DIAS-STP-85-20

THE PROBLEM OF "GLOBAL COLOR" IN GAUGE THEORIES

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ABSTRACT

The problem of "global color" (which arose recently in monopole theory) is generalized to arbitrary gauge theories: a subgroup K of the "unbroken" gauge group G is implementable iff the gauge bundle reduces to the centralizer of K in G. Equivalent implementations correspond to equivalent reductions. Such an action is an internal symmetry for a given configuration iff the Yang-Mills field reduces also. The case of monopoles is worked out in detail.

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June 1985

CPT-85/P.1776.

1. INTRODUCTION

One of the most exciting problems which arose recently in monopole theory is that of global color [1-7]. We formulate it in two steps [7]: first, we would like to define the action of a fixed element g of the (unbroken) gauge group. Under usual conditions this presents no problem. In topologically non-trivial situations, however, this may not be posssible. This is the problem of implementability. Next, if we are able to define such an action, when do we get a symmetry (in the sense of Schwarz [8] and Forgács and Manton [9]) for a given field configuration? The importance of these notions is seen, for example, from their role played in deriving conserved charges in gauge theories [7,35]. In this paper we give the mathematical solution to these problems. Our theory (formulated in fibre-bundle terms [8,10-12]) is valid for any classical gauge theory. Notice that the problem studied here is a special case of dimensional reduction [9,15].

Our starting point is Prop. 2.2, which states that a "rigid" internal action of a subgroup K of G on P exists if and only if P reduces to an $H = Z_G(K)$ - bundle Q. Furthermore, there is a (1-1) correspondence between such equivalent actions and isomorphic reductions (Props. 2.4 - 2.5).

An action of K on the principal bundle P induces an action of K also on the YM field. Similarly, we can study the action of K on matter fields - sections of bundles associated to P. The condition for such an action to exist is expressed again in terms of bundle-reduction (Theorem 3.1).

When is an action a symmetry for a given field configuration? Prop.4.3 tells us, that the action of K on (P,G) defined by (Q,H) is an internal symmetry for a Yang-Mills connection A if and only if A reduces to a connection on Q. This happens if and only if H contains the holonomy group of A [16]. The implementation of an internal symmetry-subgroup is nescessarily unique. There is an analogous statement (Prop. 4.4) for matter fields.

These theorems provide us with a complete solution of the color problem - when we are able to construct a the corresponding reductions. A first illustration is given by the non-Abelian Bohm-Aharonov experiment of Wu and Yang [33-35], where $G = SU(2)$ admits two inequivalent implementations.

The principal application of our theory is to non-Abelian monopoles [1-7]. Their basic properties [17-23] are geometrically reformulated in Section 5. The reduction of monopole bundles is worked out in Section 6. (As a by-product, we obtain also the topological theory of the "fate" of Grand Unified monopoles under successive symmetry breakings [24-28]).

The results are summarized as follows: denote by G the residual symmetry group of a monopole having $[P] \in \pi_1(G)$ as fundamental topological invariant. A subgroup K of G is **implementable** iff [P] belongs to the image of $i_*: \pi_1(Z_G(K)) \to \pi_1(G)$ induced by the inclusion $i: Z_G(K) \to G$. Furthermore, the **inequivalent implementations** are labelled by the elements of $\pi_2(G/Z_G(K))$. In particular, the implementation of the full G is unique (when it does exist).

These results are conveniently expressed in terms of the "non-Abelian charge" H of Goddard, Nuyts and Olive [17]: let us decompose H as $H = z(H) + H'$, where $z(H) \in Z(\mathfrak{g})$ and $H' \in [\mathfrak{g},\mathfrak{g}]$. We prove that G is implementable iff either (i) $[\exp 4\pi t H] \in \pi_1(G)_{free}$, and z(H), the projection of the non-Abelian charge onto the centre, is quantized: $\exp 4\pi z(H) = 1$. Equivalently, iff (ii) $\exp 4\pi H' t$, $0 \le t \le 1$, is a contractible loop. G is a symmetry for a monopole given by H iff $H \in Z(\mathfrak{g})$.

The general results are illustrated on SO(3) monopoles [18,17].

2. INTERNAL ACTIONS ON PRINCIPAL BUNDLES

Let P be a right principal G-bundle over a connected manifold M. A subgroup K of G acts internally on P, if we are given a left action $p \longrightarrow k\cdot p$ of K on P, which preserves each fibre and commutes with the (right) action of G, $k\cdot(pg) = (k\cdot p)g$, $\forall\, k \in K$, $g \in G$, $p \in P$, cf. [1-7]. If so, define the map $\tau_p : K \longrightarrow G$ by $k\cdot p = p(\tau_p(k))$. τ_p is well-defined, since $k\cdot p$ belongs to the same fibre as p, and G acts on each fibre transitively and freely. $k \longrightarrow \tau_p(k)$ is a homomorphism of K into G, which satisfies $\tau_{pg} = \mathrm{Ad}g^{-1}\circ\tau_p$, $g \in G$. In what follows we consider only the case when K acts on P freely; i.e. the homomorphism $\tau_p : K \longrightarrow G$ is injective for each p. This can always be assumed without loss of generality for symmetries (see Section 4).

Choosing a local section $s_\alpha : V_\alpha \longrightarrow P$, τ is given by $\tau^\alpha : V_\alpha \longrightarrow \mathrm{Hom}(K,G)$, where $\tau^\alpha{}_x = \tau_{s_\alpha(x)}$. If $h_{\alpha\beta} : V_\alpha \cap V_\beta \longrightarrow G$ denotes the transition function of P, then $\tau^\alpha{}_x = \mathrm{Ad}h_{\alpha\beta}\, \tau^\beta{}_x$, $x \in V_\alpha \cap V_\beta$.

An internal action of K on P is called rigid, if there exists a local trivialization $\{V_\alpha, s_\alpha\}$ of P such that each τ^α is constant [1-7]. If so, there is no loss of generality in assuming that $\tau^\alpha{}_x(k) = k$ for each x. In such a gauge

$$k\cdot s_\alpha(x) = s_\alpha(x)k, \ \forall\, x \in V_\alpha,\ k \in K. \tag{2.1}$$

PROPOSITION 2.1

An internal action of K on P is rigid if and only if the image of the associated map τ: P $\longrightarrow$ Hom(K,G) is the orbit of the inclusion map $i : K \hookrightarrow G$ under the adjoint action of G on Hom(K,G).

Proof: Suppose the action of K on P is rigid in a gauge $\{V_\alpha, s_\alpha\}$. If $p \in P$ is such that $\pi(p) \in V_\alpha$, where π is the projection $\pi : P \dashrightarrow M$, then $p = s_\alpha(\pi(p))g$ for some $g \in G$. By (2.1)

$$p\tau_p(k) = k\cdot p = k\cdot(s_\alpha(\pi(p))g) = (k\cdot s_\alpha(\pi(p)))g =$$
$$= s_\alpha(\pi(p)k)g = p\,(g^{-1}kg).$$

Hence $\tau_p(\kappa) = \mathrm{Ad}\, g^{-1}k$ and so $\tau_p = \mathrm{Ad}\, g^{-1}i$, and thus the image is the orbit of i. Conversely, if τ has a single orbit as its image, we can always choose local gauges s_α so that τ^α is constant, equal to a base point, which in this case is the inclusion map $i : K \hookrightarrow G$. In this gauge the action of K is rigid.

Requiring rigidity is seen easily to be the same as to require that, for each p, τ_p is the restriction to K of an automorphism of G [1,2,6,7].

Let us now consider a rigid internal action of $K \subset G$ and let H denote the stabilizer of $i : K \hookrightarrow G$ under the adjoint action of G,

$$H = Z_G(K) = \{g \in G \mid \mathrm{Ad}gk = k,\ \forall\, k \in K\}. \tag{2.2}$$

The orbit of i is identified with G/H, and τ can be viewed as a section of the associated bundle with fibre G/H. Any such section defines a reduction of P to an H-bundle. The reductions to H bundles are in known to be in (1-1) correspondence with sections of the associated bundle $P\times_G(G/H) \simeq P/H$, and so with rigid actions of K on P. Hence we have proved

PROPOSITION 2.2

A rigid internal action of a subgroup K of G on P exists if and only if P reduces to H = $Z_G(K)$.

It is easy to see directly that the existence of a rigid action forces the bundle to reduce by using the special gauge (2.1), for, if $s_\beta = s_\alpha h_{\alpha\beta}$, then

$$s_\alpha(kh_{\alpha\beta}) = k\cdot s_\alpha h_{\alpha\beta} = k\cdot s_\beta = s_\beta k = s_\alpha h_{\alpha\beta}k$$

showing that $h_{\alpha\beta}$ commutes with K, and hence (P,G) reduces to an H-bundle Q.

COROLLARY 2.3

G itself acts on P internally and rigidly if and only if (P,G) reduces to Z(G), the centre of G. In this case the transition functions take their values in Z(G) in a suitable gauge.

Now we turn to the question of uniqueness of the action of K. To be able to discuss several actions at once we use the following notation: we denote by $\mu : K\times P \dashrightarrow P$ the map

$\mu(k,p) = k\cdot p$. Two actions given by μ_1 and μ_2 should be regrded as equivalent if they differ only by a gauge transformation, i.e. if there exists a bundle automorphism $\sigma: P \dashrightarrow P$ preserving fibres, commuting with the G-action, and such that $\mu_2(k,\sigma(p)) = \sigma(\mu_1(k,p))$ $\forall$ $k \in K$, $p \in P$.

Equivalence preserves rigidity, since if $s_\alpha: V_\alpha \dashrightarrow P$ is a gauge in which μ_1 is constant then $\sigma \circ s_\alpha$ is a gauge in which μ_2 is constant.

An equivalence σ determines a map $\gamma: P \dashrightarrow G$ by

(2.3) $\sigma(p) = p\gamma(p)$,

which satisfies $\gamma(pg) = Adg^{-1}\gamma(p)$, so γ is a section of the bundle associated to P with G acting on itself by internal automorphisms. If we have two actions μ_1, μ_2 of K with corresponding maps τ^1, τ^2: Hom(K,G) then

$$\sigma(p)\tau^2_{\sigma(p)}(k) = \mu_2(k,\sigma(p)) = \sigma(\mu_1(k,p)) = \sigma(p)\tau^1_p(k)$$

so

(2.4) $\tau^1_p(k) = \tau^2_{\sigma(p)}(k) = Ad\gamma(p)^{-1}\tau^2_p(k)$.

Conversely, given a section γ of $P\times_G G$ satisfying (2.4) then (2.3) defines an equivalence of the two actions.

In the rigid case we may assume i: $K \hookrightarrow G$ is a subgroup and the actions of K correspond with reductions Q P to $H = Z_G(K)$-bundles. Here τ has values in the G-orbit of i and $Q = \{ p \in P | \tau_p = i \}$.

If we have two rigid actions μ_1 and μ_2 with corresponding reductions Q_1 and Q_2 then an equivalence σ of μ_1 and μ_2 implies $\tau^1{}_p = \tau^2{}_{\sigma(p)}$ by (2.4), so $Q_2 = \sigma(Q_1)$. This suggest a notion of equivalence of reductions: two reductions Q_1 and Q_2 of a principal G-bundle P are equivalent if there is an automorphism σ of P preserving fibres with $Q_2 = \sigma(Q_1)$. Then we have

PROPOSITION 2.4

Two rigid actions μ_1, μ_2 are equivalent if and only if their corresponding reductions Q_1 and Q_2 are equivalent.

Obviously, if two reductions Q_1, Q_2 of P are equivalent as reductions then they are isomorphic as H-bundles. This is in fact the only condition to be satisfied as the next result shows:

PROPOSITION 2.5

There is a (1-1) correspondence between isomorphisms of Q_1 and Q_2 as H-bundles and equivalences of Q_1 and Q_2 as reductions of P.

Proof: It remains only to show how to extend an isomorphism $\sigma_0: Q_1 \dashrightarrow Q_2$ to an isomorphism of P. For any $p \in P$ we find $g \in G$ (not unique) with $pg \in Q_1$ (since Q_1 is a reduction of P). Any other choice of g has the form gh with $h \in H$, but

$$\sigma_0(pg)g^{-1} = \sigma_0(pgh)(gh)^{-1},$$

since σ_0 is an H-map, so $\sigma(p) = \sigma_0(pg)g^{-1}$ gives a well-defined map $\sigma: P \dashrightarrow P$ with $\sigma(Q_1) = \sigma_0(Q_1) = Q_2$. It is easy to check σ is an automorphism of P.

3. INTERNAL ACTIONS ON ASSOCIATED BUNDLES

A matter field Φ is specified by giving a unitary representation $U:G \longrightarrow U(E)$ (the set of unitary transformation of a linear space E), and by selecting, in each V_α, a local representative $\Phi^\alpha : V_\alpha \longrightarrow E$ such that

$$\Phi^\alpha(x) = U(h_{\alpha\beta}(x))\Phi^\beta(x), \quad x \in V_\alpha \cap V_\beta.$$

Suppose a subgroup K of G acts on Φ pointwise and linearly. This means that $k \in K$ sends Φ to an object we denote by $(k\cdot\Phi)$, expressed locally as

$$(3.1) \qquad (k\cdot\Phi)^\alpha(x) = U^\alpha{}_x(k)\Phi^\alpha(x)$$

where each $U^\alpha{}_x$ is a representation of K on E. For (3.1) to be well-defined we need the consistency condition

$$(3.2) \qquad U^\alpha{}_x(k) = U[h_{\alpha\beta}(x)]U^\beta{}_x(k)U[h_{\alpha\beta}{}^{-1}(x)].$$

Φ is a section of the associated bundle $\mathcal{A} = P\times_G E$. (3.2) requires therefore that the $U^\alpha{}_x(k)$'s piece together to give sections of the bundle associated to P with fibre U(E), where G acts by conjugation by U(g). If the action of K on the fibre at x is denoted by $U_x(k)$, the $U^\alpha{}_x(k)$'s are local representatives of this action. U(k) is a section of $U(\mathcal{A})$.

If the representation U is not faithful, denote by N its kernel. N is a normal subgroup of G, and $G^* = G/N$ is a group to which U descends to give a faithful representation U^*. $P\times_G E$ is naturally isomorphic to $P^*\times_{G^*}E$, where $P^* = P/N = P\times_G(G/N)$ is the principal G^*-bundle associated to the homomorphism $G \dashrightarrow G^*$. In this way we may reduce to the case where U^* is faithful but note that now K need not be a subgroup of G^*. This defect can be avoided if we assume that the action of K is induced locally by gauge transformations, i.e., if we assume that, for each k, there exist functions $k_\alpha : V_\alpha \dashrightarrow G$ such that $U^\alpha{}_x(k) = U(k_\alpha(x))$. These k_α must satisfy $U[k_\alpha(x)] = U[h_{\alpha\beta}(x)k_\beta(x)h_{\alpha\beta}{}^{-1}(x)]$, so

$$(3.3) \qquad k_\alpha(x)^{-1}h_{\alpha\beta}(x)k_\beta(x)h_{\alpha\beta}{}^{-1}(x) \in N.$$

If we denote by g^* the projection of $g \in G$ into G^*, then

$$(3.4) \qquad k^*{}_\alpha(x) = h^*{}_{\alpha\beta}(x)k^*{}_\beta(x)h^*{}_{\alpha\beta}(x)^{-1}.$$

It follows that k^* defines a section of $P^*\times_{G^*}[G^*]$ with G^* acting by the adjoint representation on itself. Since each $U^\alpha{}_x$ is a homomorphism, then, although $k \longrightarrow k_\alpha(x)$ need not be a homomorphism, $k \longrightarrow k^*{}_\alpha(x)$ is a homomorphism. Thus, by (3.4), we obtain a section $\tau^*{}_x(k) = k^*{}_\alpha(x)$ of $P^*\times_{G^*}[\mathrm{Hom}(K,G^*)]$, and hence an action of K on P^* which commutes with the G^*-action. If further there are local gauges where the k_α are K-valued, then $K\cap N$ acts trivially so we get an action of $K^* = K/(K\cap N)$ which is a subgroup of G^*. Hence we get the situation studied in Section 2, with K^* acting on P^*.

Finally, if we restrict $k_\alpha(x)$ to be constant (and equal to k) in some gauge, then K^* acts rigidly on P^*. This can happen, as we have seen, if and only if P^* reduces to $Z_{G^*}(K^*)$, the centralizer of K^* in G^*. If

$$(3.5) \qquad H' = \{g \in G \mid U(g)U(k) = U(k)U(g) \ \forall\, k \in K\},$$

then this centralizer is $H^* = H'/N$. Since $P^* = P/N$, $P^*/H^* = P/H'$ and thus P^* reduces to H^* if and only if P reduces to H'. We summarize:

THEOREM 3.1

If K acts pointwise on a generic matter field Φ transforming under a unitary representation U of G on E so that there are local gauges (V_α, s_α) where this action is rigid,

$$(3.6) \qquad (k\cdot\Phi)^\alpha(x) = U(k)\Phi^\alpha(x),$$

then P reduces to the subgroup (3.5). Conversely, any reduction of P to H' induces an action of K on Φ.

This is the case in particular when K acts internally on P. Indeed, H in (2.2) is a subgroup of H'. Alternatively, observe that if the action of K on P is associated to $\tau : P \longrightarrow \mathrm{Hom}(K,G)$, then

$$\tau^*{}_{p^*}(k) = (\tau_{p^*}(k))^*$$

defines an action of K (and thus of K^*) on P^*. Φ can be viewed alternatively as a section of $P^* x_{G^*} E$, or as an equivariant function $P^* \longrightarrow E$,

$$\Phi(p^* g^*) = U(g^{*-1})\Phi(p^*), \; g^* \in G^*, p^* \in P^*.$$

Observe that the action of K on Φ is deduced from that of K^* on P^*,

$$(3.7) \quad (k\cdot\Phi)(p) = U(\tau^*{}_p(k))\Phi(p),$$

(since $\Phi(p) = \Phi(p^*)$), whose local form is

$$(3.8) \quad (k\cdot\Phi)^\alpha(x) = U(\tau^{*\alpha}{}_x(k))\Phi^\alpha(x).$$

The results of this section apply, besides monopoles (Section 7), to classical particles in external Yang-Mills fields [33].

4. INTERNAL SYMMETRIES

Let us now assume that our principal G-bundle P carries a connection form A, and let K be a subgroup of G acting on P internally. Let this action be given by τ. This allows us to define the action of a $k \in K$ on the Yang-Mills connection A, $(k\cdot A) = (k^{-1})^* A$, where * denotes the pullback of a differential form. We shall call K an internal symmetry group for the Yang-Mills field A if this action preserves the connection,

$$(4.1) \quad (k\cdot A) = A.$$

cf. [7-16]. If K is a compact, connected Lie group with Lie algebra $\mathfrak{k}$, any $k \in K$ can be written as $k = \exp \kappa$. (4.1) implies that the vectorfield

$$(4.2) \quad \tilde{\kappa}(p) = \frac{d}{dt}\Big|_{t=0} (\exp -t\kappa)\cdot p$$

is invariant under the right action of G on P, and

$$(4.3) \quad L_{\tilde{\kappa}} A = 0.$$

On the other hand, $\tilde{\kappa}(p) = \hat{\xi}_p$, the fundamental vectorfield at p associated to the infinitesimal right action of G at $p \in P$ for some $\xi \in \mathfrak{g}$. Denote $(\omega_\kappa(p)^\wedge)_p = \tilde{\kappa}(p)$. Alternatively, consider

$$(4.4) \quad \omega_\kappa(p) = \frac{d}{dt}\Big|_{t=0} \tau_p(\exp -t\kappa).$$

For each $p \in P$, the map $\mathfrak{k} \ni \kappa \dashrightarrow \omega_\kappa(p) \in \mathfrak{g}$ satisfies

$$(4.5) \quad \omega_{[\kappa_1,\kappa_2]}(p) = [\omega_{\kappa_1}(p), \omega_{\kappa_2}(p)] \text{ and } \omega_\kappa(p_0) = \kappa.$$

$\omega_\kappa(pg) = Adg^{-1}\omega_\kappa(p)$ for each κ. ω_κ is therefore an adjoint "Higgs - type" field.

To express (4.3) another way, observe that $L_{\tilde{\kappa}}A = dA(\tilde{\kappa},.) + d(A(\tilde{\kappa}))$ by the Cartan lemma. But $dA(\tilde{\kappa},.) = DA(\tilde{\kappa},.) - [A(\tilde{\kappa}),A(.)] = [A(.),A(\tilde{\kappa})]$ because $DA = F$ is *horizontal*. Finally, $A_p(\tilde{\kappa}_p) = A_p(\omega_\kappa(p)^\wedge) = \omega_\kappa(p)$, since $A(\hat{\xi}) = \xi$ for any connection A on P. So $L_{\tilde\kappa} A = D\omega_\kappa$. This yields:

PROPOSITION 4.1

A connected subgroup K of G acting internally on P is an internal symmetry for the connection A if and only if the adjoint "Higgs" field ω_κ associated to its infinitesimal action is covariantly constant for each κ,

$$(4.6) \qquad D\omega_\kappa = 0.$$

cf. [5,7,16]. When expressed in a local gauge $\{V_\alpha, s_\alpha\}$ (4.1) and (4.6) become [7-16]

$$(4.7) \qquad A^\alpha(x) = \mathrm{Ad}\, \tau^\alpha_x(k)\, A^\alpha(x) - d\tau^\alpha_x(k)[\tau^\alpha_x]^{-1}$$

and

$$D\omega^\alpha_\kappa = d\omega^\alpha_\kappa + [A^\alpha, \omega^\alpha_\kappa] = 0$$

respectively, where $A^\alpha = s^*_\alpha A$, and

$$(4.8) \qquad \omega^\alpha_\kappa(x) = \frac{d}{dt}\Big|_{t=0} \tau^\alpha_x(\exp -t\kappa),$$

$\omega^\alpha{}_\kappa(x)$ is just a local representative for ω_κ, $\omega^\alpha{}_\kappa(x) = \omega_\kappa(s_\alpha(x))$, as anticipated by the notation.

Conversely, if we can find a bracket-preserving linear map $\kappa \longrightarrow \omega_\kappa$ satisfying (4.5) which associates covariantly constant adjoint Higgs-type fields ω_κ to each $\kappa \in \mathfrak{k}$, $(\exp -\kappa)\cdot p = \exp(\omega_\kappa(p)^\wedge)$ (exponential of a vectorfield) provides us with an internal action of $k = \exp(-\kappa) \in K$ on P. In fact, $\tau_p(\exp -\kappa) = \exp(\omega_\kappa(p))$. (exponential in the group).

All solutions of (4.6) are found by parallel transport [18,7]. $\omega_\kappa(p)$ belongs therefore, for all $p \in P$, to a single adjoint orbit of G. Hence we get

PROPOSITION 4.2

The action of an internal symmetry group K on P is rigid.

As we have seen in Section 2, to have a rigid internal action of K is equivalent to requiring that the bundle (P,G) reduces to $H = Z_G(H)$. In terms of ω this reduction is obtained as $Q = \{p \in P \mid \omega_\kappa(p) = \kappa\}$ [10]. This implies

PROPOSITION 4.3

The action of K on (P,A) defined by the (Q,H) (where $H = Z_G(K)$) is an internal symmetry if and only if the connection form A reduces to Q. This happens if and only if $H = Z_G(K)$ contains the holonomy group of A. In this case Q contains the holonomy bundle and so is unique.

In particular, the <u>full gauge group G</u> is a group of internal symmetries if and only if the connection reduces to the Z(G)-bundle which characterizes the left action of G on P. This is equivalent to requiring that the generators of the holonomy group lie in the centre of [23].

Similarly, let us consider a matter field Φ, and assume there is an internal action of K on Φ determined by a section τ^* of $P^* \times_{G^*} [\mathrm{Hom}(K^*, G^*)]$. We shall say that this action of K is an <u>internal symmetry</u> for the <u>matter field</u> Φ, if

$$(4.9) \qquad (k\cdot\Phi)(p) = U(\tau^*_p(k))\Phi(p) = \Phi(p).$$

In a local (in particular in a rigid) gauge this reads

$$(4.10) \qquad U(\tau^{*\alpha}_x(k))\Phi^\alpha(x) = \Phi^\alpha(x) \text{ and } U(k)\cdot\Phi^\alpha(x) = \Phi^\alpha(x)$$

respectively. We can work also infinitesimally:

$$(4.11) \qquad \tilde\kappa^*(p^*) = \frac{d}{dt}\Big|_{t=0} (\exp -t\kappa)^* \cdot p^*$$

and

$$\omega_{\kappa^*}(p^*) = \frac{d}{dt}\Big|_{t=0} \tau^*_{p^*}(\exp -t\kappa)^*$$

provide us with a vertical vectorfield κ^* and an adjoint "Higgs" field $\omega^*{}_\kappa$ on $P^* = P/N$. The infinitesimal action of $\mathfrak{k}^*$ (the Lie algebra of K^*) on Φ reads

$$(4.12) \qquad (\tilde\kappa^*\cdot\Phi)(p) = L_{\tilde\kappa^*}\Phi(p) - \omega^*_\kappa(p)\cdot\Phi(p)$$

where the dot $\cdot$ denotes the action of the Lie algebra induced by U. The definition of a symmetric matter field reads hence infinitesimally

$$(4.13)\qquad \omega^*{}_{\kappa}{}^*\cdot\Phi(p) = 0.$$

or in a local gauge

$$(4.14)\qquad \omega^{*\alpha}_{\kappa\,*}(x)\cdot\Phi^{\alpha}(x) = 0.$$

Let us now assume that we have a Yang-Mills potential A and a covariantly constant Higgs field Φ and that we are interested in their simultaneous symmetries. First, K implementable on P implies that K is implementable also on $P\times_G E$. Furthermore, $\omega^*{}_{\kappa}(x) = (\omega_{\kappa}(p))^*$. In particular if N is discrete, $\mathfrak{k}^* = \mathfrak{k}$ and $\mathfrak{g}^* = \mathfrak{g}$, so the star can be dropped. Both ω_{κ} and Φ are now found by parallel transport, $\Phi^{\alpha}(x) = \mathrm{Ad}g^{\alpha}(x)\Phi_0$ and $\omega_{\kappa}{}^{\alpha}(x) = \mathrm{Ad}g^{\alpha}(x)\kappa$, where g^{α} is the non-integrable phase factor. (4.13) reduces hence to

$$(4.15)\qquad \kappa\cdot\Phi_0 = 0.$$

<u>PROPOSITION 4.4</u>

K is a symmetry group for a covariantly constant matter field Φ if and only if K belongs to the little group of a basepoint Φ_0 from the orbit where Φ takes its values.

As a first illustration, consider the <u>non-Abelian Bohm-Aharonov experiment</u> [7,33-35] proposed by Wu and Yang to test the existence of gauge fields. Here one considers a principal G-bundle P over the punctured plane $M = R^2\setminus\{0\}$ endowed with a flat connection A. Such bundles are classified by classes [P] in $\pi_0(G)$. The homotopy exact sequence

$$(4.16)\qquad \ldots \longrightarrow \pi_1(G) \xrightarrow{\mathrm{Ad}_*} \pi_1(G/Z(G)) \xrightarrow{\delta} \underset{[Q]}{\pi_0(Z(G))} \xrightarrow{i_*} \underset{[P]}{\pi_0(G)} \longrightarrow \ldots$$

analogous to (6.3) and induced by $Z(G) \longrightarrow G \longrightarrow G/Z(G)$ shows that G is implementable iff [P] belongs to Im i_*. If this condition is satisfied, there may be still an <u>ambiguity</u>: the different implementations - the inequivalent reductions to Z(G) - are classified by Ker i_*. For G=SU(2) for example, P is trivial because SU(2) is connected. $\pi_1(SU(2)) = 0$; $Z(G) = Z_2$, $G/Z(G) = SO(3)$ so there exist <u>two</u> gauge-inequivalent implementations corresponding to the reductions of P to a trivial or to a twisted bundle respectively [7].

When do we get an internal symmetry? Let us consider more generally a principal G-bundle P over a connected manifold M carrying a flat connection A.
The horizontal distribution of A is integrable by the Frobenius theorem. Let us choose a reference point p_0 in P, and denote by Q the leaf of the horizontal distribution through p_0. Q is a covering of M which is a reduction of P with a discrete subgroup Γ of G as structure group. As a matter of fact, Γ is just the holonomy group of A at p_0. According to Prop. 4.3 G acts as a symmetry for (P,A) iff Γ is in Z(G).

In the non-Abelian Bohm-Aharonov experiment Γ consists of powers of Φ, the non-integrable phase factor calculated along a loop which winds once around the origin [33-35]. Consequently, SU(2) acts as a group of internal symmetries only for $\Phi = 1$ (for the first implementation) or for $\Phi = -1$ (for the second implementation) [7]. The physical consequences are explained in [35].

5. MONOPOLE BUNDLES

The asymptotic properties of monopoles are determined by a principal G-bundle P over the S^2 at infinity, where G, the residual symmetry group, is compact and connected. In Grand Unified Theories one starts in general with a trivial "unifying" bundle $\tilde{P} = R^3 x \tilde{G}$, where $\tilde{G}$ - a compact and connected Lie group - is the "unifying group". At large distances the $\tilde{G}$-symmetry is spontaneously broken to a subgroup G of $\tilde{G}$. Geometrically, this means that over S^2 $(\tilde{P},\tilde{G})$ is reduced to a principal G-bundle P. Any such reduction is produced by an equivariant "reducing map" [10]. Choosing a global trivialisation of $\tilde{P}$, the reducing map can be identified with a map $\Phi: S^2 \longrightarrow \tilde{G}/G$ - the physical <u>Higgs field</u>. Φ defines a homotopy class $[\Phi] \in \pi_2(\tilde{G}/G)$, and the homotopy class [P] is $\delta[\Phi]$, where $\delta: \pi_2(\tilde{G}/G) \longrightarrow \pi_1(G)$ is the connecting homomorphism. δ is an isomorphism if $\tilde{G}$ is simply connected. Both [P] and $[\Phi]$ will be referred to as the Higgs charge in the sequel.

$$\pi_1(G) \simeq \pi_1(G)_{free} + \pi_1(G_{ss}),$$

where $\pi_1(G)_{free} \simeq Z^p$, p is the dimension of $Z(\mathfrak{g})$, and G_{ss} is the semisimple subgroup of G generated by $[\mathfrak{g},\mathfrak{g}]$. $\pi_1(G_{ss})$ is a finite Abelian group. The free part - which plays a particularly important role - is described as follows [23]: denote by $\Gamma = \{\xi \in \mathfrak{g} \mid \exp 2\pi\xi = 1\}$ and let $z: \mathfrak{g} \longrightarrow Z(\mathfrak{g})$ be the projection of the Lie algebra of G onto its centre. $z(\Gamma)$, the image of Γ under z, is a lattice whose dimension is the same as that of $Z(\mathfrak{g})$. In [23] we proved the following theorem: Define, for any loop γ in G,

$$(5.1) \qquad \rho(\gamma) = \frac{1}{2\pi} \int_\gamma z(\Theta) \in Z(\mathfrak{g}),$$

where $\Theta = g^{-1}dg$ is the canonical (Maurer-Cartan) 1-form of G. ρ defines an isomorphism of $\pi_1(G)_{free}$ with $z(\Gamma)$.

Any loop in G is known to be homotopic to one of the form $\gamma(t) = \exp 2\pi\xi t$. For this γ $\rho(\gamma) = z(\xi)$. If $\zeta_1,..,\zeta_p$ is a Z-basis for $z(\Gamma)$, then $\rho(\gamma) = \sum m_i\zeta_i$ provides us with p "quantum" numbers $m_1,..,m_p$.

In [23] we gave also a second characterization of ρ, namely that $\rho(\Phi) = \rho(\delta[\Phi])$ can also be calculated as the <u>integral of a 2-form</u> over the 2-sphere at infinity,

$$(5.2) \qquad \rho(\Phi) = \frac{1}{2\pi} \int_{S^2} \Phi^*\Omega,$$

where Ω is the projection to $\tilde{G}/G$ of the $Z(\mathfrak{g})$-valued 2-form $z(d\tilde{\Theta})$ on $\tilde{G}$.

Here we give <u>third</u> construction, adapted from Chern-Weil theory [10], Vol.II.: let us consider an arbitrary connection form A on P, and denote by F = DA its curvature form. z(F) is a $Z(\mathfrak{g})$-valued 2-form on P, which is horizontal and basic, since

$$r_g^*z(F) = z(r_g^*F) = z(Adg^{-1}F) = z(F),$$

so z(F) descends to S^2 to a $Z(\mathfrak{g})$-valued 2-form Ω^A, $z(F) = \pi^*\Omega^A$. This two-form is closed,

$$d(z(F)) = z(dF) = z(DF - [A,F]) = z(DF) = 0,$$

since z vanishes on the derived algebra and DF = 0 by the Bianchi identity. The class [z(F)] is known [10] to be independent of the choice of connection. This proves:

<u>**PROPOSITION 5.1**</u>

The cohomology class $[\Omega^A] \in H^2{}_{dR}(S^2) x Z(\mathfrak{g})$ is independent of the choice of the connection A on P. Consequently,

$$(5.3) \qquad \rho(P) = \frac{1}{2\pi} \int_{S^2} z(F) \quad \in \quad Z(\mathfrak{g})$$

depends only on the bundle P.

Let us now assume that (P,G) is the reduction of the trivial unifying bundle $(\tilde{P},\tilde{G})$ defined by $\Phi: S^2 \dashrightarrow \tilde{G}/G$ and so can be identified with the pullback by Φ of $\tilde{G}$, viewed as a principal G-bundle over the orbit $\tilde{G}/G$ [10]. The $\mathfrak{g}$ - component of the Maurer - Cartan 1-form $\tilde{\Theta}$ defines a connection on the principal G-bundle $\tilde{G}$ whose pullback by Φ is a connection form A on P such that $z(DA) = \Phi^*\Omega$, where Ω is the 2-form defined above. Thus we have established the equivalence of our new construction with those given before.

Monopole fields must also satisfy the <u>Yang-Mills-Higgs equations</u>. Assuming a sufficiently rapid fall-off at infinity, the Yang-Mills-Higgs equation on S^2 reduces to $D^*F = 0$. The solution has been found by Goddard, Nuyts and Olive [17-20,28-29]): Let us assume that P is a non-trivial

G-bundle over S^2, carrying a connection form A which satisfies the YM equation $D^*F = 0$. Then there is a vector Π in $\mathfrak{g}$ generating a homomorphism $U(1) \to G$ such that P is associated to the Hopf bundle over S^2 and the field is $F = \mathcal{F}\Pi$ with $\mathcal{F}$ the area form on the 2-sphere. Π is quantized, $\exp 4\pi\Pi = 1$. The vector Π can be chosen without loss of generality in any given Cartan subalgebra of $\mathfrak{g}$. The transition function h of a monopole is thus homotopic to $h(t) = \exp 4\pi\Pi t$, $0 \leq t \leq 1$, so $\rho(P)$ is simply

$$\rho(P) = z(2\Pi). \tag{5.4}$$

This theorem can also be reformulated by saying that the holonomy group of asymptotic monopole bundles is a U(1), generated by the "non-Abelian charge" vector Π [29]. Conversely, given Π we are able to construct an asymptotic monopole configuration, see the following section.

6. REDUCTION OF MONOPOLE BUNDLES

THEOREM 6.1

The monopole bundle (P,G) is reducible to an H-bundle Q (where H is a closed subgroup of G) iff

$$[\Phi] \in \operatorname{Im} \sigma_*, \tag{6.1}$$

where $\sigma_*: \pi_2(\tilde{G}/H) \to \pi_2(\tilde{G}/G)$ is induced by the natural projection $\sigma: \tilde{G}/H \to \tilde{G}/G$. Equivalently, iff

$$[P] = \delta[\Phi] \in \operatorname{Im} i_*, \tag{6.2}$$

where $i_*: \pi_1(H) \to \pi_1(G)$ is induced by the inclusion $i: H \hookrightarrow G$. The inequivalent reductions from G to H are parametrized by the elements of $\pi_2(G/H)$.

Proof: Suppose first that there exists a reduction (Q,H) of (P,G). (Q,H) is a reduction of $(\tilde{P},\tilde{G})$ also, and is thus determined by a reducing map ("Higgs field") $\Psi: S^2 \to \tilde{G}/H$. This reduces P if and only if each H-coset is contained in a corresponding G-coset. That is, if and only if $\Phi = \sigma(\Psi)$. But this implies $[\Phi] = \sigma_*[\Psi]$.

Conversely, if $[\Phi] \in \operatorname{Im}\sigma_*$, then $[\Phi] = \sigma_*[\Psi_1]$ for some $\Psi_1: S^2 \to \tilde{G}/H$. Φ and $\sigma(\Psi_1)$ are hence homotopic and thus gauge-equivalent [8,9,10], so there exists a map g(x) such that $\Phi(x) = g(x)\cdot\sigma(\Psi_1(x))$. But σ is a G-map, so putting $\Psi(x) = g(x)\cdot\Psi_1(x)$, $\Phi(x) = \sigma(\Psi(x))$, and hence we get a reduction Q of P defined by Ψ. Consider

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \pi_2(G/H) & & & & \\
 & & & & \downarrow \delta & & & & \\
0 & \longrightarrow & \pi_2(\tilde{G}/H) & \xrightarrow{\delta} & \pi_1(H) & \xrightarrow{(j\circ i)_*} & \pi_1(\tilde{G}) & \longrightarrow & \cdots \\
 & & \downarrow \sigma_* & & \downarrow i_* & & \| & & \\
0 & \longrightarrow & \pi_2(\tilde{G}/G) & \xrightarrow[\delta]{} & \pi_1(G) & \xrightarrow[j_*]{} & \pi_1(\tilde{G}) & \longrightarrow & \cdots \\
 & & & & \downarrow & & & &
\end{array} \tag{6.3}$$

It follows from this diagram, that $[\Phi] \in \operatorname{Im}\sigma_*$ if and only if

(6.2) holds.

Finally, by Theorem 2.5 two reductions (Q_1,H) and (Q_2,H) of (P,G) are equivalent iff $[Q_1] = [Q_2] \in \pi_1(H)$. The inequivalent reductions are hence labelled by Ker i_* which is, according to the diagram, just $\pi_2(G/H)$.

It follows from (6.2) that if Im $i_* = 0$, (in particular when H is simply connected), then (P,G) reduces to a subbundle Q with structure group H if and only if the Higgs charge of Φ is zero and so the G-bundle P is trivial [25,30].

PROPOSITION 6.2

The structure group G of P can be reduced to H if and only if h_P is homotopic to a loop in H. In this case

$$(6.4) \qquad \rho_G([P]) = z_G(\rho_H([Q])).$$

If $\pi_1(G)$ is free and H is Abelian, then (6.4) is also sufficient.

Indeed, the reductions (P,G) and (Q,H) of $(\tilde{P},\tilde{G})$ are compatible if and only if the transition function are homotopic in G, $[\exp 4\pi\Pi_P t] = [\exp 4\pi\Pi_Q t] \in \pi_1(G)$. Next, the projection maps $z_G\colon \mathfrak{g} \to Z(\mathfrak{g})$ and $z_H\colon \mathfrak{h} \to Z(\mathfrak{h})$ satisfy $z_G(z_H(\eta)) = z_G(\eta)$, $\eta \in \mathfrak{h} \hookrightarrow \mathfrak{g}$. This follows from $z_G([\mathfrak{g},\mathfrak{g}]) = 0$, observing that $\eta = z_H(\eta) + \eta'$, where $\eta' \in [\mathfrak{h},\mathfrak{h}] \subset [\mathfrak{g},\mathfrak{g}]$. Notice that any H-connection on Q extends naturally to a G-connection on P. Let F denote the curvature. On Q $z_G F = z_G(z_H F)$, since F is H-valued. (6.4) follows then from the definition.

Finally, let $h_P(t) = \exp 2\pi\xi t$ and $h_Q(t) = \exp 2\pi\eta t$ be the transition functions of the bundles P and Q. Then $\rho_G([P]) = z_G(\xi)$ and $\rho_H([Q]) = \eta$. Thus $\rho_G(h_Q) = z_G(\eta) = z_G(\xi)$. Therefore, if $\pi_1(G)$ is free and H is Abelian, h_P and h_Q are homotopic loops in G since ρ_G is now an isomorphism.

An interesting insight is gained by proceeding backwards. Let us start with a (right) principal H-bundle Q over the two-sphere, and assume $i\colon H \hookrightarrow G$ is a subgroup of G. $(q,g) \simeq (gh, h^{-1}g)$ is an equivalence relation on QxG and the set of equivalence classes (denoted by $\{q,g\}$ here) yields the associated bundle $P^Q = Q \times_H G$. P^Q is a principal G-bundle with G-action $\{q,g\} \to \{q,gg'\}$, $g' \in G$. (Q,H) is furthermore a reduction of (P^Q,G), $Q \simeq (\{q,e\} | q \in Q)$.

If (Q,H) has isomorphism class $[Q] \in \pi_1(H)$, the class of P^Q is $[P^Q] = i_*[Q] \in \pi_1(G)$. P^Q is thus isomorphic to a given G-bundle P iff

$$(6.5) \qquad i_*[Q] = [P].$$

If (6.5) is satisfied, (Q,H) is a reduction of (P,G) by construction. This shows also that the different reductions of P are parametrized by Ker $i_* \simeq \pi_2(G/H)$.

This proceedure allows us also to construct asymptotic monopole configurations [28]. The non-Abelian charge vector Π is written as $\Pi = (n/2)\xi$, n an integer and ξ a minimal U(1) generator, because Π is quantized. Denote by Y^n the Hopf-bundle S^3/Z_n. $H = \{\exp 2\pi t\xi \,|\, 0 \le t \le 1\}$ is a U(1) subgroup of G. Y^n can be viewed also as a principal H-bundle with H-action $y \to yh = h(e^{2\pi t})$ for $h = \exp 2\pi t\xi$. The associated bundle $P^{(\Pi)} = Y^n \times_H G$ is a principal G-bundle having transition function $h(\theta) = \exp 2\theta\Pi$, $0 \le \theta \le 2\pi$ being the angle parametrizing the equatorial circle of S^2. The natural connection $A^{(n)} = n\bar{y}dy/i$ of Y^n extends to a connection $A^{(\Pi)}$ on $P^{(\Pi)}$; as a mater of fact, $Y^{(\Pi)} = (\{y,e\} | y \in Y^n)$ is the holonomy bundle of $(P^{(\Pi)}, A^{(\Pi)})$. This latter is an asymptotic monopole bundle iff $[P^{(\Pi)}] \in \mathrm{Ker}\, j_*$ for $j\colon G \hookrightarrow \tilde{G}$ [20,28]. Under suitable conditions such asymptotic solutions can be extended to the interior region [31,32].

The connection on (P,G) determined by the non-Abelian charge vector reduces to a subbundle (Q,H) iff this latter contains the holonomy bundle. We conclude:

THEOREM 6.3

The Yang-Mills connection A of a principal G-bundle P over S^2 defined by the non-Abelian charge Π reduces to a subbundle (Q,H) if and only if $\Pi \in \mathfrak{h}$.

For example, (P,G) reduces to a U(1) subgroup $H = \{\exp 2\pi\eta t \,|\, 0 \le t \le 1\}$, where η is a minimal generator, iff $[\exp 2\pi t\eta] =$

$[\exp 4\pi t\Pi] \in \pi_1(G)$. A necessary condition for this is $z(\eta) = 2z(\Pi)$. This is also sufficient if $\pi_1(G)$ has no finite part. The YM connection A reduces also iff the reduced H-bundle Q contains $Y^{(\Pi)}$ which happens iff $2\Pi = n\eta$ for a suitable integer n.

These results provide us with topological informations concerning the "fate" of monopoles under successive symmetry breakings $G \longrightarrow H$ [24-27]. The topological condition found in [24,25] for its survival means exactly that the G-bundle P reduces to an H-bundle Q. On the other hand, the second condition given in [25-27] requires that the Yang-Mills connection reduces also.

7. THE COLOR PROBLEM IN MONOPOLE THEORY

Let us consider a Grand Unified monopole (A_j, Φ) with "residual" symmetry group G. G is the little group of a basepoint Φ_0 in the orbit where the Higgs field takes asymptotically its values. Our previous results imply

THEOREM 7.1

A subgroup $K \subset G$ is (rigidly) implementable if and only if, over S^2, the monopole bundle (P,G) reduces to a $Z_G(K)$-bundle Q. The rigid actions of K are in (1-1) correspondence with reductions to $Z_G(K)$. The necessary and sufficient condition of implementablity of K is

(7.1) $\delta[\Phi] \in \mathrm{Im}\, i_*$,

where i_* is the homomorphism between homotopy groups induced by the inclusion map $i: Z_G(K) \hookrightarrow G$. The inequivalent reductions are in (1-1) correspondance with the elements of $\pi_2(G/Z_G(K))$.

An alternative proof is obtained using the inverse technique of the previous section. Denote in fact $Z_G(K)$ by H and consider an H-bundle Q over S^2. Such bundles exist for each element in $\pi_1(H)$. Form the associated bundle $P^{(Q)}$. $p = qg$ with $q \in Q$ and $g \in G$ for each $p \in P^{(Q)}$, because Q is a reduction of $P^{(Q)}$. Set

(7.2) $k \cdot p = pg^{-1}kg$.

If $p = q'g'$ with $q' \in Q$ and $g' \in G$, then $q' = qh$, $g' = h^{-1}g$ for some $h \in H$. Thus

$$p(g')^{-1}kg' = pg^{-1}hkh^{-1}g = pg^{-1}kg,$$

because h is in $Z_G(K)$. (7.2) is thus a well-defined action of K on $P^{(Q)}$. Furthermore,

(7.3) $\tau_p = \mathrm{Ad}\, g^{-1} i$,

showing that this implementation is rigid. Notice also that $\tau | Q = i$.

As we have seen in Section 6, $P^{(Q)}$ is isomorphic to a given monopole bundle (P,G) iff (6.5) is satisfied, and the different possibilities are parametrized by $\pi_2(G/H) = \pi_2(G/Z_G(K))$.

For K = G we have some more results: [1,2]:

PROPOSITION 7.2

G is implementable if and only if the transition function $h(t) = \exp 4\pi t\Pi$, $0\le t\le 1$ of the bundle P is homotopic to a loop in the centre of G. This happens iff

$$(7.4) \qquad \delta[\Phi] \in \pi_1(G)_{free}$$

and

$$(7.5) \qquad \exp 2\pi\rho_G(P) = \exp 4\pi z(\Pi) = 1.$$

The implementation of G is unique.

Indeed, for $i : Z(G) \longrightarrow G$ Im i_* belongs to $\pi_1(G)_{free}$. On the other hand, a Z(G) bundle Q is represented by a loop $\exp 2\pi\zeta t$, $0\le t\le 1$, where ζ is in $Z(\mathfrak{g})$. Then $\rho_H(Q) = z_H(\zeta) = \zeta$. By $\rho_G(P) = \zeta$ by 6.4. However, $\rho_G(P) = z(2\Pi)$.

Conversely, (7.4) and (7.5) are also sufficient : $\gamma(t) = \exp 2\pi\rho_G(P)t$ is now a loop in Im i whose homotopy class is $\delta[\Phi]$, because $\rho_G(\gamma) = \rho_G(P)$ and ρ_G, when restricted to the free part, is an isomorphism. G admits at most one implementation, since G/Z(G) is a Lie group and has thus trivial second homotopy. So i_* is now injective.

To express this result another way, decompose the non-Abelian charge vector as $\Pi = z(\Pi) + \Pi'$, where Π' belongs to $[\mathfrak{g},\mathfrak{g}]$. Denote by G_{ss} the semisimple subgroup of G whose Lie algebra is $[\mathfrak{g},\mathfrak{g}]$, and let $G_{ss}{}^*$ be the simply connected covering group of G_{ss}.

PROPOSITION 7.3

G is implementable if and only if $\exp 4\pi t\Pi'$, $0\le t\le 1$ is a contractible loop in G_{ss}. This happens iff

$$(7.6) \qquad \exp^* 4\pi\Pi' = 1,$$

where $\exp^*$ is the exponential map in $G_{ss}{}^*$.

Proof: $z(\Pi)$ commutes with everything, and thus

$$\exp 4\pi\Pi' = (\exp 4\pi\Pi).(\exp 4\pi z(\Pi))^{-1} = (\exp 4\pi z(\Pi))^{-1}.$$

(7.5) and (7.6) are thus equivalent; in particular, $\exp 4\pi t\Pi'$, $0\le t\le 1$ is a loop in G_{ss}. [P] is now decomposed as

$$[P] = [\exp 4\pi t z(\Pi)] + [\exp 4\pi t\Pi'] \in \pi_1(G)_{free} + \pi_1(G_{ss}),$$

and hence (7.4) is equivalent to $\exp 4\pi t\Pi'$ contractible in G_{ss}. But $\pi_1(G_{ss})$ is known to be Γ/Γ^*, where Γ (respectively Γ^*) are the unit lattices of G_{ss} (respectively of $G_{ss}{}^*$). Thus $\exp 4\pi t\Pi'$ contractible means exactly (7.5).

This is seen alternatively by noting [1,7] that, according to the diagram (6.3), $[P] \in \mathrm{Im}\, i_*$ exactly when $\mathrm{Ad}_*[P] = 0$, i.e. the transition function Adh is contractible in $(\mathrm{Aut}G)_0 \cong \mathrm{Int}G \cong G/Z(G)$. But the condition for this is just (7.6), since G/Z(G) has $[\mathfrak{g},\mathfrak{g}]$ for Lie algebra.

(7.5) can be translated into numbers: let $\zeta_1,..,\zeta_p$ be a Z-basis for $z(\Gamma)$ (assumed non-empty), then $\rho(P) = \sum m_j\zeta_j$. On the other hand, there exist least positive integers such that $M_j\zeta_j \in \Gamma$ [23]. Thus (7.5) can hold only if, for each j, m_j/M_j is an integer, say n_j. Consequently

PROPOSITION 7.4

G is implementable if and only if (7.4) is valid and

$$(7.8) \qquad m_j = n_j M_j \text{ for suitable integers } n_j.$$

The case $K \neq G$ is similar but more complicated, cf.[7].

Next, Prop. 4.4 and Theorem 6.3 imply

THEOREM 7.5

K G is an internal symmetry group if and only if the loop $h_P(t) = \exp 4\pi t\Pi$ lies in $Z_G(K)$. This happens iff

$$(7.9) \qquad \mathrm{Ad}k\Pi = \Pi, \ \forall\, k \in K.$$

In particular, G is an internal symmetry iff Π lies in the centre. The action is then unique.

Indeed, the holonomy group of a monopole-bundle is generated by the non-abelian charge Π and (7.9) means exactly that $K \subset Z_G(\Pi)$. $[\zeta,\Phi_0] = 0$ is automatically satisfied, since $\zeta \in \xi$ stabilizes Φ_0. Alternatively, the implementation defined by a reduction (Q,H) is a symmetry iff (Q,H) contains the holonomy bundle $Y(\Pi)$.

As an illustration, consider a GUT with residual group G= SO(3). Such a situation arises, e.g., when G= SU(3) is broken by a Higgs <u>6</u> [18,27]. Choose in so(3) the Cartan algebra

$$(7.10) \qquad \mathcal{T} = aL_3 = \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \in R.$$

The non-Abelian charge vector can be gauge-rotated into $\mathcal{T}$. Then $\Pi = (m/2)L_3$ where m is an integer. $\pi_1(SO(3)) = Z_2$, $[\exp 2\pi m L_3] = m$ (modulo 2). Topologically non-trivial solutions arise therefore if <u>m is odd</u>. Denote by P the corresponding SO(3)-bundle.

G=SO(3) is not implementable on P: (7.6) would require, in fact, m to be even.

Consider now a U(1) subgroup K with minimal generator ξ, K = $\{\exp 2\pi\xi t\}$. ξ is conjugate to L_3, and hence $\pi_1(SO(3))$ $[\exp 2\pi\xi n t] = [\exp 2\pi n L_3] = n$ (modulo 2). $Z_{SO(3)}(K)=K$, so the interesting part of Diagramm (6.3) becomes

$$(7.11) \qquad \begin{array}{ccccc} \longrightarrow & \pi_1(K) & \xrightarrow{i_*} & \pi_1(SO(3)) & \longrightarrow \\ & \| & & \| & \\ & Z & \longrightarrow & Z_2 & \\ & n & \longrightarrow & n \text{ (modulo 2)} & \end{array}$$

Theorems 7.1 and 7.5 tell us therefore that

(i) K is implementable on P iff n is odd;

(ii) For n = 2k+1 $\mathrm{Ker}\, i_* \cong Z$: there is a different implementation for each k, corresponding to the different reductions to K;

(iii) K is a symmetry iff Π and ξ are parallel, $\Pi = (n/2)\xi$ for some integer n.

We are able to construct the bundles explicitly: choose an integer n and consider the Hopf bundle $Y^n = S^3/Z_n$ where S^3 is viewed as sitting in C^2. Y^n is a two-sided U(1) bundle with actions $z:y \longrightarrow z\cdot y = (zy_1, zy_2)$ and $z': y \longrightarrow y\cdot z' = (y_1 z', y_2 z')$, $y \cong (y_1,y_2) \in C^2$, $z,z' \in U(1))$. Y^n can be viewed alternatively as a two-sided principal K-bundle with $k = \exp 2\pi\xi a$ acting as $k\cdot y = (e^{2\pi i a})y$ and $y\cdot k = y(e^{2\pi i a})$ respectively.

The associated bundle $P^{(n)} = Y^n \times_K SO(3)$, is a right principal SO(3) bundle. Y^n is identified with $Y^\xi = (\{y,e\} \mid y \in Y^n)$ and so is a reduction of $P^{(n)}$.

The right action of K on Y^n was used to construct $P^{(n)}$. However, we still have a left action of K on Y^n which extends to a left action of K on $P^{(n)}$ according to

$$(7.12) \qquad k\{y,g\} = \{k\cdot y, g\} = \{y, k^{-1}g\} = \{y,g\}\cdot \mathrm{Ad}g^{-1}k,$$

where $k = \exp 2\pi\xi a$. Hence, for $p = \{y,g\}$,

$$(7.13) \qquad \tau_p(k) = \mathrm{Ad}g^{-1}k,$$

The transition function of the principal K-bundle $Y^{(\xi)}$ is $h(\theta) = \exp\theta\xi n$. Hence $\pi_1(SO(3))$ $[P_n] = n$ (modulo 2): $P^{(n)}$ is the trivial bundle for n even and is isomorphic to P for n odd. Our construction provides us hence with a rigid action of K on P for each odd integer n, as expected. These actions are obviously inequivalent.

The action of K as constructed above is a symmetry for the monopole field A given by the non-Abelian charge vector Π iff $Y^{(\xi)}$ contains the holonomy bundle, which happens iff $\Pi = (n/2)\xi$.

ACKNOWLEDGEMENTS

We are indebted to J-M. Souriau for hospitality in Marseille. P.A.H. would also like to thank the SERC of Great Britain for financial support during part of this work. Discussions with L. O'Raifeartaigh and correspondance with P. Nelson and C.Nash are gratefully acknowledged.

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