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Explicit Solution of the Corrigan-Goddard Conditions for N Monopoles for Small Values of the Parameters.

by

L. O'Raifeartaigh, S. Rouhani and L.P. Singh*,

Dublin Institute for Advanced Studies

10 Burlington Road, Dublin 4, Ireland.

Abstract

The Corrigan-Goddard conditions for the parameters describing a system of n SU(2) monopoles in static equilibrium are solved for small values of the parameters. The 4n independent parameters are exhibited explicitly, and turn out to have a simple group theoretical interpretation.

*Permanent address: Physics Department, Utkal University, Bhubaneswar 751 004, India.

As shown by Ward and Atiyah[1] the solutions to the Bogomolnyi equations for static SU(2) monopoles may be formulated in terms of an SL(2,c) transition matrix, and recently, building on some earlier results[2], Corrigan and Goddard[3] have proposed an Ansatz for the transition matrix to describe any possible configuration of n monopoles. The Ansatz is non-singular for at least small monopole separations. But it contains n(n+2) parameters satisfying n(n-2) conditions (most of them transcendental) and so it is not easy to extract a set of 4n independent parameters[4] or to identify the parameters with the physical properties of the system. Indeed it has not even been shown that the n(n-2) conditions really do admit solutions with 4n independent parameters. Accordingly, it may be of interest to simplify the conditions so that they can be examined in more detail. In a previous paper[5] the simplifications that arise by imposing special symmetries on the system were considered, and in the present note we wish to consider the simplifications that arise for small values of the parameters, that is, for linear perturbations of the superimposed (axisymmetric) solutions. It turns out that for small values of the parameters the n(n-2) conditions can be completely solved and the transition matrix exhibited explicitly. Furthermore, all 4n independent parameters and all the symmetries of the system are already present, and so the linear approximation gives some insight into the meaning of the parameters. The group-theoretical meaning becomes immediately clear. They are the excitations belonging to the (1-dimensional) irreducible representations of the extended group of rotations around the z-axis $C_\infty \times$ parity i.e. $(m,\pm)$ where $|m|=n-1,n$. The relationship of the parameters to the coordinates of the monopoles (zeros of the Higgs field $\Phi(x)$) requires further study, but apparently all 4n parameters appear in the gauge-invariant quantity $\Phi^2(x)$ and so have a direct spatial interpretation.

To obtain our results we use the Ward[2] form of the solutions to the Bogomolnyi equation, in which the transition matrix is determined by a single function f=N/H where N and H are defined as follows. Let $x_\pm = x \pm iy$, be the space coordinates, ζ the Ward-Atiyah variable and $\gamma = 2\zeta + x_+\zeta - x_-\zeta^{-1}$. Then

$$H = \gamma^n + \sum_{k,m} a_k^{(m)} \zeta^m \gamma^{k-1} \qquad \text{where} \quad k = 1 \cdots n, \quad |m| \leqslant n+1-k, \tag{1}$$

the $a_k^{(m)}$ are constant, and N is chosen so that f is entire in x,y and z. The CG Ansatz for N is that

$$N = e^{\frac{\pi}{2}K} + (-1)^n e^{-\frac{\pi}{2}K} \qquad \text{where } K = \sum_{r=1}^{n} n_r \prod_{s\neq r} \left(\frac{\gamma-\omega_s}{\omega_r-\omega_s}\right), \quad H = \prod_{r=1}^{n} (\gamma - \omega_r) \tag{2}$$

the roots ω_r are assumed distinct and n_r are the values taken by ω_r for the axisymmetric Ansatz[2] $in_r=0,\pm 2, \ldots \pm(n-1)$ for odd n and $in_r=\pm 1,\pm 3 \ldots \pm(n-1)$ for even n. The only parameters in (1) and (2) are the n(n+2) parameters and these are required to satisfy the n(n-2) conditions[3]

$$\oint \frac{d\zeta}{\zeta}\, b_\alpha\, \zeta^m = 0 \qquad \text{where} \quad K = b_n + b_{n-1}\gamma + \cdots + b_1 \gamma^{n-1} \quad , \tag{3}$$

and $\alpha = 1,\cdots,n-2$, $\quad -(n-1-\alpha) \leqslant m \leqslant (n-1-\alpha)$, leaving 4n independent parameters.

Our results will be based on the following two properties of K. First, because of the identity

$$\sum_{r=1}^{n} \omega_r \prod_{s\neq r}\left(\frac{\gamma-\omega_s}{\omega_r-\omega_s}\right) - \gamma \equiv 0 \quad , \tag{4}$$

K can be written in the form

$$K = \gamma + \sum_{r=1}^{n}(n_r-\omega_r)\prod_{s\neq r}\left(\frac{\gamma-\omega_s}{\omega_r-\omega_s}\right) \quad , \tag{5}$$

(A simple method to establish (4) is to note that the left-hand-side is a polynomial of degree n-1 in γ and vanishes at the n distinct points $\gamma=\omega_r$).

Second, if we expand K as

$$K = \sum_{r,s} b_s^r(\omega)\, \gamma^{n-s}\, n_r \Big/ \prod_{q\neq r}(\omega_r-\omega_q) \quad , \quad r,s=1,\ldots,n \tag{6}$$

then the nxn matrix $b_s^r(\omega)$ can be written in the form

$$b_s^r(\omega) = \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ -e_1^{(1)} & -e_1^{(2)} & \cdot & \cdot \\ e_2^{(1)} & e_2^{(2)} & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdot\;\cdot & 0 \\ -e_1 & 1 & 0 & \cdot\;\cdot & 0 \\ e_2 & -e_1 & 1 & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot\;\cdot & 1 \\ \omega_1 & \omega_2 & \cdot\;\cdot & \omega_n \\ \omega_1^2 & \omega_2^2 & \cdot\;\cdot & \omega_n^2 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \omega_1^{n-1} & \omega_2^{n-1} & \cdot\;\cdot & \omega_n^{n-1} \end{bmatrix} = T_{sq}(\omega_r)^{q-1} \tag{7}$$

where the e_p are the elementary symmetric functions[6] of order p of the ω_r and $e_p^{(s)}$ the elementary symmetric functions of order p obtained by omitting ω_s. Note that the matrix elements of Tsq vanish for q>s. The first equality in (7) follows directly from (2) and the second equality from

$$e_p^{(r)} = e_p - \omega_r\, e_{p-1}^{(r)} \tag{8}$$

So far there is no question of small separations, or even small variations in the $a_k^{(m)}$. But let us now consider such small variations. On comparing (1) and (2) we see that small variations in the $a_k^{(m)}$ induce small variations in the roots ω_r given by

$$\delta\omega_r = -\sum_{k,m} \delta a_k^{(m)} (\omega_r)^{k-1}\, \zeta^m \Big/ \prod_{s\neq r}(\omega_r-\omega_s) \quad . \tag{9}$$

In particular if the small variations are around the axisymmetric values of the $a_k^{(m)}$ (and therefore correspond to small separations) we have

$$\delta\omega_r = -\sum_k \delta a_k^{(m)} (n_r)^{k-1} \zeta^m / \sigma_r \qquad \text{where} \qquad \sigma_r = \prod_{s\neq r}(n_r-n_s) \tag{10}$$

Furtheremore, we see from (5) that in this case

$$K = \gamma - \sum_{r=1}^{n} \frac{\delta\omega_r}{\sigma_r} \prod_{s\neq r}(\gamma-n_s) = \gamma - \sum_{r=1}^{n} \frac{\delta\omega_r}{\sigma_r^2}\, b_s^r(n)\, \gamma^{n-s} \tag{11}$$

and the n(n-2) conditions (3) become

$$\sum_{r=1}^{n} b_\alpha^r(n) \frac{1}{\sigma_r^2} \oint \frac{d\zeta}{\zeta}\, \delta\omega_r\, \zeta^m = 0 \,, \quad \alpha = 1\cdots n-2 \,, \quad |m| \leqslant n-\alpha-1 \quad . \tag{12}$$

But then from (10) we see that the n(n-2) conditions reduce to the purely <u>algebraic</u> conditions

$$\sum b_\alpha^{(r)}(n) \frac{1}{\sigma_r^2} (n_r)^{k-1}\, \delta a_k^{(m)} = 0 \,, \tag{13}$$

Furthermore by using (11) (and the triangular nature of Tsq) we see that (13) can be written in the more symmetrical form

$$\sum_k \rho_{\alpha k}\, \delta a_k^{(m)} = 0 \qquad \text{where} \qquad \rho_{ik} = \sum_r (n_r)^{i+k-2} / \sigma_r^2 \tag{14}$$

and

$$\alpha = 1 \ldots n-1-|m| \;;\quad i,k = 1,\cdots,n+1-|m| \quad \text{for} \quad m \neq 0$$
$$\alpha = 1 \cdots n-2 \qquad\quad i,k = 1,\cdots,n \quad \text{for} \quad m = 0$$

(Note the modification in the range for m=0). We thus see that for small separations the n(n-2) conditions on the parameters reduce to the set of linear conditions (14). In particular we see that the conditions are independent for each m. In other words if we array the $a_k^{(m)}$ as in Fig. 1a the conditions apply to each column separately. Indeed since the matrices

ρ_{ik} in (14) are manifestly non-singular, we may solve (14) explicitly and write

$$\delta a_k^{(m)} = (\rho^{-1})_{k\lambda}\, \varepsilon_\lambda^{(m)} \quad , \tag{15}$$

where λ takes only the two end-values $\lambda = n-|m|, n-|m|-1$ for the generic case $1 \leq m \leq n-1$ and the values $\lambda = n-1$, n and $\lambda = 1$ for the special cases m=0 and m=n, respectively. Thus in each column the vector $\delta a_k^{(m)}$ is completely determined by the 2-vectors $\varepsilon_\lambda^{(m)}$ in the 2-spaces at the foot of the column (Fig. 1b). Since there are just 4n parameters $\varepsilon_\lambda^{(m)}$ in (15) we see that, even for small variations we obtain the full set of independent parameters allowed by the index theorem. Using (15) and (9) we also have

$$H = \prod_r (\gamma - \omega_r) \qquad \text{where} \qquad \omega_r = \eta_r - \sum_{m=1}^{n} \sum_{k=1} \frac{(\eta_r)^{k-1}}{\sigma_r} (\rho^{-1})_{k\lambda}\, \varepsilon_\lambda^{(m)} \zeta^m \tag{16}$$

for small variations.

One can actually go a little further with (15) by noting that ρ_{ik} in (14) has no elements connecting even and odd values of i,k. It then follows that (15) can be decomposed into

$$\delta a_{2i}^{(m)} = (\rho^{-1})_{2i,\tau_+}\, \varepsilon_+^{(m)} \qquad \text{and} \qquad \delta a_{2i+1}^{(m)} = (\rho^{-1})_{2i+1,\tau_-}\, \varepsilon_-^{(m)} \tag{17}$$

where $\tau_\pm$ are even and odd integers taking only a single end-value each. In other words the n(n-2) conditions (14) not only apply separately to each column m, but also apply separately to even and odd components within each column (Fig. 2). Thus the effect of the conditions is to fix the directions of the vectors $\delta a_{2i}^{(m)}$ and $\delta a_{2i+1}^{(m)}$, leaving only their magnitudes as free parameters.

A simple group theoretical interpretation of these results may be obtained by considering the group $C_\infty \times P$ of rotations around the z-axis and parity transformations. In fact one sees at once that the $\varepsilon_\pm^{(m)}$ and $\delta a_{k\pm}^{(m)}$ belong to the irreducible representations (m,$\pm$) of this group for m=n,n-1. Note that the $\delta a_{k\pm}^{(m)}$ belong to the same irreducible representation (m,$\pm$) for all k, because the axisymmetric solutions, and hence the ρ_{ik}-matrices, are $C_\infty \times P$-invariant. One sees therefore that the parametrization of H according to the 4n parameters corresponds to a decomposition of H according to $C_\infty \times P$.

One might then ask how the other symmetries of the superimposed case, such as reflexions in the coordinate planes, affect the solutions (16). Clearly the other symmetries cannot decompose the $\varepsilon_\pm^{(m)}$ further, since the latter are one-dimensional, but what they do is to relate the $\varepsilon_\pm^{(m)}$ and $\delta a_{k\pm}^{(m)}$ for different m. Since the ρ_{ik}-matrices are invariant with respect to these symmetries, the relationships will be the same for the $\varepsilon_\pm^{(m)}$ and the $\delta a_{k\pm}^{(m)}$ for all k, and so may be formulated in terms of the $\varepsilon_\pm^{(m)}$ alone. The relationships for the general $a_k^{(m)}$ have been tabulated in ref. (5) and it is trivial to transfer them to the $\varepsilon_\pm^{(m)}$. For example, hermiticity and invariance under reflexions in the (yz) and (xz)-planes imply that

$$\bar{\varepsilon}^{(m)} = (-1)^m \varepsilon^{(-m)} \quad , \quad \varepsilon_\pm^{(m)} = \varepsilon_\pm^{(-m)} \qquad \text{and} \qquad \varepsilon_\pm^{(m)} = (-1)^m \varepsilon_\pm^{(-m)} \quad , \tag{18}$$

respectively. In particular, hermiticity and xz-reflexion invariance together require that the $\varepsilon_\pm^{(m)}$ be real.

An important question is how the 4n parameters appear in the Higgs field $\Phi(x)$ and how they are related to the 3n monopole coordinates (zeros of $\Phi(x)$). Unfortunately, the answer to this question requires a computation of $\Phi(x)$ from the transition matrix, to order x^n at least, and such a computation is very difficult. The best we can obtain at present are some qualitative features. First, we note that if all the $\varepsilon_\pm^{(m)}$ are zero except one, then that one breaks the continuous axial symmetry of the function H down to the discrete axial symmetry C_m given by rotations through $2\pi r/m$, r=1...m. Thus each $\varepsilon_\pm^{(m)}$ separately describes a ring (or rings) of monopoles as discussed in ref. (5) (although the ensemble of $\varepsilon_\pm^{(m)}$ will not describe a superposition of rings). It is then clear that, to first order in $\varepsilon^{(m)}$, $\Phi^2(x)$ takes the form

$$\Phi^2(x) = f(\rho^2 z^2) + \sum_m \varepsilon_\pm^{(m)} \sum_{s=1} g_{ms}(\rho^2 z^2)(\rho e^{i\varphi})^{ms} + z \sum_m \varepsilon_\mp^{(m)} \sum_{s=1} h(\rho^2 z^2)(\rho e^{i\varphi})^{ms} + c.c. \tag{19}$$

where the coefficients $f(\rho^2 z^2), g(\rho^2 z^2), h(\rho^2 z^2)$ are independent of $\varepsilon^{(m)}$. If the rings do not collapse to points to first order in $\varepsilon^{(m)}$ (which we have proved for m=n in ref. (5) and is plausible for all m) then the coefficient of none of $\varepsilon_\pm^{(m)}$ in (19) is zero. For this reason, and also because there is no reason to discriminate against any particular set of $\varepsilon_\pm^{(m)}$ it is plausible that all 4n parameters, $\varepsilon_\pm^{(m)}$ will actually occur in (19). If this is the case, the gauge-invariant quantity $\Phi^2(x)$ will depend not only on the 3n parameters describing *the positions* of the monopoles, but also on the n 'internal' parameters[4]. With regard to the 3n monopole coordinates themselves, a preliminary study of (19) indicates that the separations between the monopoles will not be of uniform order in ε. More precisely, if all the $\varepsilon_{\pm}^{(m)}$ are of order ε, the study indicates that the separations are of all orders $\varepsilon^{r/n}$, for r=1...n.

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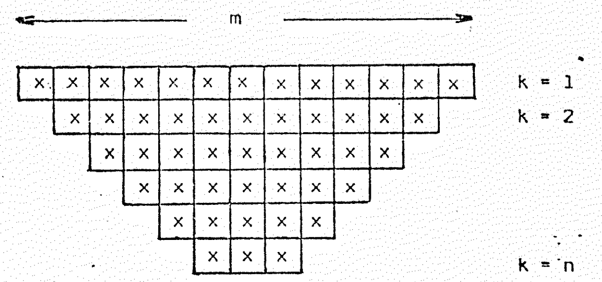

Fig. 1a

Diagramatic representation of the n(n+2) parameters which occur in the definition of H in (1). Note that the range of m for each k is limited to $-(n-k+1) \leqslant m \leqslant (n-k+1)$.

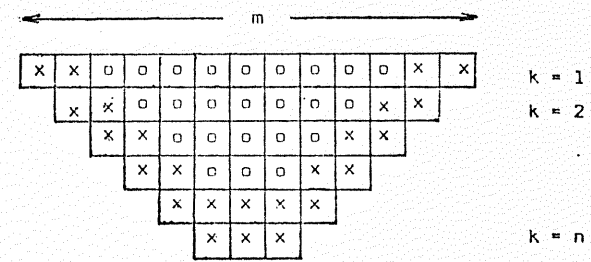

Fig. 1b

Diagrammatic representation of the 4n independent parameters $\varepsilon_{\lambda}^{(m)}$ in (15), where $\delta a_{k}^{(m)} = (\rho^{-1})_{k\lambda}\varepsilon_{\lambda}^{(m)}$ and all the $\varepsilon_{j}^{(m)}$ are zero except the last two in each column.

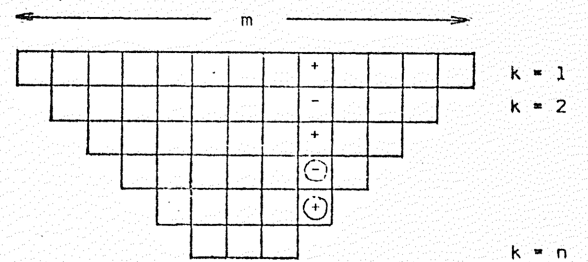

Fig. 2

The manner in which each column in Fig. 1a splits into entries of alternating parity is illustrated. The ρ-matrix in $\delta a_{k}^{(m)} = (\rho^{-1})_{k\lambda}\varepsilon_{\lambda}^{(m)}$ connects only entries of same parity. The non-zero ε-entries are circled and it is evident that there is just one ε for each column (each m) and each parity.