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A FERMI ALGEBRA FOR THE ISING MODEL ON AN INFINITE LATTICE

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Fermi algebra methods are applied to the two-dimensional Ising model on an infinite lattice. The way in which the phase transition manifests itself is discussed.

The two-dimensional Ising model in zero field has been treated algebraically by many authors, notably [1–3, 6, 7, 12, 15]. They consider an array of spins on a finite lattice, compute correlations using either the Clifford algebra [1, 2, 6, 7] or the Fermi algebra [2, 15] and then pass to the thermodynamic limit. The beauty of the algebraic formalism becomes more apparent if we use from the outset the Clifford or Fermi algebras on an inifinite lattice. This can be done using the C*-algebraic methods developed in [4, 14]. A full account of this is contained in [16]. We give here the main results; the proofs will appear elsewhere.

For each inverse temperature β we have a Fock representation of the Canonical Anticommutation Relations $[b_\beta^*(p), b_\beta(p')]_+ = \delta(p-p')$ with vacuum vector Ω_β. It is convenient to make use of the smoothed operators defined by $b_\beta^*(\varphi) = \int_{-\pi}^{\pi}\varphi(p) b_\beta^*(p)\,\mathrm{d}p/2\pi$, $\varphi \in L^2(-\pi,\pi)$ and to introduce operators J and Λ on $L^2(-\pi,\pi)$ defined by $(J\varphi)(p) = \mathrm{i}\varphi(p)$, $(\Lambda\varphi)(p) = \overline{\varphi(-p)}$. Every such representation is related to the infinite temperature representation $[a^*(p), a(p')]_+ = \delta(p-p')$ with vacuum vector Ω, by a Bogoliubov transformation

$$\mathcal{S}_\beta = \exp\left\{\mathrm{i}\int_0^\pi \theta(p)\{a^*(-p)a^*(p) + a(p)a(-p)\}\frac{\mathrm{d}p}{2\pi}\right\}, \tag{1}$$

so that

$$b_\beta^*(\varphi) = \mathcal{S}_\beta a^*(\varphi)\mathcal{S}_\beta^{-1} = a^*(\cos\Theta_\beta\varphi) + a(\sin\Theta_\beta J\Lambda\varphi), \tag{2}$$

$\Omega_\beta = \mathcal{S}_\beta\Omega$, where $(\Theta_\beta\varphi)(p) = \theta(p)\varphi(p)$, $2\theta(p) = \delta^*(p) + p - \pi$, $\delta^*(p)$ the usual Onsager angle as given in [1]. The Jordan–Wigner transformation [5] connects even products of the spin variables σ_{mn} with even products of the a's and a^*'s. Calculation of correlations involves evaluating the expectation value of these products in the vacuum state Ω_β. Consequently it is convenient to use $\Gamma(\varphi) = (a^*(\varphi) + a(\varphi))/2$ so that $\mathcal{S}_\beta\Gamma(\varphi)\mathcal{S}_\beta^{-1} = \Gamma(S_\beta\varphi)$ where $S_\beta = \exp(J\Lambda\Theta_\beta)$ and the state ω_β on the algebra $\mathcal{U}$ generated by the $\Gamma(\varphi)$ given by $\omega_\beta(X) = \langle\Omega_\beta, X\Omega_\beta\rangle$, $X \in \mathcal{U}$. This vanishes on the odd monomials and is given on the even monomials by the Wick formula

$$\omega_\beta(\Gamma(\varphi_1)\ldots\Gamma(\varphi_{2n})) = \sum_{s=2}^{2n}(-1)^s\omega_\beta(\Gamma(\varphi_1)\Gamma(\varphi_s)) \times \omega_\beta(\Gamma(\varphi_2)\ldots\widehat{\Gamma(\varphi_s)}\ldots\Gamma(\varphi_{2n})), \tag{3}$$

and so is determined by the two-point function

$$\omega_\beta(\Gamma(\varphi)\Gamma(\psi)) = s(\varphi,\psi) + \mathrm{i}s(A_\beta\varphi,\psi),\quad A_\beta = S_\beta J S_\beta^+, \tag{4}$$

(S_β^+ is the adjoint of S_β with respect to the inner product $s(\varphi,\psi) = \mathrm{Re}\int_{-\pi}^{\pi}\overline{\varphi(p)}\psi(p)\,\mathrm{d}p/2\pi$).

From the mathematical point of view the phase transition at $\beta = \beta_c$ shows itself in the following way: the operator A_β determines a Fredholm operator‡ D_β whose index, $\mathrm{ind}\,D_\beta$, is given by

$$\mathrm{ind}\,D_\beta = \begin{cases} 1 & \beta < \beta_c \\ 0 & \beta > \beta_c\,. \end{cases} \tag{5}$$

Physical manifestation of the phase transition is shown in the calculated values of the correlations and these all depend on this jump in the index. Even correlations can be calculated directly in this formalism using the shift operator W on $L^2(-\pi,\pi)$, $(W\varphi)(p) = \exp(\mathrm{i}p)\varphi(p)$, which corresponds to translation orthogonal to the transfer direction and the Onsagerian P_∞ (see [10])

‡ A Fredholm operator T on a Hilbert space is one such that the null spaces of T and its adjoint T^* are both finite-dimensional and $\mathrm{ind}(T) = \dim(\text{null space } T) - \dim(\text{null space } T^*)$.

which corresponds to translation along the transfer direction. (The Onsagerian can be thought of as the infinite volume limit of the transfer matrix normalised by dividing out the largest eigenvalue).

We have

$$P_\infty = \exp - \left\{ \int_{-\pi}^{\pi} \gamma(p) b_\beta^*(p) b_\beta(p) \frac{\mathrm{d}p}{2\pi} \right\}, \tag{6}$$

so that

$$P_\infty^n b_\beta^*(\varphi) P_\infty^{-n} = b_\beta^*(v^n \varphi), \quad P_\infty \Omega_\beta = \Omega_\beta, \tag{7}$$

where $(v\varphi)(p) = \exp\{-\gamma(p)\} \varphi(p)\} \varphi(p)$. For example

$$\langle \sigma_{00} \sigma_{10} \rangle = \omega_\beta(-\mathrm{i}\Gamma(Je_0)\Gamma(We_0)),$$

$$\langle \sigma_{00} \sigma_{20} \rangle = \langle \sigma_{00} \sigma_{10}^2 \sigma_{20} \rangle$$
$$= \omega_\beta((-\mathrm{i}\Gamma(Je_0)\Gamma(We_0))(-\mathrm{i}\Gamma(JWe_0)\Gamma(W^2 e_0))),$$

$$\langle \sigma_{00} \sigma_{10} \sigma_{20} \sigma_{30} \rangle \tag{8}$$
$$= \omega_\beta((-\mathrm{i}\Gamma(Je_0)\Gamma(We_0))(-\mathrm{i}\Gamma(JW^2 e_0)\Gamma(W^3 e_0))),$$

$$\langle \sigma_{00} \sigma_{10} \sigma_{0n} \sigma_{1n} \rangle$$
$$= \omega_\beta((-\mathrm{i}\Gamma(Je_0)\Gamma(We_0)) P_\infty^n (-\mathrm{i}\Gamma(Je_0)\Gamma(We_0)) P_\infty^{-n}),$$
$$e_0(p) \equiv 1, \quad p \in [-\pi, \pi].$$

In this formulation the treatment of translations in the two basic lattice directions appears to be asymmetric, in contrast to the Pfaffian approach [11]. This connection is seen as follows: Sz-Nagy's theorem [17] states that if T is a contraction ($\|T\| \leqslant 1$) on a Hilbert space $\mathcal{K}$, then $\mathcal{K}$ can be embedded in a larger Hilbert space $\mathcal{H}$ on which there is a unitary operator U in such a way that, on $\mathcal{K}$, $T^n = \pi U^n$ $n > 0$, where π is the projection of $\mathcal{H}$ on $\mathcal{K}$. The operator v is a strict contraction on $L^2(-\pi, \pi)$ for $\beta \neq \beta_c$. We can apply Sz-Nagy's theorem and show that $\mathcal{H}$ in this case can be taken to be $L^2([-\pi,\pi] \times [-\pi,\pi]; \rho(\psi)\,\mathrm{d}\psi\,\mathrm{d}\theta/2\pi)$ with $(UF)(\psi,\theta) = \exp(\mathrm{i}\theta) F(\psi,\vartheta)$, $(\pi F)(p) = \int_{-\pi}^{\pi} F(p,\vartheta) \times \mathrm{d}\theta/2\pi$ and $\rho(\psi) = \tanh\gamma(\psi)$. With this construction

$$(W^n v^m \varphi)(p) = \int_{-\pi}^{\pi} \exp(\mathrm{i}np) \exp(\mathrm{i}m\theta) \times \frac{\sinh\gamma(p)}{\cosh\gamma(p) - \cos\theta} \frac{\mathrm{d}\theta}{2\pi}. \tag{9}$$

A comparison of [2] and [11] yields the connection.

We emphasise that the states ω_β are determined by the even correlations and hence are independent of boundary conditions [9]. For a pure state the clustering property holds [8] so that for such a state the odd correlations can be computed from the even. Every translationally invariant Gibbs state is a mixture of the two pure states [9].

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