vol.2, No. 3(s), pp. 1540-1545 ISSN 1805-3602

Economic design of $\bar{X}$-control charts under generalized exponential shock models with uniform sampling intervals

D. Aghabeig[1], M. B. Moghadam[2]

[1]Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran; [2]Department of Statistics, Allameh Tabataba'i University, Tehran, Iran

Abstract

Economic design of $\bar{X}$ -control chart was first presented by Duncan's model (1956) model and then followed by Hu (1984) with fixed sampling intervals and failure mechanism of Poisson and Weibull distributions, respectively. For the sake of convenience, in many situations, it may be desirable to have frequency of sampling fixed with age of the system and to avoid certain drawbacks of the mentioned distributions. It appears that the Generalized Exponential model can be used as an alternative to the Poisson or Weibull models in many situations. We, here, proposed a cost model based upon Generalized Exponential with uniform sampling intervals.

Keywords: Weibull distribution; Generalized Exponential distribution; Control Charts.

Introduction

Economic design of $\bar{X}$ -control chart was first presented by Duncan's model in 1956 to control the averages of a normal process. Limit control of a $\bar{X}$ -control chart was considered in the $\pm L$ standard deviation limit from the average. A sample is taken from the process output with the interval of h per hour. Thus, the sample average drawn on the chart is shiftted from μ_0 to $\mu_0 \pm \delta\sigma$ for the sake of the occurrence of an assignable cause which μ_0, σ and δ are process average, process standard deviation and shift parameter respectively. The occurrence of sample averages are remarked out of the control limit as an indication and called an out of control process. The purpose of determining the designing parameters of n, h and L in $\bar{X}$ -control charts is to minimize the total average cost in per time unit.

Duncan assumed that the occurrence time of an assignable cause is an exponential distribution with λ parameter and the assignable causes of deviations are a Poisson process. Although the Poisson based of a failure chart is directed into simplifying the model, this is not always suitable (see e.g. Montgomery and Heikes (1976)). In 1984, Hu followed Duncan's results and presented an economic design of an $\bar{X}$ -control chart under non-Poisson process shift. He considered conditions when failure mechanism follows Weibull distribution. He assumed that sampling intervals are fixed within production process. He noted that the exponential distribution model can be used for the Weibull distribution model without any significant effect.

The three-parameter Gamma and three- parameter Weibull distribution are usually applied to analyze any type of lifetime data and skewed data. Both of these distributions have remarkable properties and physical interpretations. These distributions are applied as the most common distributions to analyze the failure mechanism of a productive process. Both of these topics are studied in the literature for example Alexander (1962), Jackson (1969) and Van Kin Ken (1961). These parameters have proper specifications and perfect physical interpretations. Location, scale and shape are the parameters of these two distributions which are a bit of flexible to analyze skewed data. Based on their shape parameter, they might be exposed to either decreasing or increasing hazard rate; as a very easy to investigate and study them in comparison to Gamma distribution. Furthermore, in most of positive data, it is observed that Weibull distribution is perfectly adjusted with data. Weibull distribution has some drawbacks, though. For instance, Bain (1978) pointed out that the maximum likelihood function estimators of Weibull distribution may not behave well for all amounts of parameters. When the shape parameter is more than 1, the hazard function of Weibull distribution

Corresponding author: M.B. Moghadam, Department of Statistics, Allameh Tabataba'i University, Tehran, Iran. E-mail: bamenimoghadam@gmail.com.

and Gamma distribution are both increasing functions. But in case of Gamma distribution, this hazard rate increases from zero to a certain number while in Weibull distribution, its hazard function raises from zero to infinity and such condition may not be suitable. This property is much more appropriate than exponential distribution where hazard rate is constant. Unfortunately, there are drawbacks in both of these distributions. One main form of Gamma distribution is that its distribution function, survival function or Gamma hazard function cannot be easily calculated in case the parameter is not an integer. However, it is possible to obtain distribution function, survival function or Gamma hazard function by mathematical tables or statistical software. This topic makes Gamma distribution be considered less than Weibull distribution because distribution function, survival function and hazard function of Weibull distribution are simply calculated. Weibull distribution is mostly proposed to analyze lifetime data.

The three-parameter Generalized Exponential (GE) distribution have special properties which are different from the two other distributions. Since its distribution function is a closed form, the inference based on the censored data can be handled more easily than with the Gamma family. The GE model can be used for analyzing any type of skewed datasets and therefore, the GE distribution may be substituted with the Weibull and Gamma families to analyze lifetime data. The GE distribution with three parameters has increasing or decreasing hazard rate according to its shape parameter. This distribution likes very similar specifications with Gamma distribution whereas this distribution has a distribution function as Weibull one which can be simply calculated. It seems that the GE shock model as an alternative can be replaced with Weibull models and most Gamma conditions.

This paper is organized as follows: The main features of the problem are presented in section 2. The main result that is an expression for the expected length of a production cycle and the expected cost per cycle are derived in section 3. In section 4, we determine the optimal design parameters to illustrate the relative advantages of choosing the sampling intervals of fixed length.

Symbols and Primary Results

In this part, we practice the economic model of Lorenzen and Vance (1986) which is the modified model of Duncan's cost model. This cost model includes the following specifications:

1- When the process is in control, the time follows the GE distribution which its density function is as below:

$$f(t)=\frac{k}{\sigma}(1-e^{-(t-\mu)/\sigma})^{k-1}e^{-(t-\mu)/\sigma} \quad (t>\mu,\ k\geq 1,\ \sigma>0). \tag{2–1}$$

where k is a shape parameter, σ is a scale parameter and μ is a location parameter. Without loss of generality we assume that $\mu=0$.

2- The process is monitored by drawing random samples of size n at h hours every times.

3- The time to sample and chart one item is negligible.

4- Production ceases during the searches and repair.

Features 1, 2, 3 and 4 are self- explanatory.

A complete description of the model was given by Lorenzen and Vance (1986). Now, we use the following notation based on Banerjee and Rahim (1987);

n: sample size

h: the length of sampling interval

Z_0: average of time for searching a false alarm

Z_1: average time to discover assignable cause

Z_2: average time to repair process

a: fixed sample cost

b: cost per unit sampled

L: control limit coefficient

Y: cost of each false alarm

W: cost to locate and repair the assignable cause

D_0: cost of quality per hour that producing is in control.

D_1: cost of quality per hour that producing is out of control.

$\alpha=$ Pr (test result has an alarm |the process is in control)

$\beta=$ Pr (test result has no alarm |the process is out of control)

ω_j: the time until the jth sample is taken ; $\omega_j=jh$, $\omega_0=0$.

p_j: the conditional probability that the process is out of control in the jth sampling interval, given that the process is still in control before time ω_{j-1}, that is

$$p_j=\frac{\int_{\omega_{j-1}}^{\omega_j} f(t)dt}{\int_{\omega_{j-1}}^{\infty} f(t)dt}=\frac{(1-e^{-\frac{jh}{\sigma}})^k-(1-e^{-\frac{(j-1)h}{\sigma}})^k}{1-(1-e^{-\frac{(j-1)h}{\sigma}})^k} \quad j=1,2,..... \tag{2–3}$$

q_j : the probability that the process will be out of control during the *j*th sampling interval;

$$q_j = \int_{\omega_{j-1}}^{\omega_j} f(t)dt = (1-e^{-\frac{jh}{\sigma}})^k - (1-e^{-\frac{(j-1)h}{\sigma}})^k \qquad j = 1, 2, \dots \tag{2–4}$$

τ_j : the average in-control time in the j th sampling interval, given that the shock occurred in the *j*th sampling interval;

$$\tau_j = \int_{\omega_{j-1}}^{\omega_j} \frac{(t-\omega_{j-1}) f(t)dt}{q_j} \tag{2–5}$$

τ : the unconditional average in-control time in a sampling interval;

$$\tau = \sum_{j=1}^{\infty} q_j \tau_j = \sigma[\psi(k+1) - \psi(1)] - \sum_{j=1}^{\infty} q_j \omega_{j-1} = \mu_t + h\sum_{j=1}^{\infty} F(jh) \tag{2–6}$$

where $\psi(.)$ is the digamma function and *F(.)* is the distribution function of GE.

A production cycle begins when a new component is installed and ends after a transition due to component failure is detected and the process is brought back to an in-control state by replacement. Assume that *E(T)* is the average time of the cycle and *E(C)* displays the total average cost of the cycle. The purpose is to find the amounts of n , h and L which minimizes *E(C)/E(T)*.

The average cycle cost and cycle time

In this section, we state a theorem that provides the expressions for *E(C)* and *E(T)* under the uniform sampling scheme.

Theorem 3-1. The following is true:

$$E(T) = (h + \alpha Z_0)\sum_{j=1}^{\infty}[1 - F(jh)] + (Z_1 + Z_2) + \frac{h}{1-\beta} \tag{3–4}$$

And

$$E(C) = (a + bn + \alpha Y + D_0 h)\sum_{j=1}^{\infty}[1 - F(jh)] + (D_0 - D_1)\tau + W + \frac{(a + bn + D_1 h)}{1-\beta} \tag{3–5}$$

Proof:

Using lemma A.1 of Banerjee and Rahim (1988), a set of recursive systems in form of *E(T)*, *E(T₁)*, *E(T₂)*, ... are obtained as below:

$$E(T) = h + p_1(1-\beta)(Z_1 + Z_2) + p_1\beta\{Z_1 + Z_2 + (1-\beta)\sum_{i=1}^{\infty} ih\beta^{i-1}\} + \alpha(1-p_1)Z_0 + (1-p_1)E(T_1)$$

$$E(T_1) = h + p_2(1-\beta)(Z_1 + Z_2) + p_2\beta\{Z_1 + Z_2 + (1-\beta)\sum_{i=2}^{\infty} ih\beta^{i-2}\} + \alpha(1-p_2)Z_0 + (1-p_2)E(T_2)$$

$$E(T_2) = h + p_3(1-\beta)(Z_1 + Z_2) + p_3\beta\{Z_1 + Z_2 + (1-\beta)\sum_{i=3}^{\infty} ih\beta^{i-3}\} + \alpha(1-p_3)Z_0 + (1-p_3)E(T_3)$$

Then

$$\begin{aligned} E(T) = {} & h[1 + (1-p_1) + (1-p_1)(1-p_2) + (1-p_1)(1-p_2)(1-p_3) + \cdots] \\ & + (Z_1 + Z_2)[p_1 + (1-p_1)p_2 + (1-p_1)(1-p_2)p_3 + \cdots] \\ & + \frac{h\beta}{1-\beta}[p_1 + (1-p_1)p_2 + (1-p_1)(1-p_2)p_3 + \cdots] \\ & + \alpha Z_0[(1-p_1) + (1-p_1)(1-p_2) + (1-p_1)(1-p_2)(1-p_3) + \cdots] \end{aligned}$$

Further simplification will prove (3-4), and using lemma A.2 of Banerjee and Rahim (1988), a set of recursive systems in form of *E(C)*, *E(C₁)*, *E(C₂)*, ... are obtained as below:

$$E(C)=(a+bn)+(D_0-D_1)\tau_1 p_1+Wp_1+\frac{(a+bn)\beta p_1}{(1-\beta)}+D_0h(1-p_1)+D_1\{(1-\beta)\sum_{i=1}^{\infty} ih\beta^{i-1}\}\beta p_1$$
$$+\alpha Y(1-p_1)+D_1hp_1+(1-p_1)E(C_1).$$

For j=2,3,...,

$$E(C_{j-1})=(a+bn)+(D_0-D_1)\tau_j p_j+Wp_j+\frac{(a+bn)\beta p_j}{(1-\beta)}+D_0h(1-p_j)+D_1\{(1-\beta)\sum_{i=j}^{\infty} ih\beta^{i-j}\}\beta p_j$$
$$+\alpha Y(1-p_j)+D_1hp_j+(1-p_j)E(C_j).$$

Then

$$\begin{aligned}
\mathrm{E}(C)=&(a+bn)[1+(1-p_1)+(1-p_1)(1-p_2)+\cdots]\\
&+(D_0-D_1)[p_1\tau_1+(1-p_1)p_2\tau_2+(1-p_1)(1-p_2)p_3\tau_3+\cdots]\\
&+W[p_1+(1-p_1)p_2+(1-p_1)(1-p_2)p_3+\cdots]\\
&+\frac{(a+bn)\beta}{(1-\beta)}[p_1+(1-p_1)p_2+(1-p_1)(1-p_2)p_3+\cdots]\\
&+\frac{D_1h\beta}{1-\beta}[p_1+(1-p_1)p_2+(1-p_1)(1-p_2)p_3+\cdots]\\
&+D_0h[(1-p_1)+(1-p_1)(1-p_2)+(1-p_1)(1-p_2)(1-p_3)+\cdots]\\
&+\alpha Y[(1-p_1)+(1-p_1)(1-p_2)+(1-p_1)(1-p_2)(1-p_3)+\cdots]\\
&+D_1h[p_1+(1-p_1)p_2+(1-p_1)(1-p_2)p_3+\cdots]
\end{aligned}$$

Further simplification will prove (3-5).

Determination of the Optimum Designing Parameters

This section studies the determination of control chart parameters. We determine the effects of scale parameter and shape parameter from the GE distribution on the designing parameters and we obtain the expected cost and time within every hour. This type is a renewal reward stochastic process was suggested by Ross (1970). Hence, due to its properties, the expected cost in each time unit is describes as a ratio of the expected cost in each production cycle to the expected time of the production cycle. The function of the expected cost is as below.

E(C)/E(T) = right side of Eq.(3-5) / right side of Eq.(3-4), (4-1)

The equation (4-1) can be applied to determine the optimum amounts of n , h and L decision variables. Unfortunately, there is no analytical solution determining the decision variables. A computer program has been prepared to obtain the optimum cost by statistical package *R*. For this, suppose that the amounts of the time parameters, shift parameter and the cost parameter are as table 1.

Table 1. Value of Time, Shift , and the Cost Parameters

Z_0	Z_1	Z_2	a	b	D_0	D_1	Y	W	δ
0.25hours	0.25hours	0.75hours	$20.00	$4.22	$50	$95	$500	$1100	0.5

Assume that the failure mechanism of operation process obeys the GE distribution by k=3 and σ=17.945. Therefore the design optimum parameters and the probabilities of Type I and Type II errors under this model are obtained as Table 2.

Table 2. The design optimum parameter values

n	h	L	α	1 –β
32	2.93 hours	1.63	0.1032	0.8849

From Table 3 , the expected cost per hour is $69.3288.

Sensitivity of the Designing Parameters and Average Cost

Table 3 indicates the economic design of $\overline{X}$ -control charts for the amounts of different parameters of GE distribution. As the amounts of σ increases, the expected cost decreases. Furthermore, as *k* increases, the expected cost reduces. If the amount of *k* is fixed, this table features out the effect of parameter σ on the optimum design (*n*, *h*, and *L*) and the expected cost. Table 3 indicates that the increase of σ amount is directed into decrease the expected cost. In other words, the expected cost gets increased as σ is decreased. The σ change affects less on *n* optimum sampling; in addition, it has no meaningful effect on the control limit coefficient *L*. Also, this table shows that the increase of *k* influences less on *n* optimum sampling size and has no meaningful effect on the control limit coefficient *L*.

Table 3. Economic Design of $\overline{x}$ -Control Charts Under GE Shock Models

GE Parameter				Economic design of $\overline{x}$ -control charts under GE shock model using uniform sampling scheme			
Set	σ	k	Mean	n	h	L	$E(C)/E(T)$
1	2.64	2	3.96	25	1.53	1.36	99.77
2	5.906	2	8.86	27	1.94	1.49	79.198
3	13.206	2	19.81	28	2.71	1.56	56.30
4	41.78	2	62.67	31	4.73	1.65	32.121
5	2.258	3	4.14	25	1.56	1.38	95.148
6	3.867	3	7.09	27	1.86	1.47	77.201
7	8.329	3	15.27	30	2.33	1.55	73.514
8	17.945	3	32.9	32	2.93	1.63	69.328
9	1.3776	4	2.87	23	1.18	1.35	166.58
10	2.0544	4	4.28	25	1.58	1.39	92.696
11	20.57	4	42.86	30	3.25	1.62	63.172

Conclusion

In this current article, we proposed an economic model for the optimum design of $\overline{X}$ -control chart. In this model, it was assumed that the process-failure mechanism is to follow a Generalized Exponential-typerandom shock model. It was also considered that the characteristic of the product's quality is normal distribution. In addition, we applied the technique of uniform sampling design to determine the parameters of optimum design and lead to a certain amount of convenience at industry.

References

Alexander, G.N. (1962). The use of Gamma distribution in estimating the regulated output from the storage. *Trans. Civil Engineering, Institute o Engineers, Australia*. 4, 29-34.

Bain, L.J. (1978). Statistical Analysis of Reliability and Life Testing Models. New York: Marcel Dekker, Inc.

Banerjee, P.K, & Rahim, M.A. (1987). The economic design of control charts: A renewal theory approach, *Engineering optimization*, *12*, 63-73.

Duncan, A.J. (1956). The economic design of $\overline{x}$ charts used to maintain current control of a process, *Journal of the American statistical association, 51,* 228-242.

Gibra, I.N. (1975), Recent developments in control charts techniques, *Journal of quality technology*, *7,* 183-192.

Hu, P.W. (1984). *Economic design of an* $\overline{x}$ *-control chart under non-poisson process shift,* Abstract, TIMS/ORSA joint national meeting, San Francisco, May 14-16, p. 87.

Jackson, O.A.Y. (1969). Fitting a Gamma or log-normal distribution to fibre-diameter measurements of wool tops. *Appl. Statist. 18,* 70-75.

Lorenzen, T.J., & Vance, L.C. (1986). The economic design of control charts: A unified approach, *Technometrics*, *28*, 3-10.

Montgomery, D.C., & Heikes, R.G. (1976). Process failure mechanism and optimal design of fraction defective control charts, *AIIE Transactions*, *8,* 467-473.

Van Kinken, J. (1961). A method for inquiring whether the distribution represents the frequency distribution of industrial accident costs. *Acturielle Studien*, 3, 83-92.

Vance, L.C. (1983). A bibliography of statistical quality control chart techniques, 1970-1980, *Journal of quality technology*, *15,* 59-62.