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BOSE-EINSTEIN CONDENSATION IN DEPENDENCE OF THE MEAN ENERGY DENSITY

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Abstract: The infinite volume limit of the thermodynamic functions of an ideal Bose gas with respect to a grand-canonical equilibrium is taken such that the mean energy density is fixed. Above the critical mean energy density the macroscopically occupied ground state contributes to the mean entropy density while the mean density of the particle number has an infrared divergence. A thermodynamic stability result is derived; for a photon gas it means that, if condensation can be achieved, the condensed state should persist in the presence of a black perturbation.

1. Introduction

In calculating the infinite volume limit of thermodynamical systems it is common to assign the mean particle number density ρ a fixed value. This amounts to take ρ as an independent variable of the infinite system. For an infinite free boson gas in the grand-canonical equilibrium, with ρ and inverse temperature β as independent variables, it turns out that the macroscopically occupied ground state does not contribute to the mean energy density u and the mean entropy density s [1]. However this is no longer true if the mean energy density itself is selected as an independent thermodynamical variable.

It is the aim of this note to describe the infinite free boson gas in terms of β and u, thereby demonstrating the difference, in the condensation region, to the familiar description in terms of β and ρ [1,2]. We show that there exists a critical mean energy density $u_c(\beta)$, such that for $u > u_c(\beta)$ the ground state is macroscopically occupied. Above $u_c(\beta)$ the pressure p is a constant with respect to u, $p(\beta,u) = p_c(\beta)$, while the mean particle number density ρ diverges. By this latter fact, a description of Bose-Einstein condensation in terms of β and u is only relevant for relativistic particles.

We will refer to a free photon gas [3] assuming that conditions can be achieved where the mean energy density and the frequency distribution are manipulated independently - for example by lasers of different frequencies focussing into a reflecting cavity. An infinity of "soft" photons, i.e. photons of an infinitesimally small energy each, giving rise to a finite total energy density integral, seems physically reasonable.

The non-zero contribution of the macroscopically occupied ground state to the entropy density, for $u > u_c(\beta)$, provides a thermodynamical stability argument: A grand-canonical state (u,β) with $u > u_c(\beta)$ has a higher entropy density than an unconstrained canonical state with β' where $u_c(\beta) < u =: u_c(\beta')$. For $u < u_c(\beta)$, the canonical state for β', with $u_c(\beta) > u =: u_c(\beta')$, is thermodynamically preferred.

Hence a photon gas should be described by a grand-canonical ensemble with $\mu = 0$ which, because of the phase transition, is mathematically not equivalent to the unconstrained canonical ensemble. - In the usual textbook treatment this subtlety is not taken into account. Either the unconstrained canonical ensemble is used a priori such that μ is not defined [4], or μ is introduced and set equal to zero, but the infinite volume limit is incorrectly calculated (see e.g. [5]).

However, strictly speaking, a satisfactory discussion of thermodynamical stability of photon condensation needs a mathematically reliable theory of Einstein condensation of interacting bosons which, up to date, does not yet exist except for some extreme idealizations [6]. So a complete theoretical analysis appears at this stage more difficult than a possible empirical approach.

Let h_R be a sequence of self-adjoint operators on $L_2(\Omega_R,\mu_R)$ where, for each R, (Ω_R,μ_R) is a bounded measure space with $\mu_R(\Omega_R) =: V_R$, and assume e^{-h_R} to be trace classe. Hence

$$\frac{1}{V_R}\, tr\,[e^{-\beta h_R}] = \frac{1}{V_R}\sum_{k=1}^{\infty} e^{-\beta\varepsilon_k^R}, \quad \beta\in[0,\infty), \tag{1.a}$$

exists where $0 \le \varepsilon_1^R \le \varepsilon_2^R \le \dots$ are the eigenvalues of h_R; they are of finite multiplicity. Introducing the energy differences $\lambda_k^R := \varepsilon_k^R - \varepsilon_1^R$ we will use the normalized partition function $\Phi_R(\beta)$, and its Laplace transformed $F_R(\lambda)$:

$$\Phi_R(\beta) := \frac{1}{V_R}\sum_{k=1}^{\infty} e^{-\beta\lambda_k^R}, \tag{1.b}$$

$$\Phi_R(\beta) =: \int_0^\infty e^{-\beta\lambda}\, dF_R(\lambda). \tag{2}$$

For concreteness we assume the spectral density $dF_R(\lambda)$ to have the form

$$dF_R(\lambda) = \alpha\lambda^q d\lambda + O\left(\frac{\lambda^{q_1}}{R^{q_2}}\right) d\lambda, \tag{3}$$

$$q,\ q_1 > 1, \quad q_2,\ \alpha > 0,$$

and we impose the specification

$$\begin{aligned} &\varepsilon_1^R > 0, \quad \text{for all } R, \\ &\varepsilon_1^R \sim V_R^{-q_3} \quad \text{for some } q_3 > 0, \quad \lim_{R\to\infty}\lambda_{g+1}^R = 0, \end{aligned} \tag{4}$$

where g counts the multiplicity of the ground state. This covers the non-relativistic Dirichlet hamiltonian of a massive particle [7],

$$h_R = -\frac{\hbar^2}{2m}\Delta, \tag{5.a}$$

$$dF_R(\lambda) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\lambda^{1/2}d\lambda - \frac{A_R}{16\pi V_R}\left(\frac{2m}{\hbar^2}\right)d\lambda + O\left(\frac{\lambda^{-1/2}}{R^2}\right)d\lambda \tag{5.b}$$

as well as the single photon hamiltonian for Dirichlet data [3],

(6.a) $$h_\alpha^R = \hbar c\,(-\Delta)^{1/2},$$

(6.b) $$dF_R(\lambda) = \frac{1}{\pi^2}(\hbar c)^{-3}\lambda^2 d\lambda - \frac{A_R}{4\pi V_R}(\hbar c)^{-2}\lambda d\lambda + O(R^{-2})\,d\lambda$$

where we refer to a smooth region $\Omega_R \subset \mathbb{R}^3$, sphere or cube say, with characteristic length R (radius, edge length), and surface area A_R. The index $\alpha = +1,-1$ in (6.a) denotes the two-fold helicity degeneracy of the eigenvalues of the photon hamiltonian.

2. Infinite volume limit of the thermodynamic functions

In this section we compute the infinite volume limit of the thermodynamic functions. For an ideal gas of bosons with single particle hamiltonian h_R, confined in a bounded region Ω_R, the grand-canonical expectation values for particle number density, energy density, pressure, and entropy density are, respectively, given as

(7) $$\rho^R(\beta,\mu) = \frac{1}{V_R}\sum_{k=1}^{\infty}\left(e^{\beta(\lambda_k^R-\mu)}-1\right)^{-1}$$

(8) $$u^R(\beta,\mu) = \frac{1}{V_R}\sum_{k=1}^{\infty}\left(\lambda_k^R+\varepsilon_1^R\right)\left(e^{\beta(\lambda_k^R-\mu)}-1\right)^{-1}$$

(9) $$p^R(\beta,\mu) = \frac{1}{\beta V_R}\sum_{k=1}^{\infty}\log\left(1-e^{-\beta(\lambda_k^R-\mu)}\right)^{-1}$$

(10) $$s^R(\beta,\mu) = k\beta\,\{u^R(\beta,\mu) - \mu\,\rho^R(\beta,\mu) + p^R(\beta,\mu)\}.$$

Here the range of the chemical potential μ is $\mu \leq 0$. Let us define

(11) $$f^R(\beta,\mu) =: f_g^R(\beta,\mu) + f_e^R(\beta,\mu), \quad \beta, -\mu \geq 0,$$

$$f = \rho,\ u,\ p,\ s,$$

$$\rho_g^R(\beta,\mu) := g\,V_R^{-1}\left(e^{-\beta\mu}-1\right)^{-1},$$

$$u_g^R(\beta,\mu) := g\,\varepsilon_1^R V_R^{-1}\left(e^{-\beta\mu}-1\right)^{-1},$$

$$p_g^R(\beta,\mu) := g\,(\beta V_R)^{-1} \log\left(1-e^{\beta\mu}\right)^{-1},$$

$$s_g^R(\beta,\mu) := k\beta\left\{ u_g^R(\beta,\mu) - \mu\,\rho_g^R(\beta,\mu) + p_g^R(\beta,\mu)\right\}.$$

Then we have

$$(12)\qquad \rho_e^R(\beta,\mu) = \frac{1}{V_R}\sum_{k=g+1}^{\infty}\sum_{n=1}^{\infty} e^{-n\beta(\lambda_k^R-\mu)},$$

$$(13)\qquad u_e^R(\beta,\mu) = \frac{1}{V_R}\sum_{k=g+1}^{\infty}\sum_{n=1}^{\infty}\lambda_k^R\, e^{-n\beta(\lambda_k^R-\mu)} + \varepsilon_1^R\,\rho_e^R(\beta,\mu),$$

$$(14)\qquad p_e^R(\beta,\mu) = \frac{1}{\beta V_R}\sum_{k=g+1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\, e^{-n\beta(\lambda_k^R-\mu)}.$$

The functions f_g^R represent the contribution of the ground state, and the functions f_e^R the contribution of the excited states. The following lemma implies the infinite volume limit of the functions f_e^R. It is based on the assumptions (3), (4).

<u>Lemma 1</u>: Given $\beta, -\mu \geq 0$; let $r+q > -1$, $r+q_1 > -1$, $r+q+s > 0$, $r+q_1+s > 0$. Then

$$(15)\qquad \lim_{R\to\infty}\frac{1}{V_R}\sum_{k=g+1}^{\infty}\sum_{n=1}^{\infty}(\lambda_k^R)^r\, n^{-s}\, e^{-n\beta(\lambda_k^R-\mu)}$$

$$= \alpha\,\Gamma(r+q+1)\,\beta^{-(r+q+1)}\sum_{n=1}^{\infty}\frac{e^{n\beta\mu}}{n^{r+q+s+1}}.$$

<u>Proof</u>:

Since $\lambda_k^R > 0$ for $k > g$,

$$\frac{1}{V_R}\sum_{k=g+1}^{\infty}\sum_{n=1}^{\infty}(\lambda_k^R)^r\, n^{-s}\, e^{-n\beta(\lambda_k^R-\mu)}$$

$$= \sum_{n=1}^{\infty} n^{-s} e^{n\beta\mu}\,\frac{1}{V_R}\sum_{k=g+1}^{\infty}(\lambda_k^R)^r\, e^{-n\beta\lambda_k^R}.$$

By means of the spectral density (3) we rewrite the summation as an integral:

$$\lim_{R\to\infty}\frac{1}{V_R}\sum_{k=g+1}^{\infty}(\lambda_k^R)^r\, e^{-n\beta\lambda_k^R}$$

$$= \lim_{R\to\infty}\int_{\lambda_{g+1}^R}^{\infty}\lambda^r e^{-n\beta\lambda}\, dF_R(\lambda)$$

$$= \lim_{R\to\infty}\left\{\alpha\int_{\lambda_{g+1}^R}^{\infty}\lambda^{r+q} e^{-n\beta\lambda}\, d\lambda + \frac{1}{R}\, O\left(\int_{\lambda_{g+1}^R}^{\infty}\lambda^{r+q_1} e^{-n\beta\lambda}\, d\lambda\right)\right\}$$

$$= \alpha\,\Gamma(r+q+1)\,(n\beta)^{-(r+q+1)}.$$

In the last equation we used $\lim_{R\to\infty}\lambda_{g+1}^R = 0$, from (4).

qed.

To state the limit results we choose the notation

$$(16)\qquad \lim_{R\to\infty} f_e^R =: f_e, \qquad f = \rho, u, p, s,$$

$$(17)\qquad g_\alpha(z) := \sum_{n=1}^{\infty}\frac{z^n}{n^\alpha}.$$

<u>Corollary 2</u>: For $\beta, -\mu \geq 0$, we have

$$(18)\qquad \rho_e(\beta,\mu) = \alpha\,\frac{\Gamma(q+1)}{\beta^{(q+1)}}\, g_{q+1}(e^{\beta\mu}),$$

$$u_e(\beta,\mu) = \alpha \frac{\Gamma(q+2)}{\beta^{(q+2)}} g_{q+2}(e^{\beta\mu}), \tag{19}$$

$$p_e(\beta,\mu) = \alpha \frac{\Gamma(q+1)}{\beta^{(q+2)}} g_{q+2}(e^{\beta\mu}). \tag{20}$$

The infinite volume limit of the thermodynamic functions (7)-(10) can be obtained by the following lemma.

<u>Lemma 3</u>: Given two sequences $\{f^R_g\}_{R\in\mathbb{N}}$, $\{f^R_e\}_{R\in\mathbb{N}}$ of real functions

$$f^R_j : [-\infty, 0] \to [0,\infty], \qquad j = g, e,$$

each f^R_j being strictly monotone increasing, $f^R_j(-\infty) = 0$, and such that

(i) each f^R_g is surjective, and $\lim_{R\to\infty} f^R_g(x) = 0$, $x < 0$;

(ii) the sequence $\{f^R_e\}$ converges to a strictly monotone, continuous function,

$$\lim_{R\to\infty} f^R_e(x) = f_e(x), \quad x \leq 0.$$

For $f^R := f^R_g + f^R_e$, and any $\bar f \geq 0$, let $\mu_R(\bar f)$ be the unique root of

$$f^R(\mu) = \bar f.$$

Then, for $\bar f < f_c := f_e(o)$, the sequence $\mu_R(\bar f)$ converges, and

$$\mu(\bar f) := \lim_{R\to\infty} \mu_R(\bar f)$$

is the unique root of

$$f_e(\mu) = \bar f.$$

For $\bar f \geq f_c$, $\mu_R(\bar f)$ converges to zero, and

$$\lim_{R\to\infty} f^R_g(\mu_R(\bar f)) = \bar f - f_c.$$

The proof of Lemma 3 is an exercise in elementary analysis.

Now we observe in (12)-(14) that for fixed β the functions

$$\mu \in [-\infty, 0] \mapsto f^R_j(\beta,\mu) \in [0,\infty], \quad j = g, e; \quad f = \rho, u, p, s,$$

satisfy the assumptions of Lemma 3 where we define

$$f^R_g(\beta, 0) = \infty, \qquad f = \rho, u, p, s.$$

Therefore the infinite volume limit of the thermodynamic functions is an immediate consequence of Lemma 3. We choose the mean energy density u as an independent variable.

<u>Theorem 4</u>: Let $\bar u, \beta \geq 0$. Define

$$u^R(\beta, \mu^R(\beta,\bar u)) := \bar u, \tag{21}$$

$$u_e(\beta, \mu(\beta,\bar u)) := \bar u, \qquad \bar u \leq u_c(\beta) := u_e(\beta, 0), \tag{22}$$

$$\mu(\beta,\bar u) := 0, \qquad \bar u > u_c(\beta).$$

Then we have

(23) $$\lim_{R\to\infty} \mu^R(\beta,\bar u) = \mu(\beta,\bar u);$$

(24) $$\lim_{R\to\infty} u_g^R(\beta,\mu^R(\beta,\bar u)) = (\bar u - u_c(\beta))^+;$$

(25) $$\lim_{R\to\infty} \rho^R(\beta,\mu^R(\beta,\bar u)) = \begin{cases} \rho_e(\beta,\mu(\beta,\bar u)), & \bar u \leq u_c(\beta), \\ \infty, & \bar u > u_c(\beta); \end{cases}$$

(26) $$\lim_{R\to\infty} p^R(\beta,\mu^R(\beta,\bar u)) = p_e(\beta,\mu(\beta,\bar u));$$

(27) $$\lim_{R\to\infty} s^R(\beta,\mu^R(\beta,\bar u)) = \begin{cases} s_e(\beta,\mu(\beta,\bar u)), & \bar u \leq u_c(\beta), \\ k\beta(\bar u + p_e(\beta,0)), & \bar u > u_c(\beta); \end{cases}$$

$$x^+ := \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Proof:

What remains to be proofed is the divergence of the mean particle number density for $\bar u > u_c(\beta)$, and the limits of pressure and entropy density. The first follows by

$$\rho^R(\beta,\mu^R(\beta,\bar u)) \geq g\, V_R^{-1}(e^{-\beta\mu^R(\beta,\bar u)} - 1)^{-1}$$

$$= \frac{1}{\varepsilon_1^R}\, g\, \varepsilon_1^R\, V_R^{-1}(e^{-\beta\mu^R(\beta,\bar u)} - 1)^{-1}$$

$$= \frac{1}{\varepsilon_1^R}\, u_g^R(\beta,\mu^R(\beta,\bar u))$$

$$\longrightarrow \lim_{R\uparrow\infty} \left(\frac{1}{\varepsilon_1^R}\right)(\bar u - u_c(\beta)) = \infty, \quad \text{for } R\to\infty.$$

The pressure behaviour is obtained as follows: For $\bar u < u_c(\beta)$, $\mu(\beta,\bar u) < 0$, hence $\lim_{R\to\infty} p_g^R(\beta,\mu^R(\beta,\bar u)) = 0$. For $\bar u > u_c(\beta)$,

$$p_g^R(\beta,\mu^R(\beta,\bar u))$$

$$= g(\beta V_R)^{-1} \log(1 - e^{\beta\mu^R(\beta,\bar u)})^{-1}$$

$$= g(\beta V_R)^{-1} \log\left\{\frac{V_R (V_R)^{q_3}}{g\,\varepsilon_1^R (V_R)^{q_3}}\, e^{-\beta\mu^R(\beta,\bar u)}\, u_g^R(\beta,\mu^R(\beta,\bar u))\right\}$$

$$= \frac{g}{\beta}(1+q_3)\, V_R^{-1} \log V_R$$

$$+ \frac{g}{\beta} V_R^{-1} \log\left\{(g\,\varepsilon_1^R V_R^{q_3})^{-1} e^{-\beta\mu^R(\beta,\bar u)}\, u_g^R(\beta,\mu^R(\beta,\bar u))\right\}$$

$$\xrightarrow[R\uparrow\infty]{} 0$$

where we used (4), (23), (24), and $\lim_{x\to\infty}(x^{-1}\log x) = 0$. Now (26) follows from Corollary 2.

To prove the entropy density limit we observe

$$-\mu^R(\beta,\bar u)\left(V_R^{-1}(e^{-\beta\mu^R(\beta,\bar u)} - 1)^{-1} + V_R^{-1}\sum_{k=2}^{\infty}(e^{\beta(\lambda_k^R - \mu^R(\beta,\bar u))} - 1)^{-1}\right)$$

$$\leq V_R^{-1}\frac{-\mu^R(\beta,\bar u)}{-\beta\mu^R(\beta,\bar u)} - \mu^R(\beta,\bar u)\, V_R^{-1}\sum_{k=2}^{\infty}(e^{\beta(\lambda_k^R - \mu^R(\beta,\bar u))} - 1)^{-1}$$

$$= V_R^{-1}\beta^{-1} - \mu^R(\beta,\bar u)\, V_R^{-1}\sum_{k=2}^{\infty}(e^{\beta(\lambda_k^R - \mu^R(\beta,\bar u))} - 1)^{-1}$$

$$\xrightarrow[R\uparrow\infty]{} \mu(\beta,\bar u)\, \rho_e(\beta,\mu(\beta,\bar u)).$$

qed.

3. Thermodynamic stability

Let us assume that we control an ideal boson gas by two independent restrictions giving rise to a grand-canonical distribution with an inverse temperature value β, and a mean energy value $\bar{u}$. Now we go on to fix the value $\bar{u}$ of the mean energy density, but we remove the β-restriction. The state which maximizes the entropy subject to one restriction only is the canonical state. For the infinite boson gas the inverse temperature β' of the canonical state corresponding to a mean energy density $\bar{u}$ is defined as

(28) $$u_c(\beta') := \bar{u}$$

This suggests to compare the value

(29) $$s(\beta, \bar{u}) := \lim_{R \to \infty} s^R(\beta, \mu^R(\beta, \bar{u}))$$

of the entropy of the grand-canonical state with the value

(30) $$s_c(\beta') := s_e(\beta', 0)$$

of the canonical state.

Proposition:

(31.a) $$s_c(\beta') > s(\beta, \bar{u}), \qquad \bar{u} < u_c(\beta),$$

(31.b) $$s_c(\beta') < s(\beta, \bar{u}), \qquad \bar{u} > u_c(\beta).$$

Proof:

a) $\bar{u} =: u_c(\beta') < u_c(\beta)$:

$$s(\beta, \mu) - s_c(\beta')$$

$$= k\beta\alpha\left\{(\Gamma(q+2)+\Gamma(q+1))\beta^{-(q+2)} g_{q+2}(e^{\beta\mu}) - \mu\Gamma(q+1)\beta^{-(q+1)} g_{q+1}(e^{\beta\mu})\right\}$$

$$- k\beta'\alpha(\Gamma(q+2)+\Gamma(q+1))\beta'^{-(q+2)} g_{q+2}(1)$$

$$= k\alpha\Gamma(q+1)(q+2)\left\{\beta^{-(q+1)}\sum_{n=1}^{\infty}\frac{e^{n\beta\mu}}{n^{q+2}}\left(1-\frac{n\beta\mu}{q+2}\right) - \beta'^{-(q+1)} g_{q+2}(1)\right\}$$

$$\leq k\alpha\Gamma(q+1)(q+2)\left\{\beta^{-(q+1)}\sum_{n=1}^{\infty}\frac{1}{n^{q+2}} - \beta'^{-(q+1)} g_{q+2}(1)\right\}$$

$$= k\alpha\Gamma(q+1)(q+2)\, g_{q+2}(1)\left\{\beta^{-(q+1)} - \beta'^{-(q+1)}\right\}$$

$$< 0,$$

where we used $q+2 > 1$, $x+1 < e^x$, and $\beta'/\beta = (u_c(\beta)/u_c(\beta'))^{1/(q+2)} > 1$.

b) $\bar{u} =: u_c(\beta') > u_c(\beta)$:

$$s(\beta, \bar{u}) - s_c(\beta')$$

$$= k\beta\bar{u} + k\beta\frac{\Gamma(q+1)}{\Gamma(q+2)} u_c(\beta) - k\beta'\left(1+\frac{\Gamma(q+1)}{\Gamma(q+2)}\right) u_c(\beta')$$

$$= k\beta\, u_c(\beta')\left\{1 + \frac{1}{q+1}\left(\frac{\beta'}{\beta}\right)^{q+2} - \frac{q+2}{q+1}\frac{\beta'}{\beta}\right\}$$

$$> 0,$$

since $\beta'/\beta < 1$, and

$$f(x) := 1 + \frac{1}{q+1} x^{q+2} - \frac{q+2}{q+1} x > 0,$$

for $0 < x < 1$; the latter is implied by $f(1) = 0$,

$$f'(x) = \frac{q+2}{q+1} (x^{q+1} - 1) < 0.$$

qed.

4. Application to a photon gas

We spell out Theorem 4 for a gas of free photons in a reflecting cavity. The single particle hamiltonian has to refer to Dirichlet data and is given by (6); assumptions (3) and (4) are satisfied. We have

$$(32) \qquad \alpha = \frac{1}{\pi^2}\left(\frac{1}{\hbar c}\right)^3; \quad q = 2; \quad q_1 = 1; \quad q_2 = 1; \quad g = 2.$$

The parameter R stands for the edge length of a cube, or the radius of a sphere etc., and V_R for the volume. The involved values of the gamma function are $\Gamma(q+2)=6$, $\Gamma(q+1)=2$.

Corollary 6: Let $\bar{u}, \beta \geqq 0$ be the values of the mean energy density, and the inverse temperature resp. Let $\mu^R(\beta,\bar{u})$ be the unique root of

$$(33) \qquad u^R(\beta, x) = \bar{u},$$

and, for $\bar{u} < u_c(\beta)$, $\mu(\beta,\bar{u})$ the unique root of

$$(34) \qquad u_e(\beta, x) = \bar{u}$$

where

$$(35) \qquad u_e(\beta,x) := \lim_{R\to\infty} \frac{1}{V_R} \sum_{\substack{k=2,3,\dots \\ \alpha=+1,-1}} (\lambda^R_{k,\alpha} + \varepsilon^R_1)\left(e^{\beta(\lambda^R_{k,\alpha} - x)} - 1\right)^{-1}$$

$$= \frac{3}{\beta} \frac{2}{\pi^2} (\beta\hbar c)^{-3} g_4(e^{\beta x}),$$

$$u_c(\beta) := u_e(\beta, 0),$$

and where $\lambda^R_{k,\alpha} := \lambda^R_k$ denotes the multiplicity due to helicity. For $\bar{u} \geq u_c(\beta)$ define $\mu(\beta,u) := 0$. Then we have

(36) $$\lim_{R\to\infty} \mu^R(\beta,\bar{u}) = \mu(\beta,\bar{u});$$

(37) $$\lim_{R\to\infty} u_g^R(\beta,\mu^R(\beta,\bar{u})) = (\bar{u} - u_c(\beta))^+;$$

(38) $$\lim_{R\to\infty} \rho^R(\beta,\mu^R(\beta,\bar{u})) = \begin{cases} \rho_e(\beta,\mu(\beta,\bar{u})), & \bar{u} \leq u_c(\beta), \\ \infty, & \bar{u} > u_c(\beta), \end{cases}$$

$$\rho_e(\beta,x) := \lim_{R\to\infty} \frac{1}{V_R} \sum_{\substack{k=2,3,\ldots \\ \alpha=+1,-1}} (e^{\beta(\lambda^R_{k,\alpha} - x)} - 1)^{-1}$$

$$= 2\pi^{-2}(\beta\hbar c)^{-3} g_3(e^{\beta x});$$

(39) $$\lim_{R\to\infty} p^R(\beta,\mu^R(\beta,\bar{u})) = \frac{1}{3} u_e(\beta,\mu(\beta,\bar{u}));$$

(40) $$\lim_{R\to\infty} s^R(\beta,\mu^R(\beta,\bar{u}))$$

$$= \begin{cases} k\beta \{ \frac{4}{3} u_e(\beta,\mu(\beta,\bar{u})) - \mu(\beta,\bar{u})\,\rho_e(\beta,\mu(\beta,\bar{u}))\}, & \bar{u} \leq u_c(\beta), \\ k\beta \{ \bar{u} + \frac{1}{3} u_c(\beta)\}, & \bar{u} > u_c(\beta). \end{cases}$$

Bose-Einstein condensation in the sense of macroscopic occupation of the ground state is given by equation (37). The infrared-divergence of the mean particle number density for over-critical mean energy density is given by (38). (39) expresses the equation of state of the photon gas. Beyond the critical density an increase in energy leaves the pressure invariant. The excess energy is absorbed in the ground state which does not contribute to the pressure.

To assess the thermodynamic stability of a grand-canonical state of a photon gas we introduce, in line with the common approach [8], a perturbation by a black body which shall not affect the value $\bar{u}$ of the mean energy density. However, by absorption and emission of photons in the course of time, this perturbation removes the β-restriction. In this situation Proposition 6 states that, below the critical density, the unconstrained canonical state is thermodynamically preferred while, above the critical density the grand-canonical state is preferred. This means that the photon gas is properly described by a grand-canonical state with inverse temperature β and chemical potential $\mu=0$. Because of the phase transition the grand-canonical state with $\mu=0$ is not equivalent to the unconstrained canonical state.

5. Discussion

1. In this note we have described an infinite ideal boson gas in terms of the inverse temperature β and the mean energy density u, in contrast to the traditional description which uses the mean particle number density ρ instead of u [1-6]. In the normal regime both descriptions are equivalent. Equally any two other variables could have been singled out, including the pressure. The differences emerge in the condensation regime. For $\rho_e(\beta,0) < \rho < \infty$, the ground state neither contributes to the energy density, nor to the pressure, nor to the entropy density. For $u_c(\beta) < u < \infty$, the particle number density diverges, and the ground state contributes to the entropy density, but not to the pressure (Theorem 4). For $p_e(\beta,0) < p < \infty$, a similar calculation implies divergences in the particle number density, in the energy density, and in the entropy density; this is just another way of saying that the pressure cannot exceed the critical value $p_e(\beta,0)$.

2. In the limit where the mean particle number density is fixed the spectral requirement for Bose-Einstein condensation is $q > 0$. In the mean energy density limit it can be weakened to $q > -1$ which coincides with the requirement for the existence of the Laplace transformed of $dF(\lambda)/d\lambda$.

3. Under a Lorentz transformation with velocity v the mean energy density transforms as $u \to (1-(v/c)^2)^{-1}u$, the inverse temperature as $\beta \to (1-(v/c)^2)^{-1/2}\beta$ [9]. Since $u_c(\beta)$ is proportional to β^{-4}, the condensed density $u-u_c(\beta)$ is not invariant under Lorentz transformations. This indicates a spontaneous break down of Lorentz invariance.

4. The hypothesis of a Bose-Einstein condensation of photons is not unknown in the literature. In [10], admittedly speculatively, P.T. Landsberg suggests the photon condensate to be matter. In [11] some aspects of the problem are exposed. The same suggestion was advocated by P. Roman and C.F. v. Weizsäcker [12]; J.D. Becker and L. Castell [13] discuss it in a cosmological framework, but the basic assumption of a finite radius of the universe is mathematically incompatible with a phase transition.

From a mathematical point of view a finite cavity can be taken as an element in an increasing sequence of enclosures to spell out the meaning of the thermodynamic limit. Equally the thermodynamic limit can be held as a mean to extract the dominating behaviour of a finite system. In this view the apparently paradoxical fact that a state with non-vanishing energy is built up by photons of limit zero energy is resolved. At the same time a physical meaning for the condensate is provided: The lowest energy state in any finite reflecting cavity is given by a non-vanishing electromagnetic wave. In the case of macroscopic occupation of the ground state, a monochromatic, necessarily coherent wave is formed in this state [3]. - A priori, this view, on a cosmological scale, appears not to be in conflict with the above mentioned condensation hypothesis of matter.

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