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Generalized XY Model

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Abstract

The methods of Lie Algebras are used to construct and solve a generalization of the XY model.

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1. Introduction

In this paper we apply the methods of Lie Algebras to solve and generalize the XY Model (Lieb, Schultz and Mattis 1961, Katsura 1962). The methods employed originate in the Spectrum Generating Algebras of particle physics - which are non-symmetry algebras of the Hamiltonian, and provide elegant solutions for quantum statistical problems (Solomon 1971, 1974). We briefly outline the approach we adopt in the present context.

In equilibrium statistical mechanics the thermodynamic behaviour of a system, whose Hamiltonian is H, follows from evaluation of the partition function

$$Q = \text{trace}\,\{\exp(-\beta H)\},$$

where β is the inverse of the absolute temperature times Boltzmann's constant. Classically, the trace may be interpreted as the sum over all the allowed configurations of the system; in quantum mechanics, as the usual Hilbert space trace.

In our algebraic treatment we shall consider H to be an element of a suitable Lie algebra of rank ℓ. This means that one can find a Cartan basis for the algebra which includes the ℓ mutually commuting elements, $h_1, h_2, \ldots, h_\ell$. The solution of the problem is obtained by finding an automorphism of the algebra, implemented by U say, such that

$$H \mapsto U H U^{-1} = \sum_{m=1}^{\ell} \Lambda_m h_m \tag{1.1}$$

where the Λ_m are known scalars (elements of the underlying field). Since in principle the spectra of the h_m are known, such an automorphism effects diagonalisation and clearly leaves the partition function Q unchanged.

Therefore the strategy to be adopted is in three parts:

A. Determine a suitable Lie algebra which is to generate the spectrum of H. The Hamiltonian H will be an element of the algebra in some (usually large) representation.

B. Choose a small-dimensional, faithful representation in which to implement the automorphism (1.1).

C. Now return to the original representation, in which (1.1) remains true and in particular the values of the scalars Λ_m are unchanged, to evaluate the spectrum of H and the partition function.

In the case of the XY model, on a cyclic lattice of N points, the application of the three-part strategy gives

A. The Hamiltonian is an element of a $2^N \times 2^N$ dimensional representation of so(2N) ⊕ so(2N). This is a rank 2N algebra.

B. We implement the automorphism (1.1) in the faithful 4N x 4N dimensional representation, determining the values of the 2N constants Λ_m.

C. We return to the $2^N \times 2^N$ representation to evaluate the partition function.

The reason that the solution of the XY Model is so readily obtained, in spite of the seemingly cumbersome nature of the machinery outlined above (nobody can diagonalize even a 4N x 4N matrix in general!) is that the translational invariance of physically interesting models means that in these cases the underlying algebra is a much smaller one, and effectively reduces the computation in all such cases to the diagonalization of a small (in our case, 2 x 2) matrix. The generalized XY Model we treat is in fact the most general translationally-invariant model consistent with the so(2N) ⊕ so(2N) algebra of the original XY Model.

Since the automorphism (1.1) reduces the computation of the partition function in C to that of a system of uncoupled spins, to which the model is therefore equivalent in an algebraic sense, we now briefly treat such a Free Spin Model.

2. Free Spin Model

Consider the following Hamiltonian, representing a system of N uncoupled spins:

$$H = -\sum_{m=1}^{N} \Lambda_m Z_m \qquad (2.$$

The Λ_m are positive scalars and

$$Z_m = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \sigma_z \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

where the matrices occurring in the direct product are all 2 x 2, and σ_z, whic occurs at the mth position, is the third of the three Pauli spinors,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

The overall negative sign in (2.1) ensures that alignment along the positive z-direction lowers the energy.

The partition function

$$Q(N, \beta) = \text{trace} (\exp \sum_{m=1}^{N} \beta\Lambda_m Z_m)$$

may be evaluated as a straightforward matrix trace, using the properties of direct products, as

$$Q(N, \beta) = \prod_{m=1}^{N} (2 \cosh \beta\Lambda_m).$$

The free energy per particle f is given in the thermodynamic limit by

$$- \beta f = \lim_{N \to \infty} \frac{1}{N} \log Q(N,\beta)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log\{2 \cosh\beta\Lambda(\phi)\} d\phi$$

where we replace the discrete valued Λ_m by the continuous function $\Lambda(\phi)$, where

$$\Lambda(\phi_m) = \Lambda_m \quad \text{at } \phi_m = 2\pi m/N \quad (m = 1, 2, \ldots, N)$$

All quantities of thermodynamic interest may be calculated from f.

In the model we shall be considering, the Hamiltonian can in fact be rotated to the somewhat more general form

$$H = - \sum_{m=1}^{N} \{\tfrac{1}{2}(1 - \gamma)\Lambda_m^+ + \tfrac{1}{2}(1 + \gamma)\Lambda_m^- \} Z_m \tag{2.2}$$

where $\Lambda_m^{\pm}$ are positive scalars and

$$\gamma = Z_1 Z_2 \ldots Z_N .$$

The partition function corresponding to this Hamiltonian may be equally readily evaluated,

$$Q(N,\beta) = 2^{N-1}\{\Pi \cosh \beta\Lambda_m^+ + \Pi \cosh \beta\Lambda_m^- + \Pi \sinh \beta\Lambda_m^+ - \Pi \sinh \beta\Lambda_m^- \} .$$

The free energy per particle determined from this partition function will depend on the relative magnitudes of Λ_m^+ and Λ_m^- ; for example, when $\Lambda_m^+ > \Lambda_m^-$ the first term will dominate.

3. Algebra of the XY Model

We consider a one-dimensional lattice of N sites, labelled 1, 2, ... , N. The XY Model is given by the following Hamiltonian of nearest-neighbour type:

$$H_1 = - \sum_{m=1}^{N} \{J_1^{xx} X_m X_{m+1} + J_1^{yy} Y_m Y_{m+1} \} . \tag{3.1}$$

The notation for X_m and Y_m is analogous to that of Z_m in the previous section, so that, for example,

$$[X_m, Y_n] = 2i \delta_{mn} Z_n .$$

We may also include additionally a contribution from an external magnetic field h

$$H_o = - h \sum_{m=1}^{N} Z_m . \tag{3.2}$$

The XY Model described by $H = H_o + H_1$ is exactly solvable, and although it does not exhibit a phase transition in the thermodynamic limit for finite β, its thermodynamic behaviour has been extensively studied. Further, the XY Model is intimately related to the solution of the two-dimensional Ising Model in transfer matrix form (Suzuki 1971); this connection is even more explicit in the case of the generalized model we shall describe in the next section, of which the XY Model is only a special case.

We now implement Part A of our strategy by determining the Spectrum Generating Algebra for the XY Model. Define the following matrices γ_r

$$\left.\begin{aligned} \gamma_r &= Z_1 Z_2 \ldots Z_{r-1} X_r \\ \gamma_{N+r} &= Z_1 Z_2 \ldots Z_{r-1} Y_r \end{aligned}\right\} \quad r = 2, 3, \ldots, N$$

$$\gamma_1 = X_1 \qquad \gamma_{N+1} = Y_1 .$$

The matrices γ_r $(r = 1, 2, \ldots, 2N)$ then generate a Clifford algebra with anti-commutation relations given by

$$\{\gamma_r, \gamma_s\} = 2\,\delta_{rs} \ .$$

The transformation from $\{X_m, Y_m, Z_m\}$ to $\{\gamma_r\}$ is sometimes called the Jordan-Wigner transformation. Using the γ_r we may construct the $N(2N-1)$ matrices L_{rs}

$$L_{rs} = -i/4[\gamma_r, \gamma_s] \quad (r, s = 1, 2, \ldots, 2N) \qquad (3.3)$$

which close under the commutation relations of the Lie algebra so(2N)

$$[L_{rs}, L_{pq}] = i(\delta_{rp}L_{sq} - \delta_{sp}L_{rq} + \delta_{rq}L_{ps} - \delta_{sq}L_{pr}) \ . \qquad (3.4)$$

(We retain the i in expressions such as (3.3) and (3.4) only when we wish to maintain the hermiticity of the operators concerned. The algebras, such as so(2N), that we are interested in are of course real Lie algebras whose defining relations do not involve i.)

From the following expressions, which hold for $m = 1, 2, \ldots, N-1$,

$$X_mX_{m+1} = 2L_{N+m,\,m+1} \qquad Y_mY_{m+1} = 2L_{N+m+1,\,m} \qquad Z_m = 2L_{m,\,N+m} \ .$$

We see that in the case of the XY Model with <u>free</u> ends, where the summation in (3.1) goes from 1 to N-1, we may immediately express H as an element of so(2N). In the <u>cyclic</u> case, however, we require the additional terms X_NX_1 and Y_NY_1 and so must enlarge the algebra. This is readily done as follows.

Introduce the matrix $\gamma = Z_1Z_2 \ldots Z_N$. This obeys

$$\{\gamma, \gamma_r\} = 0 \qquad \gamma^2 = 1 \qquad [\gamma, L_{rs}] = 0 \ .$$

Then the operators

$$L^{(a)}_{rs} = \tfrac{1}{2}(1 - a\gamma)L_{rs} \qquad\qquad a = \pm$$

close on the algebra so(2N) ⊕ so(2N)

$$\left[L^{(a)}_{rs}, L^{(b)}_{pq}\right] = i\delta_{ab}(\delta_{rs}L^{(a)}_{sq} - \delta_{sp}L^{(a)}_{rq} + \delta_{rq}L^{(a)}_{ps} - \delta_{sq}L^{(a)}_{pr});$$

This enlarged algebra now contains all the previously required quantities

$$L_{rs} = L^{+}_{rs} + L^{-}_{rs}$$

as well as the cyclic terms

$$X_NX_1 = 2(L^{(-)}_{1,2N} - L^{(+)}_{1,2N}) \ ,$$

$$Y_NY_1 = 2(L^{(-)}_{N,N+1} - L^{(+)}_{N,N+1}) \ ,$$

and so we may write the XY Hamiltonian $H = H_o + H_1$, equations (3.1) and (3.2) explicitly as an element of so(2N) ⊕ so(2N). This completes Part A of our strategy.

4. Translational Invariance

The most general element of our so(2N) ⊕ so(2N) algebra may be written

$$H = \sum_{a=\pm}\ \sum_{m,n=1}^{2N} \omega^{(a)}_{mn}\, L^{(a)}_{mn} \qquad (4$$

or, more compactly,

$$H = -\,\mathrm{tr}\,\omega\mathcal{L}$$

with ω and $\mathcal{L}$ defined as blocked 4N x 4N matrices

$$\omega = \left[\begin{array}{c|c} \omega^{(+)} & 0 \\ \hline 0 & \omega^{(-)} \end{array}\right] \qquad \mathcal{L} = \left[\begin{array}{c|c} L^{(+)} & 0 \\ \hline 0 & L^{(-)} \end{array}\right]$$

where the elements of $\omega^{\pm}$ are real numbers and those of $\mathcal{L}$ are the matrices $L^{(+)}_{rs}$, $L^{(-)}_{rs}$. We now impose translational (more precisely, cyclic) invariance by demanding that H be invariant under the action of the unitary operator $\mathcal{J}$ defined by

$$X_{r+1} = \mathcal{J} X_r \mathcal{J}^{-1} \quad Y_{r+1} = \mathcal{J} Y_r \mathcal{J}^{-1} \quad Z_{r+1} = \mathcal{J} Z_r \mathcal{J}^{-1} \;;$$

that is

$$H = \mathcal{J} H \mathcal{J}^{-1} \tag{4.2}$$

The operator $\mathcal{J}$ which obeys $\mathcal{J}^N = 1$ and generates a $2^N \times 2^N$ dimensional representation of the cyclic subgroup C_N of SO(2N) ⊗ SO(2N), is implemented on $\mathcal{L}$ by

$$\mathcal{J} \mathcal{L}_{RS} \mathcal{J}^{-1} = (\mathcal{D} \mathcal{L} \tilde{\mathcal{D}})_{RS} \qquad (\mathcal{D} \tilde{\mathcal{D}} = \mathbb{1})$$

where $\mathcal{D}$ is the 4N x 4N (numerical) matrix defined by

$$\mathcal{D} = \left[\begin{array}{c|c} \mathbb{1} \otimes \Delta^{(+)} & 0 \\ \hline 0 & \mathbb{1} \otimes \Delta^{(-)} \end{array}\right]$$

and tilde denotes matrix transpose.

The cyclic N X N matrix $\Delta^{(+)}$, and the anti-cyclic $\Delta^{(-)}$, are given by

$$\Delta^{(+)} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix} \qquad \Delta^{(-)} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ -1 & & & & 0 \end{bmatrix}$$

and obey $\Delta^{(a)} \tilde{\Delta}^{(a)} = \mathbb{1}$, $\Delta^{(a)N} = a\,\mathbb{1}$ $(a = \pm)$. Imposing the condition (4.2) on the Hamiltonian (4.1) leads to the equation

$$(\mathbb{1} \otimes \Delta^{(a)})^{-1} \, \omega^{(a)} \, (\mathbb{1} \otimes \Delta^{(a)}) = \omega^{(a)} \tag{4.3}$$

whose general solution, with $\omega^{(a)}$ anti-symmetric, is

$$\omega^{(a)} = \sum_{r=0}^{N-1} \left(J_r^{(a)} \otimes \Delta^{(a)r} - \tilde{J}_r^{(a)} \otimes \tilde{\Delta}^{(a)r} \right) \tag{4.4}$$

where the $J_r^{(a)}$ are arbitrary, real 2 x 2 matrices. This set of coefficients ω in the Hamiltonian (4.1) therefore gives the most general translationally invariant model consistent with the so(2N) ⊕ so(2N) algebra.

The subscript r in the expression (4.3) for $\omega^{(a)}$ refers to an interaction which is $(r+1)$-body and of range r. For example, taking $J_r^+ = J_r^-$, we may rewrite the Hamiltonian (4.1) as

$$H = \sum_{r=0}^{N-1} H_r$$

with

$$H_o = -h \sum_{m=1}^{N} Z_m$$

and

$$H_r = -\sum_{m=1}^{N} \{ J_r^{xx} X_m Z X_{m+r} + J_r^{yy} Y_m Z Y_{m+r} + J_r^{xy} X_m Z Y_{m+r} + J_r^{yy} Y_m Z X_{m+r} \} \quad r = 1,2,\ldots,N$$

where we have put $J_r^{(a)} = \begin{bmatrix} J_r^{yx} & J_r^{yy} \\ -J_r^{xx} & -J_r^{xy} \end{bmatrix}$ and $-h = J_o^{xx} + J_o^{yy}$, and terms like $X_m Z X_{m+r}$ are shorthand for $X_m Z_{m+1} \ldots Z_{m+r-1} X_{m+r}$. In this form we see that we may recover the XY model, by choosing all the coefficients except h,

J_1^{xx} and J_1^{yy} as zero; as well as other generalizations such as that of Suzuki ($J_r^{xy} = J_r^{yx} = 0$) (Suzuki, 1971) and Dzyaloshinsky (h, J_1^{xx}, J_1^{yy}, $J_1^{xy} = -J_1^{yx}$ nonzero) (Siskens et al, 1974). However, the form (4.1) and (4.3) is most suitable for our purposes, and we now implement Part B of our general strategy, by choosing a convenient faithful representation of so(2N) ⊕ so(2N) in which to implement the automorphism (1.1).

5. Diagonalization of the Hamiltonian

A convenient representation of so(2N) ⊕ so(2N) in which to implement the automorphism (1.1) is the standard representation of the rotation algebra

$$\left[S_{rs}, S_{pq}\right] = \delta_{sp}S_{rq} + \delta_{rq}S_{sp} - \delta_{sq}S_{rp} - \delta_{rp}S_{sq}$$

obtained by setting

$$S_{rs} = e_{rs} - e_{sr} \qquad (r, s = 1, 2, \ldots, 2N)$$

where e_{rs} is the 2N x 2N matrix defined by

$$e_{rs}^{mn} = \delta_r^m \delta_s^n \qquad (m, n = 1, 2, \ldots, 2N)$$

We may use this representation for both the "+" and "-" algebras to obtain the representation for $H^{(a)}$

$$\hat{H}^{(a)} = (-2i)\,\omega^{(a)} \qquad (a = \pm)$$

as a 2N x 2N matrix.

Since the $\omega^{(a)}$ are antisymmetric, there exists an automorphism (rotation by an orthogonal matrix) which sends $\omega^{(a)}$ to the canonical form

$$\omega^{(a)} \longmapsto \left[\begin{array}{c|c} 0 & -\Lambda^{(a)} \\ \hline \Lambda^{(a)} & 0 \end{array}\right]$$

where the diagonal matrix $\Lambda^{(a)} = \mathrm{diag}\{\Lambda_1^{(a)}, \ldots, \Lambda_N^{(a)}\}$ may be chosen to have positive entries which are readily computed (Appendix). (This amounts to choosing the commuting elements h_m of the Cartan basis (1.1) proportional to $S_{m,N+m}$ in this representation.)

Since the automorphism

$$\hat{H}^{(a)} \longmapsto \sum_{m=1}^{N} 2i\,\Lambda_m^{(a)}\, S_{m,N+m}$$

holds in the 2N-dimensional representation, it also holds in the 2^N-dimensional (Hermitian, $S \sim iL^{(a)}$) representation

$$H^{(a)} \longmapsto \sum_{m=1}^{N} -2\Lambda_m^{(a)}\, L_{m,N+m}^{(a)}$$

so that the original Hamiltonian (4.1) takes the form (2.2)

$$H \longmapsto -\sum_{m=1}^{N} \{\tfrac{1}{2}(1-\gamma)\Lambda_m^{+} Z_m + \tfrac{1}{2}(1+\gamma)\Lambda_m^{-} Z_m\}$$

of the free spin model, and the partition function may immediately be evaluat[ed,] completing Part C of our strategy. The expression for the free energy is gi[ven] in the Appendix, (A5) and (A6).

6. Conclusion

We have described a generalization of the spin-½ XY model which is exactly solvable, and the most general within the context of the so(2N) ⊕ so(2N) alge[bra] of the usual XY model and translational invariance. The expression derived the free energy, (A1) and (A2), may be chosen to have particularly simple, closed forms; for in the infinite range limit the coupling constants in (A2) can be taken as the Fourier coefficients of (fairly arbitrary) functions.

The Hamiltonian considered, though of doubtful direct physical interest, has a useful interpretation as the associated Hamiltonian of the two-dimensional Ising problem; that is, an operator commuting with the transfer matrix.

Although we have only considered translationally invariant models in this note, it is a straightforward matter to proceed to the non-invariant case. For example, if the Hamiltonian is invariant under ν-translations, where ν is some positive integer dividing N,

$$T^{\nu} H T^{-\nu} = H$$

we obtain the general solution by solving the modification of (4.3)

$$(\mathbb{1} \otimes \Delta^{(a)})^{-\nu} \, \omega^{(a)} \, (\mathbb{1} \otimes \Delta^{(a)})^{\nu} = \omega^{(a)}$$

This modification will be the subject of another note.

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Appendix

We determine the elements $\Lambda_m^{(a)}$ of the canonical form of

$$\omega^{(a)} = \sum_{r=0}^{N-1} \left(J_r^{(a)} \otimes \Delta^{(a)r} - \tilde{J}_r \otimes \tilde{\Delta}^{(a)r} \right) \qquad \text{(A1)}$$

Assume that the eigenvalue equation for $\omega^{(a)}$ is of the form

$$\omega^{(a)} (\underset{\sim}{v} \otimes \underset{\sim}{e}^{(a)}) = i\mu (\underset{\sim}{v} \otimes \underset{\sim}{e}^{a}) \qquad \text{(A2)}$$

where $\underset{\sim}{e}^{(a)}$ is an eigenvector of $\Delta^{(a)}$ with eigenvalue $\lambda^{(a)}$

$$\Delta^{(a)} \underset{\sim}{e}^{(a)} = \lambda^{(a)} \underset{\sim}{e}^{(a)}$$

By substituting the expression (A1) for $\omega^{(a)}$ in (A2) we obtain

$$\sum_{r=0}^{N-1} \left(J_r^{(a)} \lambda^{(a)r} - \tilde{J}_r^{(a)} \lambda^{(a)-r} \right) \underset{\sim}{v} \otimes \underset{\sim}{e}^{a} = i\mu \, \underset{\sim}{v} \otimes \underset{\sim}{e}^{(a)}$$

so that μ is the eigenvalue corresponding to the eigenvector $\underset{\sim}{v}$ of the 2×2 Hamiltonian matrix $M^{(a)}$

$$M^{(a)} = -i \sum_{r=0}^{N-1} \left(J_r^{(a)} \lambda^{(a)r} - \tilde{J}_r^{(a)} \lambda^{(a)-r} \right) \qquad \text{(A3)}$$

Rewriting $M^{(a)}$ in terms of the four real numbers $m_\mu^{(a)}$

$$M^{(a)} = \sum_{\mu=0}^{3} m_\mu^{(a)} \sigma_\mu \qquad \text{(A4)}$$

the two eigenvalues of $M^{(a)}$ are immediately given by

$$\mu = m_0^{(a)} + \sqrt{m_1^{(a)2} + m_2^{(a)2} + m_3^{(a)2}} \qquad (A5)$$

and

$$\mu = m_0^{(a)} - \sqrt{m_1^{(a)2} + m_2^{(a)2} + m_3^{(a)2}}$$

Since the eigenvalues of $\omega^{(a)}$ occur in conjugate pairs, the N positive values $\Lambda_m^{(a)}$ are enumerated by taking the modulus of ($A5$) corresponding to each eigenvalue $\lambda_m^{(a)}$ of $\Delta^{(a)}$

$$\lambda_m^{(a)} = \exp i\,\phi_m^{(a)} \qquad (m = 1, 2, \ldots, N)$$

with

$$\phi_m^{(+)} = 2m\pi/N \quad , \quad \phi_m^{(-)} = (2m+1)\pi/N$$

Defining the energy function $\Lambda^{(a)}(\phi)$ by

$$\Lambda^{(a)}(\phi_m) = \Lambda_m^{(a)}$$

we have explicitly

$$\Lambda^{(a)}(\phi) = \left| m_0^{(a)}(\phi) + \sqrt{m_1^{(a)}(\phi)^2 + m_2^{(a)}(\phi)^2 + m_3^{(a)}(\phi)^2} \right|$$

with the functions $m_\mu^{(a)}(\phi)$ given in terms of the matrix elements of $J_r^{(a)}$ by

$$m_0^{(a)}(\phi) = \sum_{r=1}^{N-1} \left(J_r^{(a)11} + J_r^{(a)22}\right) \sin r\phi$$

$$m_1^{(a)}(\phi) = \sum_{r=1}^{N-1} \left(J_r^{(a)12} + J_r^{(a)21}\right) \sin r\phi$$

$$m_2^{(a)}(\phi) = \sum_{r=0}^{N-1} \left(J_r^{(a)12} - J_r^{(a)21}\right) \cos r\phi$$

$$m_3^{(a)}(\phi) = \sum_{r=1}^{N-1} \left(J_r^{(a)11} - J_r^{(a)22}\right) \sin r\phi$$

The magnetic field term occurs explicitly as the r = 0 component of $J_2^{(a)}(\phi)$

$$h = -\tfrac{1}{2}\left(m_2^{(+)}(0) + m_2^{(-)}(0) \right)$$

In the case $J_r^+ = J_r^-$ we have $\Lambda_m^+(\phi) = \Lambda_m^-(\phi)$ as $N \to \infty$, and so the free energy may be written

$$-\beta f = \frac{1}{2\pi} \int_0^{2\pi} \ln \left\{ 2 \cosh \beta \Lambda(\phi) \right\} d\phi$$

as in Section 2, with

$$\Lambda(\phi) = \sum_{r=1}^{\infty} a_r \sin r\phi + \left[\left(\sum_{r=1}^{\infty} b_r \sin r\phi\right)^2 + \left(\sum_{r=1}^{\infty} c_r \cos r\phi - h\right)^2 + \left(\sum_{r=1}^{\infty} d_r \sin r\phi\right)^2 \right]^{\frac{1}{2}}$$

writing

$$a_r = J_r^{yx} - J_r^{xy}$$

$$b_r = J_r^{yy} - J_r^{xx}$$

$$c_r = J_r^{xx} + J_r^{yy}$$

$$d_r = J_r^{xy} + J_r^{yx}$$

All the thermodynamic quantities may be calculated from (A6) in the usual way