ON THE λ-SINGULARITY IN THE SPECIFIC HEAT OF A FREE BOSON GAS IN A GRAVITATIONAL FIELD

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We show that the specific heat per particle (at a constant volume) of a free boson gas in a weak gravitational field has a λ-type singularity. The magnitude of the jump is proportional to the square root of the field strength for small fields.

Internal report nr. 177
July, 1981

F. London [1] attributed the λ-transition in liquid helium to the Bose-condensation of helium atoms. In an ideal boson gas having the same density as liquid helium, Bose-condensation would occur at 3.14°K which is in rough agreement with the observed λ-transition temperature of 2.19°K. However, the singularity in the specific heat at constant volume as a function of temperature is a shallow cusp in the case of the ideal boson gas. It does not resemble strongly the apparent λ-discontinuity in the curves obtained from experiments with liquid helium. Various modifications of the ideal boson gas model have been proposed in order to remedy the discrepancy. Bijl, de Boer and Michels [2] considered the effect on the singularity of a gap in the single-particle spectrum. Gunton and Buckingham [3] pointed out that a change in the power law of the single-particle energy spectrum would produce a change in the specific heat singularity. In neither case were the modifications derived from a detailed quantum-mechanical treatment of a boson gas. In recent papers [4], [5] we have considered the effect of a weak external potential on the Bose-condensation of an ideal gas. In this letter we use our results to discuss the effect of the gravitational field on the singularity, and show that it produces a discontinuity in the specific heat and an increase in the critical temperature.

Consider a gas of non-interacting bosons each of mass m, in a rectangular box of sides a, b, b, with the side of length a vertical, and let g be the acceleration due to gravity. Let $\beta = (kT)^{-1}$ be the inverse temperature. In order that the system may be described thermodynamically, it is necessary on the one hand that both a and b be large in comparison with the thermal wave-length $\lambda = (2\pi\hbar^2\beta/m)^{1/2}$, and on the other that the variation $\delta = mga$ of the potential of a particle over the sample be small in comparison with the thermal energy. The results we describe are strictly valid in the thermodynamic limit in which the number of particles and the volume of the container become infinite while the density remains fixed, and the external potential is scaled. We expect them to be good approximations for a finite system provided that both $a/_\lambda$ and $b/_\lambda$ are much larger than one and that $\beta\delta$ is much smaller than one (except very close to the critical point). We use the functions g_r, defined for $r > 1$ and $z \leqslant 1$ by

$$g_r(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^r}, \tag{1}$$

whose asymptotic expansions for $|z-1|$ small were obtained by Robinson [6] (see also [7]). The critical inverse temperature β_c is determined by the equation

$$\left(\frac{2\pi\beta_c\hbar^2}{m}\right)^{3/2} \rho = \frac{1}{\beta_c\delta}\left(g_{5/2}(1) - g_{5/2}(e^{-\beta_c\delta})\right), \tag{2}$$

where ρ is the particle density. Using [6] we have

$$\beta_c = \beta_0\left(1 - \frac{2}{3}\frac{\Gamma(-3/2)}{g_{3/2}(1)}(\beta_0\delta)^{1/2} + \ldots\right)$$

$$\simeq \beta_0\left(1 - 0.603\ (\beta_0\delta)^{1/2} + \ldots\right), \tag{3}$$

where β_0 is given by

$$\beta_0 = \frac{m}{2\pi\hbar^2}\left(\frac{g_{3/2}(1)}{\rho}\right)^{2/3}. \tag{4}$$

The equation of state is given by

$$p(z) = \frac{1}{\beta^2\lambda^3\delta}\left(g_{7/2}(z) - g_{7/2}(z\,e^{-\beta\delta})\right)$$

where $z = 1$ for $\beta > \beta_c$ and z is given for $\beta < \beta_c$ by

$$\rho = \frac{1}{\beta\lambda^3\delta}\left(g_{5/2}(z) - g_{5/2}(z\,e^{-\beta\delta})\right). \tag{5}$$

The specific heat per particle at constant volume is given by

$$\frac{C_V}{k} = \frac{21}{4}\left(\frac{\beta_c}{\beta}\right)^{5/2} \frac{g_{7/2}(1) - g_{7/2}(e^{-\beta\delta}) - \frac{2}{7}\ \beta\delta\, g_{5/2}(e^{-\beta\delta})}{g_{5/2}(1) - g_{5/2}(e^{-\beta_c\delta})}, \quad \beta > \beta_c$$

$$\frac{C_V}{k} = \frac{21}{4}\ \frac{g_{7/2}(z) - g_{7/2}(z\,e^{-\beta\delta}) - \frac{2}{7}\, g_{5/2}(z\,e^{-\beta\delta})}{g_{5/2}(z) - g_{5/2}(z\,e^{-\beta\delta})}$$

$$- \frac{3}{2}\ \frac{g_{7/2}(z) - g_{7/2}(z\,e^{-\beta\delta}) + \frac{3}{2}\, g_{5/2}(z) - \left(\frac{3}{2} + \beta\delta\right) g_{5/2}(z\,e^{-\beta\delta})}{g_{3/2}(z) - g_{3/2}(z\,e^{-\beta\delta})},$$

$$\beta < \beta_c . \tag{6}$$

So $\frac{C_V}{k}$ has a jump at $T = T_c$ of magnitude

$$\Lambda = \frac{3}{2}\ \frac{g_{7/2}(1) - g_{7/2}(e^{-\beta_c\delta}) + \frac{3}{2}\, g_{5/2}(1) - \frac{3}{2} + \beta_c\delta\ \ g_{5/2}(e^{-\beta_c\delta})}{g_{3/2}(1) - g_{3/2}(e^{-\beta_c\delta})}. \tag{7}$$

Using [6] we have for $\beta_0\delta \ll 1$

$$\Lambda = -\frac{9}{4}\ \frac{g_{3/2}(1)}{\Gamma(-1/2)}\,(\beta_0\delta)^{1/2} + \ldots = 1.658\ (\beta_0\delta)^{1/2} + \ldots . \tag{8}$$

We see from (8) that the magnitude of the jump Λ in the specific heat for weak fields is proportional to the square root of the ratio of the variation in the potential energy of a particle over the container to the thermal energy per degree of freedom. (In the limit δ approaches zero we obtain the corresponding expressions for the free boson gas given in [8]. We find also

$$\lim_{\beta\downarrow\beta_c} \frac{C_V}{k} = \frac{21}{4} \frac{g_{7/2}(1) - g_{7/2}(e^{-\beta_c\delta}) - \frac{2}{7} g_{5/2}(e^{-\beta_c\delta})}{g_{5/2}(1) - g_{5/2}\left(e^{-\beta_c\delta}\right)}$$

$$\simeq \frac{15}{4} \frac{g_{5/2}(1)}{g_{3/2}(1)} \left\{ 1 + \frac{\Gamma(-3/2)}{g_{3/2}(1)} (\beta_0\delta)^{1/2} + \dots \right\}, \tag{9}$$

and

$$\lim_{\beta\uparrow\beta_c} \frac{C_V}{k} = \lim_{\beta\downarrow\beta_c} \frac{C_V}{k} - \Delta . \tag{10}$$

It follows from (6) that all the right-hand derivatives of the specific heat with respect to the temperature are minus infinity (they become finite in the limit $\beta_0\delta$ approaches zero), whereas $\lim_{T\uparrow T_c} \frac{C_V}{k}$ is finite. In detail we have

$$\left(\lim_{T\downarrow T_c} C_V(T)\right) - C_V(T) \sim \text{constant } |T - T_c|^{1/2} , \; T > T_c, \tag{11}$$

and

$$- C_V(T) + \left(\lim_{T\uparrow T_c} C_V(T)\right) \sim \text{constant } |T - T_c| , \; T < T_c . \tag{12}$$

Thus the singularity is of the λ-type.

For liquid helium at the transition temperature we have $a/\lambda = 1.8 \times 10^9$ and $\beta_c\delta = 2.1 \times 10^{-4}$ if $a = 10$ cm and $g = 9.8 \times 10^2$ cm/s^2.

Some of the results in this paper were obtained earlier by Gersch [9]. In particular expression (8).

The work described in this letter is part of the research program of the "Foundation for Fundamental Research on Matter", which is financially supported by the "Netherlands Organization for Pure Research".

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