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ANDOYER–DEPRIT VARIABLES USE TO THE HESS GYROSCOPE PHASE TRAJECTORIES EXPLORING

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Abstract**—***The paper deals with rotation of gyroscope in Hess' conditions. Motion equations of a solid body are established on the base of Hamiltonian formalism. There are some analytical researches and computer experiments were made on the base of numeral study of phase portrait of equations, which describe gyroscope's motion. The movements of gyroscope, which is submitted to Hess' conditions in the null constant of integral of an area and a light weight of the body, are investigated more detailed. The motion equations and integrals are expressed in variables Andoyer–Deprit. The heteroclinic trajectories of the dynamical system are examined by means of the new canonical variables.*

Index Terms—Gyroscope; rigid body with a fixed point; Hamiltonian; phase portrait; numerical modeling; separatrix; variables Andoyer–Deprit.

I. INTRODUCTION AND PROBLEM STATEMENT

Hess' gyroscope has unique analytical and qualitative characteristics, so it is a major figure in the innovative dynamics of a rigid body. Two-dimensional invariant varieties, that contain the Hess' solution, determine the border of chaos in a dynamic system. This helps to examine all possible variants of transitions from regular to chaotic motions. In this solution, there are the following things that are remarkably combined: invariant torus, which carries a quasiperiodic motions, limit cycles and isolated periodic trajectories, the simplest motions of a physical pendulum, stable and unstable relative equilibria, homo- and heteroclinic motions, frequency resonances, splitting separatrix surfaces. All these facts help us to confirm, that the most significant ideas and results of the rigid body dynamic may be clearly explained with the help of such an example as the Hess' problem about the movement of the body around a fixed point.

The motion differential equations, which are related to the principal axes of inertia of a rigid body in the fixed point, look like:

$$\dot{\mathbf{G}} = \mathbf{G} \times \frac{\partial \mathbf{H}}{\partial \boldsymbol{G}} + \boldsymbol{\gamma} \times \frac{\partial \mathbf{H}}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \frac{\partial \mathbf{H}}{\partial \boldsymbol{G}}, \qquad \mathbf{H} = \frac{1}{2}(\mathbf{G}, \mathbf{IG}) + \mu(\mathbf{r}, \gamma) \tag{1}$$

here $\mathbf{G} = (Ap, Bq, Cr)$ is the vector of kinetic moment on the coordinate system, which is connected with the body; $\boldsymbol{\omega} = (p, q, r)$ is the angular velocity of the body; γ is the ort of the vertical line in the same system; $\mathbf{I} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 1/A & 0 & 0 \\ 0 & 1/B & 0 \\ 0 & 0 & 1/C \end{pmatrix}$ is the inverse inertia tensor, which is assumed to be diagonal; $\mathbf{r} = (e_1, e_2, e_3)$ is the radius vector of the mass center in the moving system; μ is the product of the body weight and the distance from the gravity center to the fixed point.

Equations (1) with any Hamilton function admits geometrical and area integrals:

$$C_1 = (\mathbf{M}, \gamma), \qquad C_2 = \gamma^2 = 1.$$

In the work there is an investigation of the body movement, a weight distribution of which satisfies the Hess' conditions [1]

$$e_1\sqrt{A(C-B)} \pm e_2\sqrt{B(A-C)} = 0, \quad e_3 = 0,\ A > C > B. \tag{2}$$

If the conditions (2) are completed then equations (1) have linear invariant relation:

$$e_1 G_1 + e_2 G_2 = 0. \tag{3}$$

The physical meaning of limits on the parameters in Hess' case is in the following. Gyratory ellipsoid, – the surface of the kinetic energy level in the moment space **G** – has two circular cross-sections, which are passed through the third axis. Conditions (2) means that the mass center lies on the axis that is perpendicular to one of the circular cross-sections of the ellipsoid. Invariant Hess' relation means that the projection of moment on this axis is equal to zero.

II. CANONICAL VARIABLES ANDOYER–DEPRIT

For analysis of the rigid body motions in the neighborhood of the integrated case of Euler the canonical variables Andoyer–Deprit are successfully applied. These variables are also used for different computer realization of the Poincare section method [2] – [4]. We use the following notations

[5]: through $OXYZ$ we denote a fixed triangle with the origin in the fixed point; $Oxyz$ is a coordinate system which is rigidly connected with the body and axes of which are directed to the main axes of the body inertia; Σ is a plane, which passes through the point of fixing and is perpendicular to the vector of the kinetic moment **G**. In the agreed notations L is a projection of kinetic moment on the fixed axis Oz, G is value of the kinetic moment, M its projection on fixed axis Oz, l is the angle between axis Ox and the line of intersection Σ with Oxy, g is the angle between line of intersection Σ with planes Oxy and OXY, m is the angle between axis Ox and line of intersection Σ with plane OXY.

In these variables the motion equations have a canonical form:

$$\dot{l}=\frac{\partial \mathbf{H}}{\partial L},\quad \dot{L}=-\frac{\partial \mathbf{H}}{\partial l},\qquad \dot{g}=\frac{\partial \mathbf{H}}{\partial G},\quad \dot{G}=-\frac{\partial \mathbf{H}}{\partial g},\qquad \dot{m}=\frac{\partial \mathbf{H}}{\partial M},\quad \dot{M}=-\frac{\partial \mathbf{H}}{\partial m}. \tag{4}$$

where the Hamiltonian

$$\mathbf{H}=\frac{1}{2}(\mathbf{G},\,\mathbf{IG})+\mu(\mathbf{r},\,\gamma)=\mathbf{H}(L,\,G,\,M,\,l,\,g)$$

is a full energy of the rigid body and doesn't depend on the variable m, so that $M=(\mathbf{G},\,\gamma)=\text{const}$ – the area integral – can be considered as the parameter.

The transition from variables G, γ the variables Andoyer–Deprit are given by formulas:

$$\begin{cases} G_1=\sqrt{G^2-L^2}\sin l,\\ G_2=\sqrt{G^2-L^2}\cos l,\\ G_3=L. \end{cases} \tag{5}$$

$$\gamma_1=\frac{M}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}\sin l+\frac{L}{G}\sqrt{1-\left(\frac{M}{G}\right)^2}\sin l\cos g+\sqrt{1-\left(\frac{M}{G}\right)^2}\cos l\sin g,$$

$$\gamma_2=\frac{M}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}\cos l+\frac{L}{G}\sqrt{1-\left(\frac{M}{G}\right)^2}\cos l\cos g-\sqrt{1-\left(\frac{M}{G}\right)^2}\sin l\sin g,$$

$$\gamma_3=\frac{LM}{G^2}-\sqrt{1-\left(\frac{L}{G}\right)^2}\sqrt{1-\left(\frac{M}{G}\right)^2}\cos g. \tag{6}$$

For the inverse conversion we get:

$$L=G_3,\ G^2=G_1^2+G_2^2+G_3^2,\quad M=\mathbf{G}\cdot\boldsymbol{\gamma},$$

$$\sin l=\frac{G_1}{\sqrt{G_1^2+G_2^2}},\quad \cos l=\frac{G_2}{\sqrt{G_1^2+G_2^2}},$$

$$l=\operatorname{arctg}\left(\frac{G_1}{G_2}\right),\quad \sqrt{1-\frac{M^2}{G^2}}\cdot\sin g=\frac{G_2\gamma_1-G_1\gamma_2}{\sqrt{G_1^2+G_2^2}}.$$

Due to the formulas (5), (6) the Hamiltonian (1) looks like

$$\mathbf{H}=\mathrm{H}_0+\mu\mathrm{H}_1. \tag{7}$$

$$\mathrm{H}_0=\frac{1}{2}\left[(G^2-L^2)(a\sin^2 l+b\cos^2 l)+L^2c\right],$$

$$\begin{aligned}\mathrm{H}_1&=e_1\left[\frac{M}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}\sin l+\frac{L}{G}\sqrt{1-\left(\frac{M}{G}\right)^2}\sin l\cos g+\sqrt{1-\left(\frac{M}{G}\right)^2}\cos l\sin g\right]\\
&+e_2\left[\frac{M}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}\cos l+\frac{L}{G}\sqrt{1-\left(\frac{M}{G}\right)^2}\cos l\cos g-\sqrt{1-\left(\frac{M}{G}\right)^2}\sin l\sin g\right]\\
&=\frac{1}{G^2}\left(e_1\sin l+e_2\cos l\right)\cdot\left[M\sqrt{G^2-L^2}+L\sqrt{G^2-M^2}\cdot\cos g\right]+\frac{\sqrt{G^2-M^2}}{G}\sin g\left(e_1\cos l-e_2\sin l\right).\end{aligned}$$

The system (4) will looks like:

$$\dot{l}=L\left(c-a\sin^2 l-b\cos^2 l\right)-\frac{\mu}{G^2}\left(\sqrt{G^2-M^2}\cdot\cos g-\frac{ML}{\sqrt{G^2-L^2}}\right)\left(e_1\sin l+e_2\cos l\right),$$

$$\begin{aligned}\dot{g}=G\left(a\sin^2 l+b\cos^2 l\right)-\frac{\mu}{G^3\sqrt{G^2-M^2}\sqrt{G^2-L^2}}\cdot\Big[GM^2\sqrt{G^2-L^2}\sin g\left(e_1\cos l-e_2\sin l\right)\\
-\Big(M\sqrt{G^2-M^2}(G^2-2L^2)+L\sqrt{G^2-L^2}(G^2-2M^2)\cos g\Big)\cdot(e_1\sin l+e_2\cos l)\Big],\end{aligned}$$

$$\dot{m} = -\frac{\mu}{G^2\sqrt{G^2-M^2}} \cdot \left[\left(\sqrt{G^2-M^2}\sqrt{G^2-L^2} - ML\cos g\right)\cdot(e_1\sin l + e_2\cos l) - MG\cdot\sin g\cdot(e_1\cos l - e_2\sin l)\right],$$

$$\dot{L} = -\frac{1}{2}(G^2-L^2)(a-b)\sin 2l + \frac{\mu}{G^2}\left[-G\sqrt{G^2-M^2}\sin g(e_1\sin l + e_2\cos l)\right.$$
$$\left. + \left(M\sqrt{G^2-L^2} + L\sqrt{G^2-M^2}\cos g\right)\cdot(e_1\cos l - e_2\sin l)\right],$$

$$\dot{G} = \frac{\mu}{G^2}\sqrt{G^2-M^2}\left[L\sin g\cdot(e_1\sin l + e_2\cos l) + G\cos g\cdot(e_1\cos l - e_2\sin l)\right],$$

$$\dot{M} = 0.$$

Analogical canonical variables L, G, M, l, g, m were used for studying of the Hess' solution many times, for instance, in works [3], [5], [6]. In this work a sequence order of major axes of inertia is changed a little in the fixed basis *Oxyz*. Redefinition of axes of inertia hadn't great influence on the Hamiltonian structure (7), but also the equation was simplified (3). A success of future researches results from the simple analytical structure of the invariant Hess' relation.

III. HESS CASE

The Hess' condition and invariant relation (3) in variables Andoyer–Deprit will look like:

$$(c-b)e_1^2 = (a-c)e_2^2,$$

$$\sqrt{G^2-L^2}\cdot(e_1\sin l + e_2\cos l) = 0, \qquad (8)$$

When $e_1\sin l + e_2\cos l = 0$ we have the Hess' condition, and when $G^2 = L^2$ and $M = 0$ – physical pendulum.

When $\mu = 0$ we have the integrated case of Euler–Poinsot. Unperturbed system

$$\frac{dl}{dt} = \frac{\partial H_0(L,l,G_0)}{\partial L},$$
$$\frac{dL}{dt} = -\frac{\partial H_0(L,l,G_0)}{\partial l}$$

looks like

$$\frac{dl}{dt} = L\left(c - a\sin^2 l - b\cos^2 l\right),$$
$$\frac{dL}{dt} = \frac{1}{2}\left(G_0^2 - L^2\right)(a-b)\sin 2l. \qquad (9)$$

In Figure 1 the phase portrait of system is shown (12).

With any $G_0 > 0$ the system (4) has fixed points

$$x_1 : (L=0,\ l=0), \quad x_2 : (L=0,\ l=\pi).$$

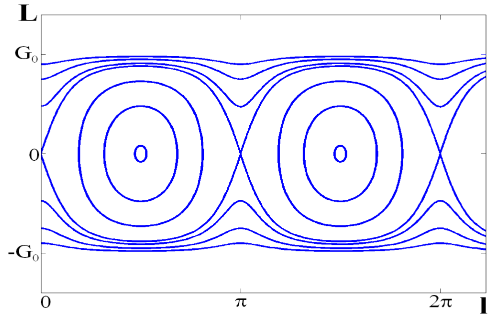


Fig. 1. Phase portrait of system (9)

Let's consider the unperturbed system with $M = 0$. In this case equations (4) will look like

$$\dot{l} = L(c - a\sin^2 l - b\cos^2 l)$$
$$-\frac{\mu}{G}\cos g\cdot\left(e_1\sin l + e_2\cos l\right),$$

$$\dot{g} = G(a\sin^2 l + b\cos^2 l)$$
$$+\frac{\mu}{G^2}L\cos g\cdot\left(e_1\sin l + e_2\cos l\right),$$

$$\dot{m} = -\frac{\mu}{G^2}\sqrt{G^2-L^2}\left(e_1\sin l + e_2\cos l\right),$$

$$\dot{L} = -\frac{1}{2}(G^2-L^2)(a-b)\sin 2l$$
$$+\frac{\mu}{G}\left[L\cos g\left(e_1\cos l - e_2\sin l\right)\right.$$
$$\left. - G\sin g\left(e_1\sin l + e_2\cos l\right)\right],$$

$$\dot{G} = \frac{\mu}{G}\left[L\sin g\left(e_1\sin l + e_2\cos l\right) + \right.$$
$$\left. + G\cos g(e_1\cos l - e_2\sin l)\right],$$

$$\dot{M} = 0.$$

Hamiltonian (7) in this case looks like

$$\mathbf{H} = \frac{1}{2}\left(\left(G^2-L^2\right)\left(a\sin^2 l + b\cos^2 l\right) + L^2c\right) + \mu\left(\frac{L}{G}\cos g\left(e_1\sin l + e_2\cos l\right) + \sin g\left(e_1\cos l - e_2\sin l\right)\right). \qquad (10)$$

Let's express the potential energy of Hess' gyroscope through variables Andoyer–Deprit:

$$\Pi = \mu\left(e_1\gamma_1 + e_2\gamma_2\right) = \frac{\mu L}{G}\cos g(e_1 \sin l + e_2 \cos l) + \mu \sin g(e_1 \cos l - e_2 \sin l).$$

We change

$$e_1 = \cos\varphi_0, \quad e_2 = \sin\varphi_0. \tag{11}$$

So the Hess' relation (8) will look like

$$\sin(l + \varphi_0) = 0 .$$

The angle φ is expressed through the components of gyratory tensor as the following

$$\varphi_0 = \mathrm{arctg}\left(\frac{\sqrt{c-b}}{\sqrt{a-c}}\right) \in \left(0, \pi/2\right)$$

and defines the fixed plane (invariant subspace) in the space $\mathbf{R}^3(G_1, G_2, G_3)$, depends on the parameters of rigid body.

Due to the replacement (11), we write the expression for the potential energy

$$\Pi = \mu\left[\frac{L}{G}\cos g \sin(l+\varphi_0) + \sin g \cos(l+\varphi_0)\right].$$

We note, that on the Hess' solution $\Pi = \mu \sin g$, on solution, which describes the physical pendulum motion $\Pi = \mu^{-1}\sin(g + l - \varphi_0)$.

The Hamiltonian (10) with the replacement (11) looks like:

$$\mathbf{H} = \frac{1}{2}\left((G^2 - L^2)(a\sin^2 l + b\cos^2 l) + L^2 c\right) + \mu\left(\frac{L}{G}\cos g \cdot \sin(l+\varphi_0) + \sin g\cos(l+\varphi_0)\right).$$

IV. The New Canonical Variables

The Hamiltonian of unperturbed system looks like

$$\mathbf{H}_0 = \frac{1}{2}\left[(G^2 - L^2)(a\sin^2 l + b\cos^2 l) + L^2 c\right] = \frac{1}{2}\left[aG_1^2 + bG_2^2 + cG_3^3\right]. \tag{12}$$

Let's consider the value

$$2T - cG^2 = aG_1^2 + bG_2^2 + cG_3^3 - cG_1^2 - cG_2^2 - cG_3^3 = (a-c)G_1^2 - (c-b)G_2^2.$$

Considering relations (10), (6), (15) we find

$$(a-c)G_1^2 - (c-b)G_2^2 = \frac{a-c}{e_1^2}\left(e_1^2G_1^2 - e_2^2G_2^2\right)$$
$$= \frac{a-c}{e_1^2}\left(e_1G_1 - e_2G_2\right)\left(e_1G_1 + e_2G_2\right)$$
$$= \frac{a-c}{e_1^2}\left(G^2 - L^2\right)\sin(l+\varphi_0)\sin(l-\varphi_0)$$
$$= \kappa \cdot \left(G^2 - L^2\right)\sin(l+\varphi_0)\sin(l-\varphi_0),$$

$$\kappa = \frac{a-c}{e_1^2}.$$

Due to these transformations we write the Hamiltonian (12)

$$\mathbf{H}_0 = \frac{1}{2}\left(2T - cG^2 + cG^2\right) = \frac{1}{2}\left[cG^2 + \kappa\left(G^2 - L^2\right)\sin(l+\varphi_0)\sin(l-\varphi_0)\right]. \tag{13}$$

We use the canonical variables [7], that are entered with the help of generating function $F_2 = 2J_1(l+\varphi_0) + J_2(g - l - \varphi_0) + mJ_3$:

$$\theta_1 = \frac{\partial F_2}{\partial J_1} \equiv 2(\varphi_0 + l), \quad \theta_2 = \frac{\partial F_2}{\partial J_2} \equiv g - l - \varphi_0,$$
$$\theta_3 = \frac{\partial F_2}{\partial J_3} \equiv m, \quad G = \frac{\partial F_2}{\partial g} \equiv J_2,$$
$$L = \frac{\partial F_2}{\partial l} \equiv 2J_1 - J_2, \quad M = \frac{\partial F_2}{\partial m} \equiv J_3$$

or

$$\theta_1 = 2(\varphi_0 + l), \quad \theta_2 = g - l - \varphi_0, \quad \theta_3 = m, \quad J_1 = \frac{1}{2}(G+L), \quad J_2 = G, \quad J_3 = M, \tag{14}$$

that assign the canonical transformation to variables (θ_i, J_i). So from (14)

$$l = \frac{1}{2}\theta_1 - \varphi_0, \quad g = \theta_2 + \frac{1}{2}\theta_1, \quad m = \theta_3, \quad L = 2J_1 - J_2, \quad G = J_2, \quad M = J_3. \tag{15}$$

We substitute (15) in the Hamiltonian (10). Considering (13) we get

$$\mathbf{H} = J_1(J_1 - J_2)(a-b)\left[\cos(\theta_1 - 2\varphi_0) - \cos 2\varphi_0\right] + \frac{1}{2}cJ_2^2 + \mu\left[J_1\sin(\theta_1+\theta_2) + (J_2 - J_1)\sin\theta_2\right] \cdot \frac{\sqrt{J_2^2 - J_3^2}}{J_2^2} + 2\mu\sin\frac{\theta_1}{2}\frac{J_3\sqrt{J_1(J_2 - J_1)}}{J_2^2}.$$

Invariant Hess' relation is from (8):

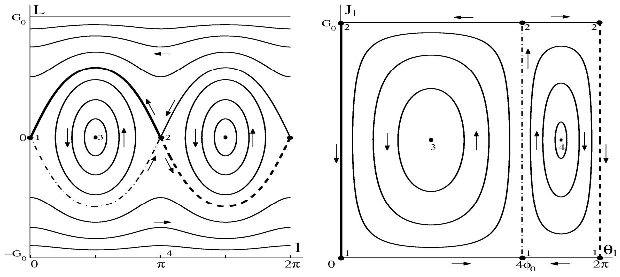
$$\sqrt{J_1(J_2-J_1)}\sin\frac{\theta_1}{2}=0.$$

On the Figure 2 the phase portrait of unperturbed system of Euler–Poinsot, which is submitted to the conditions (2), is depicted, in variables Andoyer–Deprit (in interpretation [5], [6]) and in new variables. A marked curve corresponds to separatrices. Separatrices, to which Hess' solutions are approached with $\mu\to 0$, are marked with a bold line and a dotted line. Uniform rotations are noted with digits 1–4, the motions of pendulum of unperturbed system are approached to rotations 1, 2.

On the null level of area integral $M = 0$ the Hamiltonian is equal to

$$\mathbf{H}=\frac{c}{2}J_2^2+\kappa J_1(J_1-J_2)\left[\cos(\theta_1-2\varphi_0)-\cos 2\varphi_0\right]$$
$$+\frac{\mu}{J_2}\left[J_1\sin(\theta_1+\theta_2)+(J_2-J_1)\sin\theta_2\right].$$

Then we will consider that angles $\theta_{1,2}$ are changed in modulus 2π, and changing of variables $J_{1,2}$ is bounded by inequalities $J_2>0,\ J_2\ge J_1\ge 0$.



a b

Fig. 2. Phase portrait of unperturbed Euler–Poinsot system: (a) in the variables Andoyer–Deprit; (b) in new variables

The differential equations of rigid body motion in new coordinates are the Hamilton' equations

$$\frac{d\theta_1}{dt}=\frac{\partial\mathbf{H}}{\partial J_1},\quad \frac{d\theta_2}{dt}=\frac{\partial\mathbf{H}}{\partial J_2},$$
$$\frac{dJ_1}{dt}=-\frac{\partial\mathbf{H}}{\partial\theta_1},\quad \frac{dJ_2}{dt}=-\frac{\partial\mathbf{H}}{\partial\theta_2}$$

or in an explicit form

$$\begin{aligned}
\dot\theta_1&=\kappa(2J_1-J_2)\left[\cos(\theta_1-2\varphi_0)-\cos 2\varphi_0\right]+\frac{\mu}{J_2}\left(\sin(\theta_1+\theta_2)-\sin\theta_2\right),\\
\dot\theta_2&=cJ_2-\kappa J_1\left[\cos(\theta_1-2\varphi_0)-\cos 2\varphi_0\right]+\mu\frac{J_1}{J_2^2}\left(\sin\theta_2-\sin(\theta_1+\theta_2)\right),\\
\dot J_1&=\kappa J_1(J_1-J_2)\sin(\theta_1-2\varphi_0)-\mu\frac{J_1}{J_2}\cos(\theta_1+\theta_2),\\
\dot J_2&=-\mu\frac{1}{J_2}\left((J_2-J_1)\cos\theta_2+J_1\cos(\theta_1+\theta_2)\right).
\end{aligned}\tag{16}$$

If the system solution is known (16), the cyclic coordinate θ_3 will be found by quadrature from equation

$$\dot\theta_3=2\mu\frac{\sqrt{J_1(J_2-J_1)}}{J_2^2}\sin\frac{\theta_1}{2}.$$

The projections of phase trajectories of unperturbed system (16) on the plane $\mathbf{R}^2(\theta_1,J_1)$ are shown on the Fig. 3. Area, separated by dot-dash line, is shown at the right side in the extended to three different values φ_0 form.

Calculations were made for the next values of parameters of rigid body:

$$\mu=0.01,\quad A_1=2.5,$$
$$A_2=2.0,\quad A_3=2.2.$$

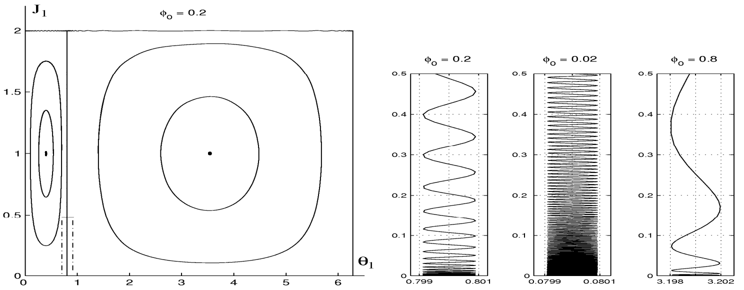


Fig. 3. Projection of phase trajectories of the dynamic system (16) on the plane $\mathbf{R}^2(\theta_1,J_1)$

Unperturbed system

$$\dot{\theta}_1 = \kappa(2J_1 - J_2)\left[\cos(\theta_1 - 2\varphi_0) - \cos 2\varphi_0\right],$$
$$\dot{J}_1 = \kappa J_1(J_1 - J_2)\sin(\theta_1 - 2\varphi_0)$$

has four fixed points

$$(\theta_1 = 0, J_1 = 0), \quad (\theta_1 = 0, J_1 = J_2^0),$$
$$(\theta_1 = 4\varphi_0, J_1 = 0), \quad (\theta_1 = 4\varphi_0, J_1 = J_2^0),$$
$$0 \le \theta_1 < 2\pi.$$

VIII. CONCLUSION

In this work, the analytical researches were performed, and also there were some computer experiments, based on the numerical analysis of the phase portrait of equations, that describe the rotation of the gyroscope. Motions of the gyroscope, which is a subject of Hess' conditions with the zero constant of integral squares and small body weight, are investigated in more details.

The equations of motion and integrals are recorded in variables Andoyer–Deprit. The heteroclinic trajectories of the dynamic system are explored with the help of the canonical variables.

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В. В. Кириченко. Використання змінних Андуайє–Депрі для вивчення фазових траєкторій гіроскопа Гесса

Проведено дослідження обертання гіроскопа за умов Гесса. Виведено рівняння руху твердого тіла на основі формалізму Гамільтона. Проведено аналітичні дослідження, а також комп'ютерні експерименти, які засновані на чисельному вивченні фазового портрета рівнянь, які описують обертання гіроскопа. Рівняння руху і інтеграли записані в змінних Андуайє–Депрі. За допомогою нових канонічних змінних вивчені гетероклінічні траєкторії динамічної системи.

Ключові слова: гіроскоп; обертання твердого тіла з нерухомою точкою; Гамільтоніан; фазовий портрет; розщеплення сепаратріси; змінні Андуайє–Депрі.

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В. В. Кириченко. **Использование переменных Андуайе–Депри для изучения фазовых траекторий гироскопа Гесса**

Проведено исследование вращения гироскопа при условиях Гесса. Выведены уравнения движения твердого тела на основе формализма Гамильтона. Проведены аналитические исследования, а также компьютерные эксперименты, основанные на численном изучении фазового портрета уравнений, описывающих вращение гироскопа. Уравнения движения и интегралы записаны в переменных Андуайе–Депри. С помощью новых канонических переменных изучены гетероклинические траектории динамической системы.

Ключевые слова: гироскоп; вращение твердого тела с неподвижной точкой; Гамильтониан; фазовый портрет; расщепление сепаратрис; переменные Андуайе–Депри.

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