta)}{\beta\left[1-\int_0^\infty e^{-\beta t-\alpha C(t)}dF(t)\right]}.$$

●

The mean and the second moment of $K(t)$ are given in the following theorem in the form of Laplace transforms.

Theorem 4

For $\beta > 0$,

a. $$\int_0^\infty E[K(t)]e^{-\beta t}dt = \frac{\int_0^\infty C(t)e^{-\beta t}dF(t)}{\beta[1-F^*(\beta)]}. \quad (9)$$

b. $$\int_0^\infty E[K^2(t)]e^{-\beta t}dt = \frac{\int_0^\infty C^2(t)e^{-\beta t}dF(t)}{\beta[1-F^*(\beta)]} + \frac{2\left[\int_0^\infty C(t)e^{-\beta t}dF(t)\right]^2}{\beta[1-F^*(\beta)]^2}. \quad (10)$$

Proof:

It is easy to check that

$$\frac{\partial}{\partial\alpha}\int_0^\infty E\left[e^{-\alpha K(t)}\right]e^{-\beta t}dt\Big|_{\alpha=0} = -\int_0^\infty E[K(t)]e^{-\beta t}dt.$$

If we take the derivative with respect to α of right side of (8) set $\alpha = 0$, then we get

$$\frac{-\int_0^\infty C(t)e^{-\beta t}dF(t)}{\beta[1-F^*(\beta)]},$$

so part (a) is proved. For part (b), take the second derivatives with respect to α on both sides of (8) and then set $\alpha = 0$. ●

To calculate the probability density function and the statistical moments of the total maintenance cost generally we have to do numerical integration. We refer to Davis & Rabinowitz (1984) and Abate & Whitt (1992) for a numerical integration method.

Example 3

Suppose that the inter-maintenance times T_i, i = 1, 2, 3, … are independent, gamma distributed with parameter n = 2 and $\lambda > 0$. The probability density function of T_i is

$$f(x) = \lambda^2 xe^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Suppose that the maintenance cost $C(t) = t > 0$. Firstly we will calculate the expected total maintenance cost. Using (9) we get the Laplace transform of $E[K(t)]$ as follows

$$\int_0^\infty E[K(t)]e^{-\beta t}dt = \frac{2\lambda^2}{\beta^2(\beta+\lambda)(\beta+2\lambda)}.$$

Inverting this transform we get

$$E[K(t)] = t + \frac{4e^{-\lambda t} - e^{-2\lambda t} - 3}{2\lambda}.$$

Now we will calculate the second moment of the total maintenance cost. Using (10) we get

$$\int_0^\infty E[K^2(t)]e^{-\beta t}dt = \frac{2\lambda^2(3\beta^2+6\lambda\beta+4\lambda^2)}{\beta^3(\beta+2\lambda)^2(\beta+\lambda)^2}.$$

Inverting this transform we get

$$E[K^2(t)] = t^2 - \frac{3t}{\lambda} + \frac{4-2(\lambda t+1)e^{-\lambda t} - (\lambda t+2)e^{-2\lambda t}}{\lambda^2}.$$

It follows that

$$Var[K(t)] = \frac{7 + (16-24\lambda t)e^{-\lambda t} - 30e^{-2\lambda t} + 8e^{-3\lambda t} - e^{-4\lambda t}}{4\lambda^2}.$$

Next we will find the probability distribution of the total maintenance cost. As an illustration we will set $\lambda = 1$.

Using (8) we get the double Laplace transform of $K(t)$ as follows

$$\int_0^\infty E\left[e^{-\alpha K(t)}\right]e^{-\beta t}dt = \frac{[(\beta+1)^2-1](1+\beta+\alpha)^2}{\beta(1+\beta)^2[(1+\beta+\alpha)^2-1]}.$$

Inverting this transform with respect to β we get

$$E\left[e^{-\alpha K(t)}\right] = \frac{\alpha^2 te^{-t}}{\alpha^2-1} + \frac{(\alpha^3-\alpha-2)e^{-t}}{(\alpha^2-1)^2} + \frac{(2-\alpha)e^{-\alpha t}}{2(\alpha-1)^2} + \frac{\alpha e^{-(\alpha+2)t}}{2(\alpha+1)^2}$$

Note that

$$E\left[e^{-\alpha K(t)}\right] = \int_0^\infty f_{K(t)}(x)e^{-\alpha x}dx$$

where $f_{K(t)}(x)$ denotes the probability density function of $K(t)$. So

$$\int_0^\infty f_{K(t)}(x)e^{-\alpha x}dx = \frac{\alpha^2 te^{-t}}{\alpha^2-1} + \frac{(\alpha^3-\alpha-2)e^{-t}}{(\alpha^2-1)^2} + \frac{(2-\alpha)e^{-\alpha t}}{2(\alpha-1)^2} + \frac{\alpha e^{-(\alpha+2)t}}{2(\alpha+1)^2}$$

.

To find $f_{K(t)}(x)$ we have to invert this transform numerically. Denote the right-hand side of this equation by $\hat{f}(\alpha)$. We will use the following numerical inversion formula for inverting this transform:

$$f_{K(t)}(x) \approx \frac{e^A}{2x}\hat{f}(A/x) + \frac{e^A}{x}\sum_{k=1}^{M}(-1)^k \operatorname{Re}(\hat{f}([A+i\pi k]/t)$$

where A is a constant (we take A = 5) and M is a large integer (we take M = 1000), see Abate and Whitt (1992). As an example if t = 20, the using this inversion formula we get the probability density function $f_{K(t)}(x)$ as shown in the following figure.

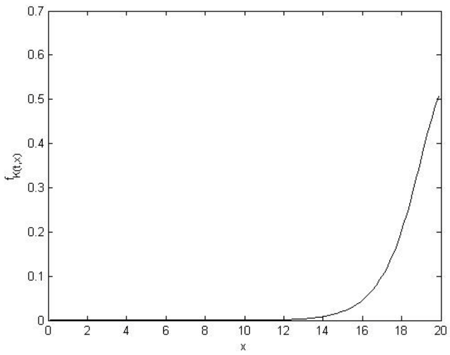

Figure 1. The probability density function of $K(20)$ when the inter-maintenance times have gamma distribution with parameters n = 2 and λ = 1, and the cost function $C(t) = t$.

CONCLUSION

In this paper we model the total maintenance cost of a repairable system where the inter-maintenance times are modeled as discrete-time and continuous-time renewal processes. The maintenance cost is assumed to be a function of the lifetime of the system.

In the discrete-time model we derive the double generating function of the total maintenance cost from which we may calculate its probability distribution. We also derive the generating functions of the first and the second moments of the total maintenance cost. Basically we may derive the generating functions of the higher moments of the total maintenance cost, but the formula will be more complicated. Two examples are given where we model the inter-maintenance costs by a binomial random variable in the first example, and by a sifted Poisson random variable in the second example. By applying a numerical inversion method of generating functions we calculate the probability distributions of the total maintenance costs. In the continuous-time model we derive the double Laplace transform of the total maintenance cost. We also derive the Laplace transforms of the first and the second moments of the total maintenance cost.

The Laplace transforms of the higher moments can also be obtained but the formulae are too complicated. We give an example where the inter-maintenance costs is modeled by a gamma random variable. Using a numerical inversion method of Laplace transforms we calculate the probability density function of the total maintenance cost

REFERENCES

Abate J & Whitt W. 1992. The Fourier-series method for inverting transforms of probability distributions, *Queuing Systems* 10: 5 – 88.

Choudhury GL, Lucantoni DM, & Whitt W. 1994. Multidimensional transform inversion with applications to the transient M/G/1 queue. *Annals of Applied Probability* **4**(3): 719 – 740.

Davis PJ & Rabinowitz P, 1984. *Methods of Numerical Integration*. 2nd Ed. Academic Press, New York.

Suyono, 2002. *Renewal Processes and Repairable System*. Disertation, Delft University Press.

Van Noortwijk JM. 1988. Optimal replacement decisions for structures under stochastic deterioration. *Proceedings of the Eight IFIP WG 7.5 Working Conference on Reliability and Optimization of Structural Systems.* Krakow. Poland: 273 – 280.

Van Noortwijk, JM. 2003. Explicit formulas for the variance of discounted life-cycle cost. *Reliability Engineering and System Safety* 80. pp. 185 – 195.

Van Noortwijk. JM & Frangopol DM. 2004. Two probabilistic life-cycle maintenance models for deteriorating civil infrastructures. *Prob. Eng. Mech. 19*, pp. 354-359.