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The General Bogoliubov Transformation

A. I. Solomon
The Open University

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The General Bogoliubov Transformation

A. I. Solomon
Open University
Milton Keynes, U.K.

A large variety of problems in many-body theory can be solved by using a canonical transformation on the creation and annihilation operators which sends the hamiltonian operator into diagonal form. This useful diagonalization trick was first introduced by Bogoliubov[1], and enabled him to solve the problem of superfluid Helium Four. For that specific problem the transformation took the following form: given annihilation and creation operators a_k and a_k^+ respectively for bosons of momentum k, and so obeying

$$[a_k, a_{k'}^+] = \delta_{k,k'}$$

we transform to new operators α_k, α_k^+ by

$$a_k \longmapsto \alpha_k = u_k a_k - v_k a_{-k}^+$$
$$a_k^+ \longmapsto \alpha_k^+ = u_k a_k^+ - v_k a_{-k}$$

This transformation is canonical - that is, the new operators obey

$$[\alpha_k, \alpha_{k'}^+] = \delta_{k,k'}$$

if $\quad u_k^2 - v_k^2 = 1,\ u_k = u_k^+,\ v_k = v_k^+.$

As I have already mentioned this procedure is by no means limited to the superfluidity problem; with only slight modification this canonical transformation can be used to diagonalize the BCS hamiltonian of superconductivity, as well as obtain the explicit solutions to the well-known exactly solvable lattice models referred to as the Ising Model and XY Model. What all these problems have in common - apart from their exact solvability - is that for each of them the hamiltonian can be thought of as an element of a Lie algebra, the spectrum-generating algebra; in each case the hamiltonian operator is, more precisely, an element of an irreducible representation of the algebra. Putting this element in diagonal form gives the spectrum of the physical system in question.

In the language of Lie algebras, what corresponds to the canonical transformation described above? In fact, what corresponds to the diagonalisation of our hamiltonian? Well, we may in general find a basis for an n-dimensional rank ℓ semi-simple Lie algebra g in the form

$$\{h_1, h_2, \ldots, h_\ell; e_1, e_2, \ldots, e_{n-\ell}\}$$

where the h_i generate a maximal abelian subalgebra of g (Cartan subalgebra). Then the diagonalisation of our hamiltonian $x \in g$ corresponds to finding an automorphism

$$\phi : g \longmapsto g$$

such that $x \longmapsto \phi(x) = \sum_{i=1}^{\ell} a_i h_i$

where the a_i are real coefficients in the applications; that is, we send the hamiltonian to a sum of mutually commuting elements. Given the values of a_i, and the spectra of the h_i, then the spectrum of the hamiltonian x is immediate; all this, of course, in the relevant representation determined by the physical problem.

In this context, the general Bogoliubov transformation is simply the automorphism ϕ above. We justify this nomenclature essentially by illustrating the process in a few selected examples. The advantage of expressing these ideas in group theoretical language is that one can bring to bear all the power of group theory; in particular, representation theory. One need not perform the diagonalisation of the hamiltonian in the original representation supplied by the physical situation; one can implement the automorphism ϕ in a smaller faithful representation (generally the defining representation of the Lie algebra). And it is generally not necessary to give explicitly the form of the general Bogoliubov transformation ϕ; it is sufficient to know the resulting diagonal form of the hamiltonian to solve the problem. This resulting diagonal form can be obtained either by an explicit matrix diagonalisation, or by use of the Killing Form and Casimir invariants.

We now illustrate the preceding ideas by some examples. Due to limitations of space we can do no more than outline the results and give references. For each example we give an algebra which generates the spectrum - not necessarily the minimal algebra in the sense of Joseph[2] - and indicate briefly the physical consequences.

Superconductivity: Using the well-known BCS reduced hamiltonian, the spectrum generating algebra turns out to be su(2); more precisely, there is one such algebra for each energy level of the many-fermion system. For one such energy level, the hamiltonian may be written

$$x = -2\epsilon J_3 + 2\Delta J_2 .$$

where the generators J_i obey $[J_i,J_j] = i\epsilon_{ijk}J_k$, and ϵ is the energy and Δ and associated energy gap. We may use the Killing Form $B(x,y) = \mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y)$ $x,y \in g$ to define a "length-squared" $B(x,x) = 4(\epsilon^2 + \Delta^2)$ for the hamiltonian. Since this is invariant under any automorphism, including the Bogoliubov transformation, diagonalisation - that is, rotating to the single Cartan generator $h_1 \equiv J_3$ - sends

$$x \longmapsto 2\sqrt{\epsilon^2 + \Delta^2}\ J_3$$

and so gives the energy spectrum $2\sqrt{\epsilon^2 + \Delta^2}$.

Superfluid Helium Four[3]: In this case the spectrum is generated by the non-compact algebra su(1,1) in an analogous manner to the superconductivity case. The reduced hamiltonian $x \in su(1,1)$ has the form

$$x = 2NV(\mu J_3 - J_1)$$

where the generators J_i obey $[J_1,J_2] = iJ_3$, $[J_2,J_3] = iJ_1$, $[J_3,J_1] = iJ_2$; and N = Number density, V = potential, $\mu = 1 + \epsilon/NV$ where ϵ = energy. Use of the Killing Form now gives $B(x,x) = (2NV)^2(\mu^2 - 1)$ which implies for a positive potential ($\mu > 1$), the energy spectrum $(2NV)(\mu^2 - 1)^{\frac{1}{2}}$.

The XY Model[4]: This is a one-dimensional lattice of n sites; at each site i there sits a spin one-half X_i or Y_i which interacts with its nearest neighbour. There may also be present a 'magnetic field' term Z_i. It turns out that this translationally invariant system has spectrum generating algebra so(2n) $\oplus$ so(2n); the rank $\ell = 2n$ and the corresponding Bogoliubov rotation sends the system to a sum of an uncoupled spins $\{h_1, h_2, \ldots, h_{2\ell}\}$.

The Ising Model[5]: The Transfer Matrix for this model is an element of the group so(2n) $\oplus$ so(2n) algebra associated with the XY model above; and this leads to a similar solution.

Superfluid Helium Three[6]: This is an anisotropic analogue of the superconductivity problem: using a BCS-type reduced hamiltonian leads to the spectrum being generated by the algebra su(4), of which superconductivity su(2) is a subalgebra. The energy spectrum is now given in terms of two energy gaps, which are the degenerate pairs of eigenvalues of a 4×4 matrix. These gaps are associated with the two main superfluid phases of Helium Three.

References

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