DIAS-STP-80-36

"On the Quantization of Multilinear Momentum Observables"

by

K. McFarlane, School of Theoretical Physics, Dublin Institute for Advanced Studies, 10, Burlington Road, DUBLIN 4, Eire.

and

K.K. Wan, Department of Theoretical Physics, University of St. Andrews. ST. ANDREWS, Fife, Scotland.

Summary: A generally accepted set of axioms of quantization is introduced and applied to the quantization of the multilinear momentum observables so as to delimit the possible forms of the corresponding quantum operators. This analysis leads to a canonical decomposition of the quantum observables into a series of symmetric operators each of which is determined by an unknown auxilliary tensor generated by the multilinear momentum. Methods of removing the residual indeterminateness in the differential operators are then critically reviewed, and a particular choice illustrated by means of examples defined on the real line.

Physics Abstracts Classification No: 03.65: Quantum Theory, Quantum Mechanics.

1. - Introduction

We develop in this paper some aspects of the quantization problem for the multilinear momentum observables, and begin our discussion with a definition: An observable A is a multilinear momentum observable (multilinear momentum) if and only if, in every chart (U_α,α) of the cotangent bundle (phase space) T^*M of a Riemannian (configuration) manifold M, A has relative to the coordinate system $\{(x^i,p_i) \mid i\in[1,m]\}$ (*) the tensor form (†)

$$A = a^{i_1\ldots i_n}(x^k)\, p_{i_1}\cdots p_{i_n} \quad , \; n\in[0,\infty] \,, \tag{1}$$

in which p_{i_k} denotes the momentum conjugate to the coordinate x^{i_k}, and $a^{i_1\ldots i_n}$ a fully symmetric contravariant tensor of order n.

Ideally we should implement the programme of geometric quantization; that is we should first identify a symmetric differential operator $Q_o(A)$, defined on the set $C_o^\infty(M)$, of infinitely differentiable functions of compact support, and naturally associated with the classical observable A ; then ascertain whether a given A is quantizable by testing $Q_o(A)$ for essential self-adjointness; and finally determine the explicit form of the quantum observable by the calculation of the (unique) adjoint $Q_o^\dagger(A)$. It is however clear that such a programme could not, in the case of the multilinear momenta, be carried out either to completion or with full rigor; for in the first part of the programme no generally agreed mathematical or physical principle is known which will determine $Q_o(A)$, and in the second the mathematical difficulties involved in the analytic manipulation, as is required, of n^{th} order partial differential operators

---

(*) The notation $[m,n]$ is a shorthand for the set of integers $\{m,m+1,m+2,\ldots,n\}$, infinity being excluded.

(†) For an explanation of the notation when $n=0$ as well as for all other implicit notational conventions, refer to appendix A.

(1) F. Bloore Colloques Internationaux C.N.R.S no 237, pp 299-303.

are all but insuperable. It is thus not surprising that there seems to have been no previous systematic and rigorous discussion of the quantization of multilinear momenta, though much work has been carried out, more especially into the problems surrounding the bilinear momenta and the Hamiltonian (BLOORE ([1]), BLOORE and ROUTH ([2]), BLOORE, ASSIMAKOPOLOUS, and GHOBRIAL ([3]), CASTELLANI ([4]), UNDERHILL and TARAVIRAS ([5]), WAN and VIASMINSKY ([6]), ([7])).

We shall therefore be more modest in our goals, both in system and in rigour and shall limit ourselves instead to the following problems: (i) the determination of the most general form of the differential operator $Q_o(A)$ so as to delimit the degree of uncertainty in $Q_o(A)$ which needs to be resolved by some further mathematical or physical principle; (ii) the discussion and review, especially in the case of the bilinear momenta, of some means of prescribing a unique differential expression for $Q_o(A)$;and finally (iii) the exemplification of some features of our discussion by means of an explicitly worked example.

([2]) F. BLOORE and L. ROUTH Il Nuovo Cimento 47B, 78-84 (1978)

([3]) F. BLOORE, M. ASSIMAKOPOULOS, and I.R. GHOBRIAL J. Math. Phys 17, 1034-1038 (1976)

([4]) L. CASTELLANI Il Nuovo Cimento 48A, 359-363 (1978).

([5]) J. UNDERHILL and S. TARAVIRAS Lecture Notes in Physics 50, pp 210-216 (New York, 1976)

([6]) K.K. WAN and C. VIASMINSKY Prog. Theor. Phys. 58. 1030-1044 (1977).

([7]) K.K. WAN and C. VIASMINSKY J. Phys. A:Math.Gen. 12, 643-647 (1979).

2. - On the general form of the symmetric operators $Q_o(A)$.

We consider in this section the effect of three axioms of quantization which, while clearly necessary, are not sufficient for the unique determination of the formal observable $Q_o(A)$, but which nevertheless yield much insight into its general form. These axioms, which will require no detailed justification here, are as follows:

axiom 1: $Q_o(A)$ has a differential expression given in terms of the coordinates $\{x^i \mid i \in [1,m]\}$ of a local chart of the configuration manifold by the partial differential operator

(2) $$Q_o(A) = (-i\hbar)^n \sum_{k=0}^{n} \eta^{i_1 \dots i_k} \partial_{i_1} \dots \partial_{i_k} ,$$

in which the coefficients $\eta^{i_1 \dots i_k}(x^\ell)$ are fully symmetric in all indices and are assumed to be real-valued, and in which in particular $\eta^{i_1 \dots i_n} = a^{i_1 \dots i_n}$, as is in accordance with the formal prescription

(3) $$Q_o(p_i) = -i\hbar(\partial_i + \tfrac{1}{2}\,\mathrm{div}(\partial/\partial x^i)) .$$

axiom 2 : $Q_o(A)$ is a symmetric operator defined on the domain $C_o^\infty(M)$ of infinitely differentiable functions of compact support.

axiom 3 : For all states ψ of $C_o^\infty(M)$, $Q_o(A)\psi$ transforms as an invariant, as does the wave-function ψ itself.

The most general differential expression compatible with the above axioms then has, as is demonstrated in appendix B, the form

(4) $$Q_o(a^{i_1 \dots i_n} p_{i_1} \dots p_{i_n}) = (-i\hbar)^n \sum_{k=0}^{n} b^{i_1 \dots i_k} \delta_{i_1} \dots \delta_{i_k} ,$$

in which $b^{i_1 \dots i_\ell}(x^m)$ is a fully symmetric contravariant tensor of order ℓ , and in which δ_{i_ℓ} denotes the action of the covariant derivative with respect to the coordinate x^{i_ℓ} . Additionally the tensors $b^{i_1 \dots i_k}$ satisfy the "initial condition"

$$b^{i_1 \dots i_n} = a^{i_1 \dots i_n} , \tag{5}$$

and the recurrence relation

$$b^{i_1 \dots i_\ell} = \sum_{k=\ell}^{n} (-1)^{n+k} \binom{k}{\ell} b^{i_1 \dots i_k}{}_{|i_{\ell+1} \dots i_k} , \ \ell \in [0, n] , \tag{6}$$

in which $\binom{k}{\ell}$ denotes a binomial coefficient, and the subscript vertical bar covariant differentiation. Note particularly that the tensors $b^{i_1 \dots i_k}$ and $b^{i_1 \dots i_\ell}$ with $k \neq \ell$ are not, in general, related (for example by contraction) other than by the recurrence (6), and that the quantities $b^{i_1 \dots i_k}$ may be functions of the parameter $\hbar$ as well as of the coordinates x^k .

Study of the recurrence (6) shows that to prescribe a unique differential expression for $Q_0(A)$ it is necessary to specify, in addition to the quantity $a^{i_1 \dots i_n} = b^{i_1 \dots i_n}$, the tensors

$$b^{i_1 \dots i_{n-2k}} , \ k \in [1, [\tfrac{1}{2}n]] , \tag{7}$$

in which $[\frac{1}{2}n]$ denotes the integer part (*) of $\frac{1}{2}n$. We may recast this result as follows: Given a classical observable $B_v = b^{i_1 \dots i_v} p_{i_1} \dots p_{i_v}$, we may define a symmetric operator on the domain $C_0^\infty(M)$ by the differential expression

$$\Xi_0(B_v) = (-i\hbar)^v \sum_{k=0}^{v} \alpha_k^v \, b^{i_1 \dots i_v}{}_{|i_{k+1} \dots i_v} \, \delta_{i_1} \dots \delta_{i_v} , \tag{8}$$

(*) Generally we define $[x]$ for any real x as the supremum of $\{n \in \mathbb{Z} \mid n < x\}$, in which $\mathbb{Z}$ denotes the integers.

in which the coefficients α_k^v are real numbers satisfying the recurrence relation

$$\alpha_\ell^v = \sum_{k=\ell}^{v} (-1)^{v+k} \binom{k}{\ell} \alpha_k^v , \quad \alpha_v^v = 1 . \tag{9}$$

Hence regarding the coefficients $b^{i_1 \dots i_k}$ in the expression (4) of $Q_0(A)$ as arising from some classical observable $B_k = b^{i_1 \dots i_k} p_{i_1} \dots p_{i_k}$, we may rewrite $Q_0(A)$ as

$$Q_0(A) = \sum_{k=0}^{[\frac{1}{2}n]} (-i\hbar)^{2k} \, \Xi_0(B_{n-2k}) , \tag{10}$$

when the coefficients of the tensorial expansion (6) assume the form

$$b^{i_1 \dots i_\ell} = \sum_{k=0}^{[\frac{1}{2}(n-\ell)]} \alpha_\ell^{n-2k} \, b^{i_1 \dots i_{n-2k}}{}_{|i_{\ell+1} \dots i_{n-2k}} , \quad \ell \in [0, n] . \tag{11}$$

Note especially that the canonical decomposition (10) exists for every choice of the coefficients α_k^v consistent with the system (9).

We elect for definiteness in the sequel that

$$\alpha_k^{2v} = \binom{v}{k-v} , \quad \alpha_k^{2v+1} = \binom{v+1}{k-v} - \tfrac{1}{2}\binom{v}{k-v} , \quad v \in [0, \infty] , \tag{12}$$

in which the undefined binomial coefficients are identically zero, since these coefficients result in the maximum number of lower order terms in the expression of $\Xi_0(B_v)$ being zero, and assist in symmetry induced surface integral calculations when $Q_0(A)$ is extended to larger domains, and since then $\Xi_0(B_v)$ has the manifestly symmetric forms

$$(13) \quad \Xi_o(B_v) = (-i\hbar)^v \delta_{i_1}\cdots\delta_{i_\mu} a^{i_1\cdots i_v}\delta_{i_{\mu+1}}\cdots\delta_{i_{2\mu}}, \quad v=2\mu,$$

$$\Xi_o(B_v) = (-i\hbar)^v\delta_{i_1}\cdots\delta_{i_\mu}\tfrac{1}{2}\{\delta_{i_{\mu+1}}, a^{i_1\cdots i_v}\}_+\delta_{i_{\mu+2}}\cdots\delta_{i_v}, \quad v=2\mu+1,$$

in which $\{\ ,\ \}_+$ denotes the anticommutator bracket of operators.
This assumption then results in the expressions

$$(14) \quad \Xi_o(a^i p_i) = (-i\hbar)(a^i\delta_i + \tfrac{1}{2}a^i_{|i}),$$

$$\Xi_o(a^{ij}p_ip_j) = (-i\hbar)^2(a^{ij}\delta_i\delta_j + a^{ij}_{|j}\delta_i),$$

$$\Xi_o(a^{ijk}p_ip_jp_k) = (-i\hbar)^3(a^{ijk}\delta_i\delta_j\delta_k + \tfrac{3}{2}a^{ijk}_{|k}\delta_i\delta_j + \tfrac{1}{2}a^{ijk}_{|jk}\delta_i),$$

$$\Xi_o(a^{ijkl}p_ip_jp_kp_l) = (-i\hbar)^4(a^{ijkl}\delta_i\delta_j\delta_k\delta_l + 2a^{ijkl}_{|l}\delta_i\delta_j\delta_k + a^{ijkl}_{|kl}\delta_i\delta_j),$$

and in the corresponding canonical decompositions

$$(15) \quad Q_o(a^ip_i) = (-i\hbar)(a^i\delta_i + \tfrac{1}{2}a^i_{|i}),$$

$$Q_o(a^{ij}p_ip_j) = (-i\hbar)^2(a^{ij}\delta_i\delta_j + a^{ij}_{|j}\delta_i + b(a^{ij})),$$

$$Q_o(a^{ijk}p_ip_jp_k) = \Xi_o(a^{ijk}p_ip_jp_k) - \hbar^2\Xi_o(b^k(a^{ijk})p_k),$$

$$Q_o(a^{ijkl}p_ip_jp_kp_l) = \Xi_o(a^{ijkl}p_ip_jp_kp_l) - \hbar^2\Xi_o(b^{ij}p_ip_j) + \hbar^4 b(a^{ijkl}),$$

in which the quantities $b(a^{ij}), b^k(a^{ijk}), b^{ij}(a^{ijkl})$, and $b(a^{ijkl})$ are undetermined tensors of the indicated type.

The canonical decomposition (10), as embodied in the lowest order examples (15), precisely circumscribes the degree of arbitrariness

remaining in the differential expression of $Q_o(A)$. The problem of the formal quantization of the multilinear momenta has thus been reduced to the determination of the tensor quantities $B_{n-2k}(A)$, $k \in [1,[\tfrac{1}{2}n]]$, a task to which we shall now turn.

3. - <u>On the determination of the quantities $B_{n-2k}(A)$.</u>

We find it convenient to divide our discussion into two parts; the first, more detailed, concerned with the special case of the bilinear observable $a^{ij}p_ip_j$; and the second, rather speculative, concerned with the most general case.

3.1 - <u>On the form of the operator $Q_o(a^{ij}p_ip_j)$.</u>

We here outline and contrast three distinct methods whereby the function $b(a^{ij})$ in the expression for $Q_o(a^{ij}p_ip_j)$ may be determined, and begin our discussion of the first of these procedures with an axiom:

<u>axiom 4</u>: Formal quantization is such as to preserve the constants of free motion, so that

$$(16) \quad Q_o(\{a^{ij}p_ip_j, g^{ij}p_ip_j\}) = -i\hbar^{-1}[Q_o(a^{ij}p_ip_j), Q_o(g^{ij}p_ip_j)],$$

in which $\{\ ,\ \}$ denotes the poisson bracket, $[\ ,\ ]$ the commutator bracket, and g^{ij} the contravariant metric tensor of the configuration space.

This axiom has, as have other more general axioms, been studied by BLOORE [1], BLOORE and GHOBRIAL [8], BLOORE and ROUTH [2], BLOORE, ASSIMAKOPOULOS, and GHOBRIAL [3], who have obtained the following results: First the assumption of linearity

(17) $$Q_o((\alpha a^{ij}+\beta b^{ij})p_ip_j)=\alpha Q_o(a^{ij}p_ip_j)+\beta Q_o(b^{ij}p_ip_j),$$

together with conservation under quantization of constants of the free motion yields for a conserved bilinear momentum the expression

(18) $$b(a^{ij}) = \tfrac{1}{4}a^{i|j}{}_{i|j} = \tfrac{1}{4}g^{jk}a^{i}{}_{i|jk},$$

as may alternatively be demonstrated (*) from (16) assuming in place of linearity that the free quantum Hamiltonian is, as in MACKEY's ([9]) scheme, simply the Laplacian, And second if the expression (18) is held to obtain for all second order observables, then the above scheme is incompatible with the DIRAC ([10]) correspondence

(19) $$Q_o(\{a^{ij}p_ip_j, b^kp_k\}) = -i\hbar^{-1}[Q_o(a^{ij}p_ip_j), Q_o(b^kp_k)],$$

else in the special case where the momentum b^kp_k is associated with a Killing vector field.

([8]) F. BLOORE and I.R. GHOBRIAL J. Phys. A: Math. Gen 8 1863 - (1975)

(*) We omit the demonstration which, while lengthy, is a straightforward application of Ricci's identities (I.S.SOKOLNIKOFF Tensor Analysis and Applications to Geometry, second edition, (New York, 1964)).

([9]) G.W. MACKEY The Mathematical Foundations of Quantum Mechanics (Reading, Mass. 1963)

([10]) P.A.M. DIRAC Principles of Quantum Mechanics (Oxford, 1958)

An alternative procedure for specifying $b(a^{ij})$ is to require that

axiom 5: in the case where the bilinear observable is the square of a momentum, quantization is in accordance with the squaring axiom ([11]),

(20) $$Q_o(b^ib^jp_ip_j) = Q_o^2(b^ip_i).$$

What is surprising is that this axiom is inconsistent (*) with the axiom of linearity (17).

As a final method of quantization we propose the following:

axiom 6: The formal observables $Q_o(a^{ij}p_ip_j)$ are such that

(21) $$b(a^{ij}) = \alpha a^{i|j}{}_{i|j} + \beta a^{ij}{}_{|ij},$$

in which α and β are real constants, and moreover satisfy the requirement that each positive-definite observable has as quantum analogue a positive operator.

This results, after substituting from various examples (appendix C), in the ideally simple form

(22) $$b(a^{ij})=0.$$

Turning now to compare the above methods of quantization we perceive that, whereas each is based upon a natural correspondence,

(11) G. TEMPLE Nature 135, 957 (1935).

(*) Consider the examples on the real line with Cartesian coordinate x $P_1 = \sin(x)p$, $P_2=\cos(x)p$, and compare the expression $Q_o(P_1^2+P_2^2)$ with $Q_o^2(P_1)+Q_o^2(P_2)$.

each is nevertheless inconsistent with all the others. There would seem, at present, to be no overwhelming reason to prefer any one of the above procedures to any other. However to be definite in the sequel we shall assume that the choice of axiom 6 is the correct one, since this results in the ideally simple canonical decomposition

(23) $$\mathcal{Q}_o(a^{ij}p_ip_j) = \Xi_o(a^{ij}p_ip_j),$$

and, as has been demonstrated by KIMURA (12), in the attractive equivalent expression

(24) $$\mathcal{Q}_o(a^{ij}p_ip_j) = g^{-1/4}(-i\hbar\partial_i)g^{1/4}a^{ij}g^{1/4}(-i\hbar\partial_j)g^{-1/4},$$

in which as usual g denotes the determinant of the metric tensor.

3.2: Some remarks upon determining the quantities $B_{n-2k}(A)$.

We here outline and briefly discuss two possible methods of determining in the general case the quantities $B_{n-2k}(A)$, $k\in[0,[\frac{1}{2}n]]$, and tentatively find in favour of the second. Consider firstly the general axiom:

axiom 7: The formal quantum operators $\mathcal{Q}_o(A)$ obey the relation

(25) $$\mathcal{Q}_o(\{g^{ij}p_ip_j, A\}) = -i\hbar^{-1}[\mathcal{Q}_o(g^{ij}p_ip_j), \mathcal{Q}_o(A)]..$$

Whilst this is a natural and physically appealing rule, it is nevertheless, as was demonstrated by BLOORE, ASSIMAKIPOULOS, and GHOBRIAL (3), inconsistent, the only cases where (25) uniquely determines the operators $\mathcal{Q}_o(A)$ (of at most second order) corresponding

(12) T. KIMURA Prog. Theor. Phys. 58, 1261-1277 (1971)

to reducible configuration manifolds, either of one-dimension, or of vanishing Ricci tensor, or of constant curvature. We may conclude therefore that this axiom has no general applicability in the quantization of multilinear momenta. Alternatively we may be less ambitious and demand instead that the system (25) be valid only when A is a constant of the free motion. The disadvantage of this scheme lies in the extreme computational difficulty involved in the explicit calculation even of the lowest order observables $\mathcal{Q}_o(A)$.

Restricting our attention for the moment to one-dimensional manifolds, we propose as our second axiom of quantization the following:

axiom 8: The formal quantization of the multilinear momenta is such that the class of quantizable momenta is, in a sense to be made explicit below, maximally large, the quantities $B_{n-2k}(A)$ being assumed to have the general form

(26) $$B_{n-2k}(A) = \epsilon^n_{2k}\, a^{i_1\dots i_n}{}_{|i_{n-2k+1}\dots i_n}\, p_{i_1}\cdots p_{i_{n-2k}},\ k\in[0,[\tfrac{1}{2}n]],$$

in which the quantities ϵ^n_{2k} are real constants.

We next observe that a necessary condition for the essential self-adjointness of $\mathcal{Q}_o(A)$ on the interval manifold $M=(a,b)$ of the real line is that the following boundary conditions be satisfied

(27) $$\epsilon^n_{2k}\, a^{i_1\dots i_n}{}_{|i_{n-2k+1-\rho}\dots i_n}(c) = 0,\ c = a \text{ or } b,\ \rho\in[0,[\tfrac{1}{2}n-\tfrac{1}{4}]],$$

a result which may be obtained (*) by computing the symmetry induced boundary conditions for the extension of $\mathcal{Q}_o(A)$ to the domain set $C^\infty(M)$ of infinitely differentiable functions on M. The system

(27) may be illustrated for the special case $n=5$ say, for which we obtain the equations

$$\text{(28)}\quad \begin{aligned} &a^{11111}(c)=0\ , \quad \epsilon^5_2 a^{11111}_{111}(c)=0\ , \quad \epsilon^5_4 a^{11111}_{11111}(c)=0\ ;\\ &a^{11111}_{11}(c)=0\ , \quad \epsilon^5_2 a^{11111}_{1111}(c)=0\ ,\\ &a^{11111}_{111}(c)=0\ , \end{aligned}$$

in which $c=a$ or b, and where we have noted that by (5) $\epsilon^n_o=1$. It is now immediate from the pattern illustrated in (28) above that independent of the choice of (a,b), the number of equations of constraint will be minimum provided only that

$$\text{(29)}\qquad \epsilon^n_{2k}=0\ , \quad k\in[1,[\tfrac{1}{2}n-\tfrac{1}{4}]] .$$

This set of equations uniquely determines the observables of odd order, and determines those of even order to within the scalar function $\epsilon^n_n a^{i_1\dots i_n}_{i_1\dots i_n}$. Preservation of positivity under quantization is then sufficient (†) to set $\epsilon^n_n=0$, so that for the case of one-dimensional manifolds axiom 8 leads to the quantization rule

$$\text{(30)}\qquad Q_o(A)=\Xi_o(A).$$

It is now natural to suppose, at least for the purposes of the sequel, that (30) holds quite generally for all manifolds and all observables A.

(*) This is a very long and somewhat technical calculation details of which may be found in K. McFarlane, Ph.D Thesis (St. Andrews University 1980)

(†) The demonstration is by means of explicitly worked examples on the real line.

4 - An illustration of the proposed quantization scheme.

To facilitate comparison with other methods of quantization, we shall develop explicit expressions for certain quantum observables defined on the real line $\mathbb{R}$ with the usual metric and endowed with the Cartesian coordinization $\{x|x\in\mathbb{R}\}$. More precisely we shall, for a representative group of infinitely differentiable functions f of the (complete) momentum xp determine the coefficients of the expression

$$\text{(31)}\qquad Q_o(f(xp))=\sum_{k=0}^{\infty} a^f_k Q^k_o(xp)\ ,$$

as will accord with the rule of quantization

$$\text{(32)}\qquad Q_o(f(xp))=\sum_{k=0}^{\infty} f^{(k)}(0)\ \Xi_o(x^k p^k)/k!\ .$$

This rule of quantization is itself obtained by performing a Taylor expression of f in the argument xp, by assuming a generalisation of the linearity equation (17), and by applying the quantization rule (30). Turning our attention first of all to the observables $x^k p^k$, we deduce after some calculation (appendix D) that

$$\text{(33)}\qquad Q_o(x^k p^k)=\sum_{j=0}^{k}(-i\hbar)^{k-j}\beta^k_j\, Q^j_o(xp)\ ,$$

the coefficients β^k_j of which are prescribed by the system

$$\text{(34)}\qquad \beta^{2\ell}_j=\beta^{2\ell+1}_{j+1}=\sum_{i=j}^{2\ell} S^i_{2\ell}\binom{i}{j}(\ell-\tfrac{1}{2})^{i-j}\ , \ell\in[0,\infty]\ , j\in[0,2\ell]\ ,$$

S^i_j denoting a Stirling number(13) of the first kind, and in particular satisfy the symmetry conditions $\beta^{2\ell}_{2j+1} = \beta^{2\ell-1}_{2j} = 0$, $\ell \in [1,\infty]$, $j \in [0,\ell]$. Concretely we obtain the lowest order decompositions

$$\mathcal{Q}_0(x^2p^2) = \mathcal{Q}_0^2(xp) + \tfrac{1}{4}\hbar^2, \tag{35}$$

$$\mathcal{Q}_0(x^4p^4) = \mathcal{Q}_0^4(xp) + \tfrac{5}{2}\hbar^2\mathcal{Q}_0^2(xp) + \tfrac{9}{16}\hbar^4,$$

$$\mathcal{Q}_0(x^6p^6) = \mathcal{Q}_0^6(xp) + \frac{35}{4}\hbar^2\mathcal{Q}_0^4(xp) + \frac{259}{16}\hbar^4\mathcal{Q}_0^2(xp) + \frac{125}{64}\hbar^6,$$

from which the corresponding odd-order expansions follow by means of the relation $\mathcal{Q}_0(x^{2\ell+1}p^{2\ell+1}) = \mathcal{Q}_0(xp)\,\mathcal{Q}_0(x^{2\ell}p^{2\ell})$. To complete (*) our discussion we calculate (appendix D) the formal expansions of certain transcendental functions to obtain

$$\mathcal{Q}_0(\sin xp) = \sin\left(2/\hbar \sinh^{-1}(\hbar/2)\,\mathcal{Q}_0(xp)\right), \tag{36}$$

$$\mathcal{Q}_0(\cos xp) = \cos\left(2/\hbar \sinh^{-1}(\hbar/2)\,\mathcal{Q}_0(xp)\right)(1+\hbar^2/4)^{-1/2},$$

$$\mathcal{Q}_0(\sinh xp) = \sinh\left(2/\hbar \sin^{-1}(\hbar/2)\,\mathcal{Q}_0(xp)\right),$$

$$\mathcal{Q}_0(\cosh xp) = \cosh\left(2/\hbar \sin^{-1}(\hbar/2)\,\mathcal{Q}_0(xp)\right)(1+\hbar^2/4)^{-1/2}.$$

5 - Conclusion.

The foregoing reactions have illuminated in some measure the problem of the quantization of the multilinear momentum observables, and have led in particular to a plausible and systematic scheme of (formal) quantization. Progress has been made in delimiting possible quantizations of the multilinear momenta, and a preferred scheme has been illustratively applied to functions of the momentum xp on $\mathbb{R}$, and this has led to rather pleasing explicit forms for the quantum analogues of $\cos(xp)$ and $\sin(xp)$.

Finally we may remark that we do not claim the above-discussed methods of quantization to be exhaustive (in particular we have not included WEYL's ([14]) rule or its generalisations due to UNDERHILL ([15]), and UNDERHILL and TARAVIRAS ([5]); a great breath of material remains unexplored by our brief summary.

*   *   *   *   *   *   *

K. McFarlane acknowledges the support of a Royal Society European Exchange Programme Fellowship, and the facilities made available to him by the Dublin Institute for Advanced Studies.

([13]) C. JORDAN Calculus of Finite Difference, second edition, (New York, 1950).

(*) Note also that the equation for $\mathcal{Q}_0(x^2p^2)$ shows, as is readily verified from the work of CASTELLANI ([4]), that our proposed quantization scheme is inconsistent with Weyl's rule, the "symmetrization" rule, and the Born-Jordan rule.

([14]) H. WEYL Theory of Groups and Quantum Mechanics, second edition (New York, 1931).

([15]) J. UNDERHILL J. Math. Phys. 19, 1932-1935 (1977).

APPENDIX A : Some notational conventions.

(i) The usual summation and variable free-index conventions hold.

(ii) The symbols " $|$ " and " δ " denote the operator of covariant differentiation; thus

$$b^{i_1\cdots i_k}_{|j_1\cdots j_\ell}, k,\ell\in[1,\infty], = \delta_{j_\ell}\cdots\delta_{j_1}(b^{i_1\cdots i_k}). \tag{A.1}$$

(iii) The symbols " $,$ " and " ∂ " denote the operation of partial differentiation thus

$$b^{i_1\cdots i_k}_{,j_1\cdots j_\ell}, k,\ell\in[1,\infty] = \partial_{j_\ell}\cdots\partial_{j_1}(b^{i_1\cdots i_k}). \tag{A.2}$$

(iv) The symbol $a^{i_1\cdots i_n}p_{i_1}\cdots p_{i_n}$ is to be interpreted when $n=0$ as the (momentum independent) scalar function $a(x^k)$.

(v) The symbol $i_{\ell+1}\cdots i_k$, $k,\ell\in[0,\infty]$ is defined recursively as $(i_{\ell+1}\cdots i_{k-1})i_k$, $k\geq\ell+2$; i_k, $k=\ell+1$; and is void when $\ell=k$. This is as illustrated below

$$\sum_{k=0}^{3} b^{i_1\cdots i_3}_{|i_1\cdots i_k} = b^{i_1i_2i_3} + b^{i_1i_2i_3}_{|i_1} + b^{i_1i_2i_3}_{|i_1i_2} + b^{i_1i_2i_3}_{|i_1i_2i_3}. \tag{A.3}$$

APPENDIX B : On the general form of the symmetric operators $Q_o(A)$.

We omit the demonstration that $Q_o(A)$ may be assumed to have the tensor form $(-i\hbar)^n\sum_{k=0}^{n} b^{i_1\cdots i_k}\delta_{i_1}\cdots\delta_{i_k}$ of equation (4), and consider only the proof of the recurrence relation (6).

We have that, by axiom 2, $\forall\phi,\psi\in C_o^\infty(M)$, $\langle\psi|Q_o(A)\phi\rangle=\langle Q_o(A)\psi|\phi\rangle$, or equivalently that

$$\int_M\Big[(-i\hbar)^n\sum_{k=0}^{n} b^{i_1\cdots i_k}\psi_{|i_1\cdots i_k}\Big]^*\phi = \int_M \psi^*(-i\hbar)^n\sum_{k=0}^{n} b^{i_1\cdots i_k}\phi_{|i_k\cdots i_1}. \tag{B.1}$$

By applying the identities

$$\int_M b^{i_1\cdots i_k}\phi_{|i_1\cdots i_k}\psi^* = \int_M(-1)^k(b^{i_1\cdots i_k}\psi^*)_{|i_1\cdots i_k}\phi, \tag{B.2}$$

and

$$(b^{i_1\cdots i_k}\psi)_{|i_1\cdots i_k} = \sum_{\ell=0}^{k}\binom{k}{\ell} b^{i_1\cdots i_k}_{|i_{\ell+1}\cdots i_k}\psi_{|i_1\cdots i_\ell}, \tag{B.3}$$

We may then deduce that

$$\int_M\psi^*(-i\hbar)^n\sum_{k=0}^{n} b^{i_1\cdots i_k}\phi_{|i_k\cdots i_1} = \int_M\Big[(-i\hbar)^n\sum_{\ell=0}^{n}\sum_{k=\ell}^{n}(-1)^{n+k} b^{i_1\cdots i_k}_{|i_{\ell+1}\cdots i_k}\psi_{|i_1\cdots i_\ell}\Big]^*\phi, \tag{B.4}$$

when comparison of(B.4)and(B.2) yields the desired result. Given this result the canonical decomposition (10) now follows by substituting from the formula defining $\Xi_o(B_{n-2k})$ into the original recurrence for $b^{i_1\cdots i_k}$.

Explicitly we obtain the following identities: first

$$Q_o(A)=\sum_{k=0}^{[\frac{1}{2}n]}(-i\hbar)^{2k}(-i\hbar)^{n-2k}\sum_{\ell=0}^{n-2k}\alpha_\ell^{n-2k}\, b^{i_1\cdots i_{n-2k}}_{|i_{\ell+1}\cdots i_{n-2k}}\delta_{i_1}\cdots\delta_{i_\ell}, \tag{B.5}$$

which upon rearrangement becomes

$$(\text{B.6}) \qquad (-i\hbar)^n \sum_{\ell=0}^{n} \Big(\sum_{k=0}^{[\frac{n-\ell}{2}]} \alpha_\ell^{n-2k}\, b^{i_1 \dots i_{n-2k}}_{\;\;|i_{\ell+1} \dots i_{n-2k}} \Big) \delta_{i_1} \dots \delta_{i_\ell},$$

from which (11) follows; and secondly

$$(\text{B.7}) \quad \sum_{k=\ell}^{n} (-1)^{n+k} \binom{k}{\ell} b^{i_1 \dots i_k}_{\;\;|i_{\ell+1} \dots i_k} = \sum_{j=0}^{[\frac{n-\ell}{2}]} \Big(\sum_{k=\ell}^{n} (-1)^{n+k} \binom{k}{\ell} \alpha_k^{n-2j} \Big) b^{i_1 \dots i_{n-2j}}_{\;\;|i_{\ell+1} \dots i_{n-2j}},$$

which upon substitution from the recurrence (9) yields

$$(\text{B.8}) \quad \sum_{k=\ell}^{n} (-1)^{n+k} \binom{k}{\ell} b^{i_1 \dots i_k}_{\;\;|i_{\ell+1} \dots i_k} = \sum_{j=0}^{[\frac{n-\ell}{2}]} \alpha_\ell^{n-2j}\, b^{i_1 \dots i_{n-2j}}_{\;\;|i_{\ell+1} \dots i_{n-2j}} = b^{i_1 \dots i_\ell},$$

as is the required symmetry condition (6).

APPENDIX C. On the determination of the quantities $B_{n-2K}(A)$.

We show that the preservation of positivity under quantization together with the assumption that $b(a^{ij}) = \alpha a^{ij}_{\;\;|ij} + \beta a^{ij}_{\;\;|ij}$ is sufficient to set $\alpha = \beta = 0$ by combining inequalities on α and β deduced from the following four special cases:

example 1 : on the manifold $M = (1,2)$ with the usual metric, consideration of the bilinear observable $A = x^2 p^2$, in which x is a Cartesian coordinate, and p its conjugate momentum, yields the general quantum analogue

$$(\text{C.1}) \qquad Q_0(x^2 p^2) = (-i\hbar)^2 (x^2 D^2 + 2xD + 2(\alpha+\beta)),$$

from which, upon noting that the operator $-(x^2 D^2 + 2xD)$ is positive, we may deduce $\alpha + \beta \geqslant 0$.

example 2 : on the manifold $M = (1,2)$ with the usual metric, the observable $A = (\sin x) p^2$. Deduce $\alpha + \beta \leqslant 0$

example 3 ; on the Euclidean space (1,2)x(1,2), the observable $A = (p_x + x p_y)^2$. Deduce $\beta \geqslant 0$.

example 4 : on the manifold (1,2)x(1,2) with the usual metric, the observable $A = (p_x + (\sin^{1/2} x) p_y)^2$. Deduce $\beta \leqslant 0$.

APPENDIX D: An illustration of the proposed quantization scheme.

We sketch the analysis, based upon contour integration as a summation technique, which leads to the results of the fourth reaction of the paper, and begin with two technical lemmata:

lemma 1: $(xD)^n = \sum_{k=0}^{n} (-\frac{1}{2})^{n-k}\binom{n}{k}(xD+\frac{1}{2})^k$, $n\in[0,\infty]$.

proof: by induction

lemma 2: $x^n D^n = \sum_{\ell=1}^{n+1} S_n^\ell (xD)^\ell$, S_n^ℓ denoting a Stirling number of the first kind.

proof: see JORDAN ([13]).

Expansion of the espressions for $\Xi_o(A)$ then yields with the aid of these lemmata the forms

$$\text{(D.1)}\qquad \beta_j^{2n} = \sum_{k=0,j-n-1}^{n}\ \sum_{\ell=1,j}^{n+k+1}\binom{n}{k}(2n)!\,(-\tfrac{1}{2})^{\ell-j}S_{n+k}^{\ell}/(n+k)!\,,$$

$$\beta_j^{2n+1} = \sum_{k=0,j-n-1}^{n}\ \sum_{\ell=1,j}^{n+k+1}\left(\binom{n+1}{k}-\tfrac{1}{2}\binom{n}{k}\right)(2n+1)!\,(-\tfrac{1}{2})^{\ell-j}S_{n+k}^{\ell}/(n+k)!\,,$$

in which the lower limits of summation are taken to be the larger of the pair of options. The residue and binomial theorems now yield the formulae

$$\text{(D.2)}\qquad S_{n+k}^{\ell} = \frac{1}{2\pi i}\oint\frac{x!\,dx}{(x-n-k)!\,x^{\ell+1}}\,,\qquad x^j/(1-x)^{j+1} = \sum_{\ell=j}^{\infty}\binom{\ell}{j}x^\ell\,,$$

which upon substitution into (D.1) ultimately yield the expressions

$$\text{(D.3)}\qquad \beta_{j+1}^{2n+1} = \beta_j^{2n} = \frac{1}{2\pi i}\oint\frac{(x+n)!\,dx}{(x-n)!\,(x+\frac{1}{2})^{j+1}}\,,$$

or equivalently, upon setting $y=x-n$ and expanding,

$$\text{(D.4)}\qquad \beta_{j+1}^{2n+1} = \beta_j^{2n} = \sum_{k=j}^{2n} S_{2n}^{k}\binom{k}{j}(n-\tfrac{1}{2})^{k-j}\,.$$

As to the transcendental functions, we deal explicitly only with $Q_o(\sin xp)$, the others following similarly. We begin with the formal expansion

$$\text{(D.5)}\qquad Q_o(\sin xp) = \sum_{n=0}^{\infty}(-1)^n Q_o(x^{2n+1}p^{2n+1})/(2n+1)!\,,$$

which upon substitution from (D.3) becomes

$$\text{(D.6)}\qquad Q_o(\sin xp) = \frac{1}{2\pi i}\oint\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\hbar^{2n}(x+n)!\,Q_o^{2k+1}(xp)}{(2n+1)!\,(x-n)!\,\hbar^{2k}(x+\frac{1}{2})^{2k+1}}\,dx\,.$$

Noting that the k-summation may, formally at least, be expressed in closed form, and upon setting $y=x-\frac{1}{2}$, we obtain

$$\text{(D.7)}\qquad Q_o(\sin xp) = \frac{Q_o(xp)}{2\pi i}\oint\sum_{n=0}^{\infty}\frac{(y-\frac{1}{2}+n)!\,\hbar^{2n}}{(y-\frac{1}{2}-n)!\,(2n+1)!}\;\frac{y\,dy}{y^2+\hbar^{-2}Q_o^2(xp)}\,.$$

Consider the final summation, which after some rearrangement may be expressed in terms of the hypergeometric function

$$\text{(D.8)}\qquad F(a,b,c;x) = \sum_{n=0}^{\infty}\frac{(a+n-1)!\,(b+n-1)!\,(c-1)!\,x^n}{(a-1)!\,(b-1)!\,(c+n-1)!\,n!}\,,$$

as simply $F(\frac{1}{2}+y,\frac{1}{2}-y,\frac{3}{2};-\hbar^2/4)$. Reference to standard tables then yields the identity

$$\text{(D.9)}\qquad F(\tfrac{1}{2}+b,\tfrac{1}{2}-b,\tfrac{3}{2};\sin^2x) = \sin(2bx)/(2b\sin x)\,,$$

from which the final result follows:

List of Hand-written Symbols

Symbol	Character	Symbol	Character
A	Bold capital A	a	lower case a
B	Bold capital B	b	lower case b
C	Bold capital C	c	lower case c
D	Bold capital D	d	lower case d
F	Bold capital F	f, ℓ	lower case f
M	Bold capital M	g	lower case g
Q	Bold capital Q	i	lower case i
S	Bold capital S	j	lower case j
T	Bold capital T	k	lower case k
U	Bold capital U	ℓ	lower case l
Z	Bold capital Z	m	lower case m
Σ	Bold Greek capital "sigma"	n	lower case n
Ξ	Bold Greek capital "xi"	p	lower case p
		x	lower case x
		y	lower case y

Symbol	Character	Symbol	Character
α	Greek lower case "alpha"	∞	infinity symbol
β	Greek lower case "beta"	$\circ$	zero subscript
δ	Greek lower case "delta"	$\dagger$	dagger
∂	Greek lower case "delta"	$\hbar$	Dirac's constant (crossed h)
ϵ	Greek lower case "epsilon"	$<$	less than
η	Greek lower case "eta"	$\leqslant$	less than, equal to
μ	Greek lower case "mu"	$\vert$	vertical bar
ν	Greek lower case "nu"	$\mathbb{R}$	standard symbol for the reals
ϕ	Greek lower case "phi"	$[\ ,\ ]$	square brackets
ψ	Greek lower case "psi"	$\{\ ,\ \}$	curly brackets
		$\{\ ,\ \}_+$	curly brackets with subscript
			plus sign
		$\binom{k}{\ell}$	tall round brackets with lower case entries(Binomial coefficient)
		*	asterix

List of Hand-written Symbols

Symbol	Character	Symbol	Character
A	Bold capital A	a	lower case a
B	Bold capital B	b	lower case b
C	Bold capital C	c	lower case c
D	Bold capital D	d	lower case d
F	Bold capital F	f, ℓ	lower case f
M	Bold capital M	g	lower case g
Q	Bold capital Q	i	lower case i
S	Bold capital S	j	lower case j
T	Bold capital T	k	lower case k
U	Bold capital U	ℓ	lower case l
Z	Bold capital Z	m	lower case m
Σ	Bold Greek capital "sigma"	n	lower case n
Ξ	Bold Greek capital "xi"	p	lower case p
		x	lower case x
		y	lower case y

Symbol	Character	Symbol	Character
α	Greek lower case "alpha"	∞	infinity symbol
β	Greek lower case "beta"	$\circ$	zero subscript
δ	Greek lower case "delta"	$\dagger$	dagger
∂	Greek lower case "delta"	$\hbar$	Dirac's constant (crossed h)
ϵ	Greek lower case "epsilon"	$<$	less than
η	Greek lower case "eta"	$\leqslant$	less than, equal to
μ	Greek lower case "mu"	$\vert$	vertical bar
ν	Greek lower case "nu"	$\mathbb{R}$	standard symbol for the reals
ϕ	Greek lower case "phi"	$[\ ,\ ]$	square brackets
ψ	Greek lower case "psi"	$\{\ ,\ \}$	curly brackets
		$\{\ ,\ \}_+$	curly brackets with subscript
			plus sign
		$\binom{k}{\ell}$	tall round brackets with lowe case entries(Binomial coeffic
		*	asterix