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SCALAR POTENTIALS FOR VECTOR FIELDS AND AN APPLICATION TO QUANTUM ELECTRODYNAMICS

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Abstract: For an arbitrary vector field $\vec{F}: \vec{x} \in \mathbb{R}^3 \mapsto \vec{F}(\vec{x}) \in \mathbb{R}^3$, the representation $\vec{F} = \nabla \Phi + \vec{L}\Psi + \nabla \wedge \vec{L}\chi$ is proved where Φ, Ψ, χ are scalar potentials. Using this decomposition in the Maxwell equations disentangles the longitudinal and transversal degrees of freedom of the electromagnetic field. As a result the electromagnetic field can be quantized restriction free.

1. Introduction

In the theory of electromagnetism it is common to decompose a three-dimensional vector field $\vec{F}: \vec{x} \in \mathbb{R}^3 \mapsto \vec{F}(\vec{x}) \in \mathbb{R}^3$ as

$$\vec{F}(\vec{x}) = \nabla \Phi(\vec{x}) + \nabla \wedge \vec{A}(\vec{x}) \tag{1}$$

thereby introducing a scalar function Φ and a vector Potential $\vec{A}$. However there exist other decompositions in terms of three scalar potentials so called Debye potentials [1,2,3]. We use the decomposition

$$\vec{F}(\vec{x}) = \nabla \Phi(\vec{x}) + \vec{L}\,\Psi(\vec{x}) + \nabla \wedge \vec{L}\,\chi(\vec{x}), \tag{2}$$

where $\vec{L}$ denotes the angular momentum operator, which can be found in [2].

It is one purpose of this note to establish rigorously the validity of decomposition (2). In particular, the uniqueness of the potentials has to be related precisely to the possible gauge transformations. As in [1] where a rigorous proof of a slightly different version of (2) was given, we will employ the Hodge decomposition [4] as a main tool.

The other purpose of this note is to show that the longitudinal and transversal degrees of freedom of the electromagnetic field disentangle if we use the scalar fields of (2) in the Maxwell equations. Without further assumptions, such as Lorentz condition or Coulomb gauge, we arrive at wave equations for the transversal potentials. Therefore the transversal potentials can be quantized canonically without the notorious restrictions which are a main cause of the difficulties in quantum electrodynamics [5]. On the other hand, up to a time derivation, the gradient potentials of the electric field and the exterior current are identical. This shows, that the longitudinal part of the Maxwell field belongs to the sources. - With regard to the separation between transversal and longitudinal, it is interesting to notice that from a group theoretical point of view, where photons are defined as irreducibel unitary rest mass zero representations of the Poincaré group, it is the introduction of the conventional vector potential which causes difficulties [6].

2. The representation theorem

We use Hodge's decomposition theorem for exterior differential forms of degree p on a closed orientable n-dimensional Riemannian manifold Ω, in the form given by [4]. It states that every C^2 p-form α can be decomposed uniquely into a sum of three forms

(3) $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

α_1 being exact, α_2 coexact, and α_3 harmonic; i.e. there exists a (p-1)-form β_1 with

(3.a) $\alpha_1 = d\beta_1$

and a (p+1)-form γ_2 with

(3.b) $\alpha_2 = \delta\gamma_2 := (d\gamma_2^*)^*$

while, for $\Delta := (-1)^{np}\delta d + (-1)^{np+n} d\delta$,

(3.c) $\Delta\alpha_3 = 0$.

If the manifold Ω is 2-dimensional and α a 1-form, the forms β_1 and γ_2^* are scalars. If in addition Ω is a sphere the first Betti number is zero, and α_3 vanishes. Hence we have the special situation that the decomposition of the 2-component vector α is given in terms of two scalar fields β_1 and γ_2^*. This implies the following lemma (see [1]):

Lemma 1: Let Ω be a 2-dimensional sphere and $\vec{F}_t$ a C^2-vector field on Ω. Let ∇_t denote the gradient on Ω, and $\hat{r}$ the unit vector at $\vec{r} \in \Omega \subset \mathbb{R}^3$ perpendicular to Ω. Then there exist functions S and T on Ω

such that

(4) $$\vec{F}_t = \nabla_t S + \hat{r} \wedge \nabla_t T.$$

$\nabla_t S$ gives the exact 1-form $d\beta_1$, and $\hat{r} \wedge \nabla_t T$ the coexact 1-form $\delta\gamma_2 = \delta(T d\Omega)$. ($d\Omega$ denotes the obvious 2-dimensional differential.)

To apply this lemma for a 3-dimensional vector field $\vec{F}: \vec{x} \in \mathbb{R}^3 \mapsto F(x) \in \mathbb{R}^3$, we use radial and tangential coordinates:

(5) $$\vec{F}(\vec{x}) =: F_1(\vec{x})\,\hat{x} + \vec{F}_t(\vec{x}).$$

So excluding the origin we relate $\vec{x} \in \mathbb{R}^3$ one-to-one to $(x, \hat{x}) \in \mathbb{R} \times \Omega$, $x := |\vec{x}|$, with Ω the unit sphere, and $\hat{x} := \vec{x}/x$. F_1 is a scalar function on the domain of $\vec{F}$, and $\vec{F}_t(\vec{x})$ a tangent vector to Ω at $\hat{x}$. In these coordinates the gradient is given as

(6) $$\nabla_{\vec{x}} =: \frac{\partial}{\partial x}\hat{x} + (\nabla_{\vec{x}})_t .$$

<u>Theorem 2</u>: Given a region $\Omega \subseteq \mathbb{R}^3 \setminus \{0\}$, with regular boundary $\partial\Omega =: \bar{\Omega} \setminus \Omega$, and a C^3-vector field $\vec{F}: \vec{x} \in \Omega \mapsto \vec{F}(\vec{x}) \in \mathbb{R}^3$. Then there exist three scalar functions Φ_F, Ψ_F, χ_F on Ω such that

$$\vec{F} = \nabla\Phi_F + \vec{L}\,\Psi_F + \nabla \wedge \vec{L}\,\chi_F .$$

Requiring Φ_F to vanish on the boundary, and Ψ_F, χ_F not to be spherically symmetric, these functions are unique.

<u>Proof</u>:
We may add always a spherically symmetric function to Ψ_F, χ_F without changing (2). Let us assume Φ_F to vanish on $\partial\Omega$, and Ψ_F, χ_F not to be spherically symmetric. We use the Poisson formula [7]

(7) $$\vec{F}(\vec{x}) = \nabla_{\vec{x}}^2 \int_\Omega K(\vec{x},\vec{y})\,\vec{F}(\vec{y})\,d^3\vec{y}$$

where the kernel K is a fundamental solution of the Laplace equation, and abbreviate $\vec{G}(\vec{x}) := \int_\Omega K(\vec{x},\vec{y})\,\vec{F}(\vec{y})\,d^3\vec{y}$. Employing the decompositions (5) and (4), and some vector identities, we have

$$\begin{aligned}
\vec{F}(\vec{x}) &= \nabla^2 \vec{G}(\vec{x}) \\
&= \nabla(\nabla \cdot \vec{G}(\vec{x})) - \nabla \wedge (\nabla \wedge \vec{G}(\vec{x})) \\
&= \nabla\Big(\nabla \cdot \big\{G_1(\vec{x})\,\hat{x} + \nabla_t S(\vec{x}) + \hat{x} \wedge \nabla_t T(\vec{x})\big\}\Big) \\
&\quad - \nabla \wedge \Big(\nabla \wedge \big\{G_1(\vec{x})\,\hat{x} + \nabla_t S(\vec{x}) + \hat{x} \wedge \nabla_t T(\vec{x})\big\}\Big) \\
&= \nabla\big(\nabla \cdot (G_1(\vec{x})\,\hat{x}) + \nabla_t^2 S(\vec{x})\big) \\
&\quad - \nabla \wedge (\nabla \wedge \hat{x}\, G_1(\vec{x})) + \nabla \wedge \Big(\nabla \wedge \hat{x}\,\frac{\partial}{\partial x} S(\vec{x})\Big) \\
&\quad - \nabla \wedge \big(\nabla \wedge \{\hat{x} \wedge \nabla T(\vec{x})\}\big) \\
&= \nabla\big(\nabla \cdot (G_1(\vec{x})\,\hat{x}) + \nabla_t^2 S(\vec{x})\big) \\
&\quad + \nabla \wedge \vec{L}\,\frac{i}{x}\Big(G_1(\vec{x}) - \frac{\partial}{\partial x} S(\vec{x})\Big) + \vec{L}\,\nabla^2\Big(i\,\frac{T(\vec{x})}{x}\Big).
\end{aligned}$$

This proves the existence of the Debye potentials.

Suppose now

$$\vec{F} = \nabla\Phi + \vec{L}\,\Psi + \nabla \wedge \vec{L}\,\chi = \nabla\Phi' + \vec{L}\,\Psi' + \nabla \wedge \vec{L}\,\chi'.$$

Then straightforward vector calculations imply

$$\nabla^2\Phi = \nabla \cdot \vec{F}, \quad \vec{L}^2\Psi = \vec{L} \cdot \vec{F}, \quad \vec{L}^2\nabla^2\chi = -(\vec{L} \wedge \nabla) \cdot \vec{F},$$

hence $\nabla^2(\Phi - \Phi') = 0, \quad \vec{L}^2(\Psi - \Psi') = 0,$

$\vec{L}^2 \nabla^2(\chi - \chi') = -x^2 \nabla_t^4(\chi - \chi') = 0,$ and therefore

$\Phi = \Phi', \quad \Psi = \Psi', \quad \chi = \chi'.$ qed.

Remark:

The "gauge freedom" associated with decomposition (2) is the freedom to add spherically symmetric scalar functions to the Debye potentials Ψ, χ. The requirement of spherical symmetry distinguishes this gauge from the usual vector potential gauge where the gradient of any scalar function can be added to the vector potential.

3. Application to Maxwell equations and Quantum Electrodynamics

The Maxwell equations are given by

(8.a) $\nabla \wedge \vec{E} = -\dot{\vec{B}}$

(8.b) $\nabla \wedge \vec{H} = \vec{J} + \dot{\vec{D}}$

with the usual assumptions

(9.a) $\nabla \cdot \vec{D} = \rho$

(9.b) $\nabla \cdot \vec{B} = 0$

(10.a) $\varepsilon_0 \vec{E} = \vec{D} - \vec{P}$

(10.b) $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

Combining (10) and (8.b) yields

(8.b') $\nabla \wedge \vec{B} = \mu_0 (\vec{J} + \nabla \wedge \vec{M} + \dot{\vec{P}}) + \varepsilon_0 \mu_0 \dot{\vec{E}}.$

Now we decompose all vector fields according to (2),

(11) $\vec{F} = \nabla \Phi_F + \vec{L} \Psi_F + \nabla \wedge \vec{L} \chi_F, \quad F = E, B, D, H, P, M,$

under the tacit assumption that the gradient potentials vanish on the boundary of the region Ω under consideration which is supposed to exclude the origin, and to be sufficiently regular. Thus (8.a) is, up to a gauge, equivalent to

(12.a.α) $\dot{\Phi}_B = 0,$

(12.a.β) $\Psi_E = -\dot{\chi}_B,$

(12.a.γ) $\nabla^2 \chi_E = \dot{\Psi}_B.$

and (8.b') equivalent to

(12.b.α) $$\dot{\Phi}_E = -\frac{1}{\varepsilon_0}\left(\Phi_J + \dot{\Phi}_P\right),$$

(12.b.β) $$\Psi_B = \mu_0\left(\chi_J + \dot{\chi}_P + \Psi_M + \varepsilon_0\,\dot{\chi}_E\right)$$

(12.b.γ) $$\nabla^2\chi_B = -\mu_0\left(\Psi_J + \dot{\Psi}_P - \nabla^2\chi_M + \varepsilon_0\,\dot{\Psi}_E\right).$$

the assumptions (9) are equivalent to

(13.a) $$\nabla^2\left(\varepsilon_0\,\Phi_E + \Phi_P\right) = \rho,$$

(13.b) $$\Phi_B = 0.$$

The remarkable result of this calculation is the separation of the gradient potentials Φ and the transversal potentials Ψ, χ. Without any further assumptions or restrictions, we arrive at a wave equation by eliminating Ψ_E and Ψ_B.

(14.a) $$\Box\,\chi_E = \mu_0\left(\dot{\chi}_J + \ddot{\chi}_P + \dot{\Psi}_M\right), \qquad \Box := \nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}$$

(14.b) $$\Box\,\chi_B = -\mu_0\left(\Psi_J + \dot{\Psi}_P - \nabla^2\chi_M\right).$$

(13.a) and (12.b.α) can be combined to give the continuity equation

(15) $$\nabla^2\Phi_J + \dot{\rho} = 0$$

which is nothing else than

(15') $$\nabla\cdot\vec{J} + \dot{\rho} = 0.$$

Equation (15) underlines that charge transport is exclusively related to the gradient potential of the current.

The formulation (12) of the Maxwell equations exhibits a structure which is crucial for the quantization procedure. By (12.b.α), the electric gradient field Φ_E coincides with a matter field. Hence its possible quantization is subject to a quantum theory of matter. In the free case Φ_E has to vanish, according to (13.a) and the boundary assumptions. But even with different boundary conditions, since Φ_E, and Φ_B are submitted to a Laplace equation, a canonical quantization would make no sense. We definitely have to exempt Φ_E and Φ_B from the quantization of the electromagnetic field.

In the wave equation (14) the inhomogeneities refer only to matter (current $\vec{J}$, polarization $\vec{P}$, magnetizetion $\vec{M}$) while the homogeneous solutions precisely exhaust the free electromagnetic field. Therefore the quantization of the free electromagnetic field is achieved by the canonical quantization of the solutions of the homogeneous wave equations

(16) $$\Box\,\chi_\alpha = 0, \qquad \alpha = E, B,$$

and presents no problem at all. General quantized solutions of (14) would involve the quantization of the inhomogeneities as well which has to originate in quantum theory of matter; but that is beyond the scope of this note.

The equations (16) do not fix multiplicative constants in the fields χ_α. For later convenience we redefine

(17) $$i\left(\frac{\vec{L}^2}{\mu_0}\right)^{1/2}\chi_B \longmapsto \chi_B,$$
$$i\left(\varepsilon_0\,\vec{L}^2\right)^{1/2}\chi_E \longmapsto \chi_E.$$

The redefined potentials are real; notice that the imaginarity of Ψ and χ in (11) comes from the definition of the angular momentum operator, $\vec{L}_{\vec{x}} := -i\vec{x}\wedge\nabla_{\vec{x}}$.

We sketch the canonical quantization of (16). $(\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2})\chi_\alpha = 0$ can be derived from the Lagrange density

$$(18)\qquad \mathcal{L}(\dot{\chi}_\alpha, \nabla\chi_\alpha) = \frac{1}{2}\sum_{\alpha=E,B}(\varepsilon_0\mu_0\dot{\chi}_\alpha^2 - \nabla\chi_\alpha\cdot\nabla\chi_\alpha)$$

by the Euler Lagrange equation

$$(19)\qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\chi}_\alpha} + \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\chi_\alpha)} = 0.$$

The canonical momentum with respect to χ_α is

$$(20)\qquad \pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\chi}_\alpha} = \varepsilon_0\mu_0\dot{\chi}_\alpha\ ,$$

and the hamiltonian density

$$(21)\qquad \mathcal{H} = \frac{1}{2}\sum_{\alpha=E,B}\left(\frac{\pi_\alpha^2}{\varepsilon_0\mu_0} + \nabla\chi_\alpha\cdot\nabla\chi_\alpha\right)$$

$$= \frac{1}{2}\sum_{\alpha=E,B}\left(\frac{\pi_\alpha^2}{\varepsilon_0\mu_0} - \chi_\alpha\nabla^2\chi_\alpha\right)$$

A Fourier transformation with respect to the time variable makes (16) assume the form of an eigenvalue equation,

$$(22)\qquad -\nabla^2 u_k = \varepsilon_0\mu_0\omega_k^2 u_k\ .$$

We specify a boundary value problem on Ω, Dirichlet data say, such that the laplacian is self-adjoint. Now let us expand $\chi_\alpha(x,t)$ with respect to an orthonormal basis $\{u_k\}$ of eigenfunctions of $-\nabla^2$:

$$(23)\qquad \chi_\alpha(\vec{x},t) = \sum_k (2\varepsilon_0\mu_0\omega_k)^{-1/2}\{b_{k\alpha}u_k(\vec{x})e^{-i\omega_k t} + b_{k\alpha}^* u_k^*(\vec{x})e^{i\omega_k t}\};$$

the orthogonality may be given with respect to the inner product

$$\langle f, g\rangle := \int_\Omega f^*(\vec{x})\,g(\vec{x})\,d^3\vec{x}\ ,$$

$$(24.a)\qquad \langle u_k, u_{k'}\rangle = \delta_{k,k'}\ .$$

We assume a non-trivial time dependence in (23), $\omega_k > 0$ for all k (see final section). Now insert (23) in (20) and (21). Observe that both u_k, and u_k^* are eigenfunctions to the same eigenvalue in (22) such that by

$$0 = \langle u_k, (-\nabla^2)u_{k'}^*\rangle - \langle -\nabla^2 u_k, u_{k'}^*\rangle$$

$$= \varepsilon_0\mu_0(\omega_{k'}^2 - \omega_k^2)\langle u_k, u_{k'}^*\rangle\ ,$$

we have

$$(24.b)\qquad \langle u_k, u_{k'}^*\rangle = 0\ ,\qquad \omega_k \neq \omega_{k'}\ .$$

Hence the hamiltonian density

$$(25)\qquad \mathcal{H}_p = \frac{1}{2}\omega_p(b_p^* b_p + b_p b_p^*),\quad p := (k,\alpha),\quad \alpha = E, B,$$

follows straightforward. Canonical quantization consists in postulating the commutation relations

$$(26)\qquad [b_p, b_{p'}^*] = \hbar\,\delta_{p,p'}\ ,\qquad [b_p, b_{p'}] = 0.$$

Therefore the hamiltonian density is

$$(27)\qquad \mathcal{H}_p = \omega_p b_p^* b_p + \frac{1}{2}\hbar\omega_p\ .$$

We can rearrange the decomposition of the electromagnetic field into the poloidal potentials χ_E, χ_B, to realize the helicity:

(28.a) $$\chi_\kappa := \frac{1}{\sqrt{2}}\left(\chi_B + i^{\kappa}\chi_E\right), \qquad \kappa = +1, -1.$$

This transformation leaves the commutation relations (26) unaffected, i.e. they are valid for

(28.b) $$q = (k, \kappa), \qquad \kappa = +1, -1,$$

(28.c) $$b_q := \frac{1}{\sqrt{2}}\left(b_{k,B} + i^{\kappa} b_{k,E}\right),$$

(29) $$[\, b_q, b^*_{q'} \,] = \hbar\, \delta_{q,q'}, \qquad [\, b_q, b_{q'} \,] = 0.$$

The hamiltonian density with respect to helicity is

(30) $$\mathcal{H}_q = \omega_q\, b^*_q\, b_q + \frac{1}{2}\hbar\omega_q .$$

4. Discussion

1. The energy density of an electromagnetic field is defined as [8]

(31.a) $$\mathcal{H} = \frac{1}{2}\left(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D}\right)$$

which, for the free field, is

(31.b) $$\mathcal{H} = \frac{1}{2}\left(\frac{1}{\mu_0}\vec{B}^2 + \varepsilon_0 \vec{E}^2\right).$$

We insert the decomposition (2) of the free $(\vec{E},\vec{B})$-field where the gradient potentials vanish. Now we distinguish the pseudo-scalar toroidal Debye potentials $\tilde{\psi}_E$, $\tilde{\psi}_B$ and the scalar poloidal Debye potentials $\tilde{\chi}_E$, $\tilde{\chi}_B$ from the redefined potentials,

$$(\psi_B, \chi_B) := i\left(\frac{\vec{L}^2}{\mu_0}\right)^{1/2}(\tilde{\psi}_B, \tilde{\chi}_B)$$

$$(\psi_E, \chi_E) := i\left(\varepsilon_0 \vec{L}^2\right)^{1/2}(\tilde{\psi}_E, \tilde{\chi}_E)$$

(32) $$\vec{F} = \vec{L}\,\tilde{\psi}_F + \nabla\wedge\vec{L}\,\tilde{\chi}_F, \qquad F = E, B.$$

The rewritten hamiltonian density (31.b) is

$$\mathcal{H} = \frac{1}{2\mu_0}\left(\vec{L}\,\tilde{\psi}_B + \nabla\wedge\vec{L}\,\tilde{\chi}_B\right)^2 + \frac{\varepsilon_0}{2}\left(\vec{L}\,\tilde{\psi}_E + \nabla\wedge\vec{L}\,\tilde{\chi}_E\right)^2.$$

By vector analysis calculations, where the involved surface integrals vanish due to assumed Dirichlet boundary conditions, we get

$$\begin{aligned}\mathcal{H} &= \frac{1}{2\mu_0}\left(-\varepsilon_0^2\mu_0^2\, \dot{\tilde{\chi}}_E\, \vec{L}^2\, \dot{\tilde{\chi}}_E + \tilde{\chi}_B\, \nabla^2 \vec{L}^2\, \tilde{\chi}_B\right) \\ &\quad + \frac{1}{2}\varepsilon_0\left(-\dot{\tilde{\chi}}_B\, \vec{L}^2\, \dot{\tilde{\chi}}_B + \tilde{\chi}_E\, \nabla^2 \vec{L}^2\, \tilde{\chi}_E\right) \\ &= \frac{1}{2}\left\{\varepsilon_0\mu_0\, \dot{\chi}_E^2 - \chi_B \nabla^2 \chi_B + \varepsilon_0\mu_0\, \dot{\chi}_B^2 - \chi_E \nabla^2 \chi_E\right\}.\end{aligned}$$

Respecting (20) this expression coincides with (21).

2. The possibility of a non-trivial field for the eigenvalue zero in the Helmholtz equation (22), which happens to emerge in the infinite volume limit, or for Neumann data, etc., presents a conceptual problem with respect to the photon. A rest mass zero particle with kinetic energy zero cannot exist. However the photon concept is restored by the gauge freedom for the potentials χ_E, χ_B. The spherically symmetric eigenfunction of the eigenvalue zero can always be subtracted. At the same time this prevents the photon to have zero helicity. - Because of the "zero point energy", the quantized hamiltonian of the electromagnetic field does not show up the zero energy difficulty.

The gauge freedom introduces the non-uniqueness of the ground state of the electromagnetic field.

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