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GRAVITATIONAL RADIATION and the EQUIVALENCE PRINCIPLE by the TECHNIQUE of VIRTUAL QUANTA

by

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Abstract

After reviewing the Weizsächer-Williams technique of virtual quanta for calculation of electromagnetic radiation in bremsstrahlung encounters, we extend the method to the domain of gravitational encounters and set up a correlation between collision problems and the corresponding problem of the generation of gravitational radiation. In the local rest frame of a relativistic test particle the gravitational field of a large mass consists predominantly of a pulse of plane-fronted gravitational waves. We Fourier-analyse this equivalent pulse and consider the scattering of the individual frequency components, virtual quanta, by the test body. The scattering occurs because of the long-range Newtonian field which gives a Rutherford-like cross section. The escape of this radiation to infinity, suitably transformed, gives us the radiative loss of gravitational energy by a rapidly moving particle. The radiation spectrum and total energy radiated are computed as an example.

We then turn to the case where one or both of the masses possess an electric charge, and calculate the total electromagnetic and gravitational energy radiated in such encounters. We consider both the case in which the deflection is principally electromagnetic in nature, and the case in which the deflection is principally gravitational. The results are interpreted by considering the predictions of the equivalence principle, for the behavior of the test particle, <u>and for the behavior of the virtual quanta</u>. As expected from the equivalence principle, the total radiation produced is larger for electromagnetic deflection than for gravitational deflection through the same angle.

I. Introduction

These lectures describe an approximation scheme which is applicable to the calculation of radiation processes involving gravitational interactions and/or the radiation of gravitational waves. This work was done in conjunction with Y. Nutku, and the basic reference[1] is: Matzner and Nutku, Proc. Roy. Soc. A336 285 (1974).

This procedure is an adaptation of the classic method of virtual quanta introduced by Weizsächer[2] and Williams [3], which they applied to classical and to quantum processes involving electromagnetic interactions and radiation. A very accesible description of this technique is found in:[4] Jackson, Classical Electrodynamics (Wiley and Sons, New York; 1962).

The classical version of the virtual quantum method is an approximation which is applicable to high speed encounters between (charged) particles. The approximation becomes accurate as the relative speeds involved approach c; i.e. as the energy parameter ($c \equiv 1$ henceforth)

$$\gamma \equiv (1-v^2)^{-\frac{1}{2}} \to \infty \quad . \tag{1.1}$$

where v is a typical velocity of the system. (In what follows we will be explicit about the particular motion to which γ refers.)

In quantum mechanics the accuracy of the virtual quantum method is poor when the kinetic energies are comparable to the mass of election, or when impact parameters become small compared to the Compton radii of the particles involved.

We shall here be primarily interested in extending the virtual quantum method to applications involving gravity. The quantum mechanics limits have little relevance in that case, but other considerations limit the validity of this scheme in the gravitational case.

We shall see that the difficulties which limit this technique's range of validity in the gravitational case are exactly the difficulties which are associated with the equivalence principle: gravitational fields can be transformed away and there is no localizable gravitational energy density. These problems make the virtual quantum method an excellent model on which to exercise mathematical and physical intuition. The results in terms of calculated radiation spectra are in the end obtained with relative ease, justification in itself of the utility of the method. Moreover, in the end we will be able to use the results of these calculations to better understand the principle of equivalence itself.

II. Electromagnetic Bremsstrahlung due to Electromagnetic Accelerations; Virtual Quanta.

The virtual-quantum method is best applied in a bremsstrahlung situation, which we henceforth assume. We also assume that one particle is essentially a test particle, i.e. it is much less massive than the other. Thus we consider the deflection of (and subsequent electromagnetic radiation by) a test charge e, with mass m, moving with energy parameter $\gamma \equiv (1-v^2)^{-\frac{1}{2}} \gg 1$ with respect to a very massive charge Q (mass M). We assume a hyperbolic encounter with (large) impact parameter b. We assume for this electromagnetic example that the gravitational interaction between the masses is negligible.

The approximation divides the calculation into three separate calculations which are then joined to find the radiation produced.

1) The small particle (m,e) as it approaches the heavy particle (M,Q), sees the Coulomb electric field of (M,Q) transformed so that it resembles a pulse of plane electromagnetic radiation, with a spectrum $d^2E/(d\omega dA)$ (Energy per unit frequency per unit area).

2) The small particle possesses a charge, and hence there is a cross section $d\sigma(\omega)/d\Omega$ for it to scatter electromagnetic waves of a particular frequency. This cross section multiplied by the incident flux of energy per frequency interval in the equivalent plane pulse gives the frequency and angular spectrum of the scattered radiation

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_m = \frac{d\sigma(\omega)}{d\Omega}\,\frac{d^2E}{d\omega dA} \qquad (2.1)$$

This scattered radiation is the radiation produced in the bremsstrahlung encounter, according to the virtual quantum method.

3) Since the heavy particle defines a more natural "laboratory frame" for the observation of the radiation, we consider the propagation of the radiation out to spatial infinity and its observation by an observer there who is at rest with respect to M. Since we neglect gravitational effects here, this simply amounts to a Lorentz transformation of the quantity (2.1) to the rest frame of M.

We now give the details of such a calculation.

In a hyperbolic encounter with large impact parameter b, the deflection of a test mass (m,e) due to electromagnetic forces arising from a large mass with charge (M,Q) goes to zero as $\gamma\to\infty$:

$$\theta_{e.m} \to \frac{2eQ}{\gamma m b} \to 0 \quad . \tag{2.2}$$

Hence the small particle moves on a straight line past the heavy one, in the limit $\gamma\to\infty$. Thus the frame of the light particle is, in this limit, a Lorentz frame, and the Coulomb field of (M,Q) can be transformed by a simple Lorentz transformation into the reference frame of (m,e). In the $\gamma\to\infty$ limit, this field appears as a pulse of radiation, with energy/area/(angular frequency) [4]

$$\frac{d^2E}{d\omega dA} = \frac{1}{\pi^2}\ \frac{Q^2}{b^2} \left(\frac{\omega b}{\gamma}\right)^2 K_1^2\left(\frac{\omega b}{\gamma}\right) \tag{2.3}$$

where K_1 is a modified Bessel function. This spectrum is approximately constant from zero frequency up to $\omega_c = \gamma/b$, and falls off exponentially for $\omega>\omega_c$ so that

$$\frac{d^2E}{d\omega dA} \approx \begin{cases} \dfrac{Q^2}{\pi^2 b^2} & \omega<\omega_c \\ 0 & \omega>\omega_c \end{cases} \tag{2.4}$$

Since we concentrate on classical systems, we consider that this spectrum of virtual photons is scattered by the Thomson cross section (instead of its quantum mechanical generalization):

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{THOMSON}} = e^2 \left(\frac{e}{m}\right)^2 (1-\sin^2\theta\cos^2\varphi) \tag{2.5}$$

where θ is the polar angle in the frame of the light particle taken with the pole opposite the direction of motion and φ is the azimuthal angle taken to be zero in the plane of the orbit. Note that this cross section is independent of ω .

The energy produced in the bremsstrahlung encounter is, in the frame of the light particle,

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_m = \frac{d^2E}{d\omega dA} \left.\frac{d\sigma}{d\Omega}\right|_{m\,\text{THOMSON}} \tag{2.6}$$

where now the subscript m denotes quantities expressed in the frame of the small particle m. To obtain the energy in the frame of the large charge, the "lab" or observer frame, we perform the appropriate Lorentz transformation. Since $dE/d\omega$ is invariant under this transformation, we must simply transform the angles involved. The result is

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_{\infty} = \gamma^{-2}(1-vn_{3\infty})^{-2}\frac{d^2E}{d\omega dA}\left.\frac{d\sigma}{d\Omega}\right|_{m\ \mathrm{THOMSON}} \tag{2.7}$$

where on the right hand side the quantity $n_{3\infty} = \cos\Theta_{\infty}$, with the pole in Θ aligned with the direction of motion of the small particle, and where (ω_m, θ_m) are expressed in terms of $(\omega_\infty, \Theta_\infty)$ by

$$\omega_m = \omega_\infty\gamma(1-vn_{3\infty}) \tag{2.8}$$

and

$$n_{3m} \equiv -\cos\theta_m = \frac{n_{3\infty}-v}{1-vn_{3\infty}} \tag{2.9}$$

Of course

$$\varphi_\infty = \varphi_e$$

At every angle in the lab frame the spectrum is a Doppler shifted copy of $d^2E/d\omega dA$.

The total energy produced in the bremsstrahlung encounter is:

$$E_{tot}\ \{\text{E.M. Rad.; E.M. Defl.}\} = \iint \left.\frac{d^2E}{d\omega d\Omega}\right|_{\infty} d\Omega_\infty d\omega_\infty$$

$$= \frac{Q^2e^2\left(\frac{e}{m}\right)^2}{\pi\gamma^2 b^3}\left(\int_0^\infty dx\, x^2\, K_1^2(x)\right)\int_{-1}^{1}\frac{dn_{3\infty}(1+n_e^2)}{(1-vn_{3\infty})^3} . \tag{2.10}$$

The integral involving K_1^2 may be evaluated using tables (e.g.[5]) and equals $3\pi^2/32$. The integral involving $n_{3\infty}$ is elementary, and equals $\frac{8}{3}\gamma^4[1+O(\gamma^{-2})]$.

From Eq. (2.10) we have then:

$$E_{tot}\left\{\text{E.M. Rad.; E.M. Defl.}\right\} = \frac{\pi}{4}\ \frac{\gamma^2 Q^2 e^2}{b^3}\ \left(\frac{e}{m}\right)^2 \tag{2.11}$$

In the course of this work we shall have to evaluate several integrals like those that appear in Eq (2.10). In all cases one factor in the integrand will be an equivalent pulse spectrum which can be approximated like (2.4) with an upper limit at $\omega_c = \gamma/b$ and smooth simple behavior for $\omega<\omega_c$. This kind of approximate analysis often yields much simpler analytic expressions, and allows very simple estimates of the energy radiated. For instance, result (2.11) could have been anticipated, up to numerics, by noting that in the frame of the small particle Eqs (2.4) give

$$\int \frac{d^2E}{dAd\omega}\, d\omega \sim \frac{\gamma Q^2}{b^3} \tag{2.12}$$

and, still in the frame of the small particle, the scattered energy approximately equals (2.12) multiplied by the total cross section:

$$E_{tot}\Big|_m \sim \frac{\gamma Q^2}{b^3}\, e^2 \left(\frac{e}{m}\right)^2 \tag{2.13}$$

and a large fraction of this radiation is blue-shifted by a factor $\sim\gamma$ in going to the lab frame, leading to an estimate in agreement with (2.11) above).

It will be seen that the radiation theory in this problem has, in the virtual quantum approach, been all pushed into the calculation of the scattering cross section. Any technical difficulty resides in that calculation. The scattering cross section used in the example just above, the Thomson cross section, describes the scattering of a plane wave by a free charge and has a straightforward physical interpretation. The incident wave causes the particle to undergo a particular oscillatory motion; this accelerated motion of a charged particle causes electromagnetic radiation with a characteristic angular distribution. The reaction to the plane wave and the reradiation of the scattered flux are local phenomena, directly associated with the particle. When we use this cross section in the virtual quantum calculation outlined above, the calculation explicitly makes use of this locality of the scattering.

Straightforward calculation shows that the radiation calculated in this manner is, for this electromagnetic case, exactly the radiation calculated by considering the total accelerated orbit of the particle (m,e).

III. The Frame Transformation for the Gravitational Case

The production of gravitational radiation, in the virtual-quantum approach, must occur by the scattering of an equivalent wave pulse of gravitational flux in the frame of the small particle. It is appropriate here to give a somewhat detailed description of the plane wave pulse seen by the small particle.

Our approach, which is similar to that of Pirani,[6] is to project the Riemann tensor of the large (uncharged for now) mass M into a Fermi-propagated frame carried with the small mass m. Fermi propagation corresponds to the physical evolution of the non-rotating frame of an observer moving with m, so this gives a direct measure of the tide-producing physical components of the Riemann tensor at the instataneous position of m. For the case of interest here (both bodies uncharged) m follows a geodesic and Fermi propagation reduces to parallel propagation. In general, however, the Fermi propagated frame will be appropriate.

Thus we consider the ultrarelativistic motion of a test body of negligible mass m in the field of a Schwarzschild mass M. We shall need to know the appearance of the Schwarzschild field due to M for an observer situated on the test body which is freely falling on a geodesic which we have arranged to be arbitrarily close to a null geodesic of the Schwarzschild geometry. These fields cannot in general be obtained by one simple Lorentz transformation, since we cannot approximate the trajectory by a straight line even in the ultrarelativistic limit owing to the fact that light itself is affected by curvature. In fact, in contrast to (2.2), we have, for gravitational deflection of the small particle as $\gamma \to \infty$:

$$\theta_{grav} \to \frac{4M}{b} \tag{3.1}$$

So the procedure of performing Lorentz transformations, as we do in electrodynamics and as Pirani does in general relativity, must be abandoned.

The specialization of Fermi-propagation[7] to the case where the observer is following a geodesic and the introduction of Fermi normal coordinates which are appropriate for the discussion of the geometry in the immediate neighbourhood of the test body is due to Synge[8] and Manasse & Misner[9]. In the following we shall apply the formalism of these authors to the problem of a test body in a hyperbolic geodesic of the Schwarzschild geometry. Using the usual local coordinates centered on M, which we shall write with capital letters, the Schwarzschild metric is

$$ds^2 = X\, dT^2 - X^{-1}\, dR^2 - R^2 (d\Theta^2 + \sin^2 d\Phi^2) \quad , \tag{3.2}$$

$$X = 1 - 2M/R$$

and for a particle following a geodesic in this geometry we have the first integrals of motion

$$1 = XT'^2 - X^{-1}\, R'^2 - R^2\, \Theta'^2 \tag{3.3}$$

$$\gamma = XT'$$

$$\ell = R^2\, \Theta'$$

where prime denotes differentiation with respect to proper time. The geodesic equations now reduce to the quadrature

$$\left(\frac{dU}{d\Theta}\right)^2 = U^3 - U^2 + \frac{U}{L^2} + \frac{\gamma^2 - 1}{L^2} \qquad (3.4)$$

with

$$U = 2M/R \quad \text{and} \quad L = \ell/2M \qquad (3.5)$$

which can be integrated [10], [11] by elliptic functions.

Since the initial 3-velocity vector and the centre of mass of M define a plane on which all subsequent motion takes place, we have without loss of generality taken this to be the polar plane Φ = const. We want to set up a parallelly propagated frame on such a geodesic followed by the test particle, which will thus be at rest in this frame. The time-like direction, defined by the tangent to the geodesic, will have the unit basis vector

$$\underset{\sim}{e}_{(o)} = T' \frac{\partial}{\partial T} + R' \frac{\partial}{\partial R} + \Theta' \frac{\partial}{\partial \Theta} = \frac{\partial}{\partial t} \qquad (3.6)$$

where t is the proper time of the particle and $\underset{\sim}{e}_{(o)}$ is manifestly Fermi-propagated. By the symmetries of the problem it is also obvious that

$$\underset{\sim}{e}_{(3)} = \frac{1}{R \sin \Theta} \frac{\partial}{\partial \Phi} \qquad (3.7)$$

is the desired unit basis vector perpendicular to the plane of motion. Rather than proceed directly to a Fermi-propagated frame, we shall introduced an intermediate step where we consider a pair of vectors

$$\bar{\underset{\sim}{e}}_{(1)} = \frac{1}{N}\left(\frac{R'}{X} \frac{\partial}{\partial T} + \gamma \frac{\partial}{\partial R}\right) \qquad (3.8)$$

$$\bar{e}_{(2)} = \frac{1}{R}\left[\frac{\ell}{N}\left(T'\frac{\partial}{\partial T} + R'\frac{\partial}{\partial R}\right) + N\frac{\partial}{\partial \varphi}\right] \qquad (3.9)$$

where $$N = (1 + \ell^2/R^2)^{1/2} \qquad (3.10)$$

which are time-dependent linear combinations of the pair $e_{(1)}, e_{(2)}$, which complete the desired set. We note that $\{\bar{e}_{(\alpha)}\} \equiv \{e_{(o)}, \bar{e}_{(1)}, \bar{e}_{(2)}, e_{(3)}\}$ form an orthonormal basis. Such a procedure is convenient because $\bar{e}_{(1)}, \bar{e}_{(2)}$ are very simply adopted to the symmetries of the problem and they can be stated independently of the particular geodesic under consideration, while the final rotation

$$e_{(1)} = \cos\vartheta\, \bar{e}_{(1)} + \sin\vartheta\, \bar{e}_{(2)} \qquad (3.11)$$

$$e_{(2)} = -\sin\vartheta\, \bar{e}_{(1)} + \cos\vartheta\, \bar{e}_{(2)} \qquad (3.12)$$

which needs to be performed to obtain the parallelly propagated set

$$\{e_{(\alpha)}\} = \{e_{(o)}, e_{(1)}, e_{(2)}, e_{(3)}\}$$

depends crucially on the specific geodesic the test body is following through the explicit solution of (3.4). That is, ϑ is determined by the equations of parallel transport: $\nabla_{e_{(o)}} e_{(1)} = 0, \nabla_{e_{(o)}} e_{(2)} = 0$ which reduce to

$$\frac{d\vartheta}{dt} = \frac{-\gamma\ell}{\ell^2 + R^2} \qquad (3.13)$$

IV. The Riemann Tensor

We shall now apply the basis of equations (3.6-3.13) to obtain the appearance of a Schwarzschild field for our observer. It will be convenient to introduce the Petrov-Pirani notation where we identify pairs of indices

$$\begin{array}{ccccccc} \mu\nu & 23 & 31 & 12 & 01 & 02 & 03 \\ A & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{4.1}$$

and deal with components in the space of bivectors. In particular the Riemann tensor is a symmetric bilinear form in this space. The raising and lowering of indices is performed by means of the metric with the components

$$g^{AB} = g^{\mu[\alpha}g^{\beta]\nu} \quad ; \qquad A \Leftrightarrow \mu\nu \, , \quad B \Leftrightarrow \alpha\beta \tag{4.2}$$

which has signature zero.

The tensor transformation law leads to an obvious transformation law for bi-vectors

$$2\bar{e}^{[\mu}_{(a)}\bar{e}^{\nu]}_{(b)} F_{\mu\nu} \leftrightarrow F_{(\bar{A})} = E_{(\bar{A})}{}^{B} F_{B} \quad , \tag{4.3}$$

and the explicit form of the transformation connecting components expressed in the Schwarzschild coordinates T,R,Θ,Φ to those expressed in the orthonormal basis (3.8) to (3.13) can be written out as a matrix:

$$
E_{(\bar{A})}{}^{B} = \begin{bmatrix}
\frac{N}{R^2 \sin\theta} & \frac{-R'\ell}{R^2 N \sin\theta} & 0 & 0 & 0 & \frac{-\ell Y}{R^2 N X \sin\theta} \\
0 & \frac{Y}{N R \sin\theta} & 0 & 0 & 0 & \frac{R'}{R N X \sin\theta} \\
0 & 0 & \frac{Y}{R} & \frac{\ell}{R} & \frac{-R'}{R X} & 0 \\
0 & 0 & \frac{Y\ell}{R^2 N} & N & \frac{-R'\ell}{R^2 N X} & 0 \\
0 & 0 & \frac{-R'}{R N} & 0 & \frac{1}{N R X} & 0 \\
-\frac{\ell}{R^3 \sin\theta} & \frac{R'}{R \sin\theta} & 0 & 0 & 0 & \frac{Y}{R X \sin\theta}
\end{bmatrix} \tag{4.4}
$$

Again in this notation, the Riemann tensor in the original Schwarzschild coordinates is given by

$$
R_{AB} = \text{diag.}\ (-2MR \sin^2\theta,\ (M/RX)\sin^2\theta,\ M/RX,\ 2M/R^3,\ -MX/R,\ -(MX/R)\sin^2\theta) \tag{4.5}
$$

and using equation (4.5) we find that the components of the Riemann tensor in the frame defined by $\bar{e}_{(\alpha)}$ become simply

$$
R = \begin{pmatrix} P & Q \\ Q & -P \end{pmatrix} \tag{4.6}
$$

where P, Q are 3 x 3 matrices

$$
P = \frac{M}{R^3} \begin{pmatrix} -2-3\frac{\ell^2}{R^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+3\frac{\ell^2}{R^2} \end{pmatrix} \qquad Q = \frac{3M\ell N}{R^4} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.7}
$$

called respectively the electric and magnetic parts of the Riemann tensor. In this frame we can recognize a time-dependent but Schwarzschild-like induction field in P. In the limit $\ell \to \infty$, a condition which holds for all non-radial null geodesics, the pieces proportional to ℓ^2 make up a type N gravitational wave with the amplitude

$$\sigma = \frac{3\,M\,\ell^2}{R^5} \quad , \tag{4.8}$$

travelling in the negative 2 direction. We shall be only interested in the ultra-relativistic limit where the geodesic which the observer is following is very close to a null geodesic. For such geodesics ℓ can be made as large as we choose and therefore in the subsequent discussion we shall keep only the leading terms which are proportional to ℓ^2 in the Riemann tensor. The physical components of the Riemann tensor which the observer measures are given by the transformation of equations (4.7), (4.8) according to the rotation in equations (3.11 - 3.13). For the ultrarelativistic limit we find

$$P = \sigma\begin{pmatrix} -\cos^2\vartheta & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cos^2\vartheta \end{pmatrix} + \sigma\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & -\sin^2\vartheta & \cdot \\ \cdot & \cdot & \sin^2\vartheta \end{pmatrix} + \sigma\begin{pmatrix} \cdot & \frac{1}{2}\sin 2\vartheta & \cdot \\ \frac{1}{2}\sin 2\vartheta & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \tag{4.9}$$

and

$$Q = \sigma\begin{pmatrix} \cdot & \cdot & \cos\vartheta \\ \cdot & \cdot & \cdot \\ \cos\vartheta & \cdot & \cdot \end{pmatrix} + \sigma\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \sin\vartheta \\ \cdot & \sin\vartheta & \cdot \end{pmatrix} \tag{4.10}$$

We observe that the time dependences entering these equations through $\vartheta(t)$ are not the same for P and Q. Hence, if the ϑ time-dependence is the dominant factor in the problem, the magnetic-like Q cannot complement the electric-like P to form a proper plane pulse of gravitational radiation. But for the problem of one particle shooting past another the time dependence of σ will turn out to be the controlling factor, because R changes so quickly that the fields are cut off before ϑ has time to change appreciably. In this case the fields will be approximately complementary (the approximation being better as $\ell \to \infty$) and we obtain pulses of linearly polarized gravitational radiation. If, on the other hand, we are dealing with circular orbits where σ is a constant and only ϑ varies with time, we do not obtain a pulse of radiation.

This situation is exactly analogous to the one which is encountered in the electromagnetic theory. There the magnetic field has the complementary algebraic structure and the same frequency as the electric field in the case of a particle rapidly shooting past another, but for circular orbits the magnetic field has zero frequency. As in the Weizsächer-Williams prescription, we let the electric-like P determine the character of the waves and insert by fiat the associated complementary part Q which is necessary to form a type N plane wave.

V. Frequency Spectrum

We shall now address ourselves to the problem of computing the effects of incident plane gravitational waves, such as the equivalent pulse of radiation that was obtained in the previous section, on the test body. We shall proceed by analysing the pulse into its frequency components and regard each component as a particle, a virtual graviton, which is scattered by the small mass as in a collision process. For this purpose we need to associate an energy density and momentum flux with these gravitational waves. Since gravitational energy cannot be localized, the definition of a Poynting vector for gravitational waves is a difficult problem which can be resolved straightforwardly only in asymptotically flat regions. To obtain a sensible definition of the energy flux of a gravitational wave we must require that an averaging be carried out over distances comparable to the wavelength of the gravitational wave but short compared to the physical sizes of the system under consideration. Furthermore we must demand that our results reduce to the well-known expressions which hold in the weak field limit. The definition for the energy flux

$$\frac{d^2E}{dtdA}(t) = \frac{1}{2} \int^t \int^t P_{AB}(t') \; P^{AB}(t'') \; dt' \, dt'' \tag{5.1}$$

where P_{AB} are the Riemann tensor components is essentially the expression proposed by Gibbons and Hawking[12]. Definition (5.1) satisfies the requirements mentioned above and we shall take it as the basis of our subsequent discussion. Equation (5.1) corresponds to taking the time average over roughly one period and it is a particularly appropriate expression for the energy flux in short bursts of radiation of which our equivalent pulse is an example. Since P_{AB} is referred to the orthonormal basis $e_{(\alpha)}$, its contravariant components are obtained by raising with the 3 x 3 positive definite flat metric. The total energy incident per unit area is then the integral of (5.1) over all time

$$\frac{dE}{dA} = \int_{-\infty}^{\infty} P(t) \, dt \tag{5.2}$$

The expression for the total energy which we just wrote as a time integral can be converted to an integral over a frequency spectrum if we analyse the equivalent pulse of radiation into its Fourier components. We may without any ambiguity carry out the Fourier decomposition using the proper time associated with an observer moving on the test body. We have the Fourier transform

$$\breve{P}_{AB}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \; P_{AB}(t) \, dt \tag{5.3}$$

with the inverse

$$P_{AB}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \; \tilde{P}_{AB}(\omega) \, d\omega \tag{5.4}$$

and by Parseval's theorem, we can write equation (5.2) as

$$\frac{dE}{dA} = \int_0^\infty \frac{d^2E}{d\omega dA} \, d\omega \tag{5.5}$$

where

$$\frac{d^2E}{d\omega dA} = \frac{1}{\omega^2} \tilde{P}_{AB}(\omega) \, \tilde{P}^{AB}(\omega) \; . \tag{5.6}$$

$\frac{d^2E}{d\omega dA}$ is the equivalent pulse frequency spectrum (energy per unit area per unit frequency interval).

VI. The Equivalent Pulse in a Distant Bremsstrahlung Encounter.

Even though Eq (3.1) shows that there is always some net gravitational deflection of the small mass, for distant encounters the orbit may be approximated by a straight line in the asymptotically flat space (at least for purposes of evaluating the equivalent pulse). Thus:

$$\begin{aligned} R^2 &= b^2 + v^2T^2 \\ &= b^2 + v^2\gamma^2t^2 \end{aligned} \tag{6.1}$$

where $b \approx \ell/\gamma$ is the impact parameter and $v \approx 1$ is the velocity of the test particle. With this expression for R(t) the solution of (3.13) is elementary and we find

$$\vartheta(t) = \frac{\ell}{\sqrt{(b^2+\ell^2)}} \arctan \frac{\gamma t}{\sqrt{(b^2+\ell^2)}} \; . \tag{6.2}$$

As we have $\ell \approx \gamma b \gg b$

$$\cos\vartheta(t) \approx (1+t^2/b^2)^{-\frac{1}{2}} \tag{6.3}$$

and from (6.1) and (4.8)

$$\sigma(t) \approx \frac{3M\ell^2}{b^5} (1+\gamma^2t^2/b^2)^{-\frac{5}{2}} \; . \tag{6.4}$$

It will be noticed that ϑ is defined so that $\vartheta = 0$ corresponds to the instant of closest approach.

Notice also that $\cos\vartheta \approx 0$ for $(t/b)^2 > 1$, while the corresponding relation for σ is $\sigma(t) \approx 0$ for $\gamma^2(t/b)^2 > 1$. Because of the explicit appearance of γ in σ, $\cos\vartheta \approx 1$ during all the time that σ is non-negligible. These time scales indicate that the highest frequency in the Fourier transform of $\cos\vartheta$ is $\sim b^{-1}$, while the highest frequency in the Fourier transform of σ is approximately

$$\omega_c = \gamma/b >> b^{-1} \quad . \tag{6.5}$$

All this becomes of more than academic interest when we realize that to obtain the frequency spectrum we must evaluate Fourier integrals which are typically of the form

$$\tilde{P}_{11}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(t)\cos^2\vartheta(t)e^{i\omega t}\, dt \tag{6.6}$$

Because of the difference in time scales, we may set $\cos\vartheta \approx 1$, so that

$$\tilde{P}_{11}(\omega) \approx \frac{3M\ell^2}{b^5\sqrt{(2\pi)}} \int_{\infty}^{\infty} e^{i\omega t}(1+\gamma^2t^2/b^2)^{-\frac{5}{2}}\, dt \quad . \tag{6.7}$$

This expression is readily recognized as the integral representation for modified Bessel functions of the second kind

$$\tilde{P}_{11}(\omega) = \sqrt{\left(\frac{2}{\pi}\right)}\, \frac{M}{\gamma}\, \omega^2 K_2\left(\frac{b\omega}{\gamma}\right) \quad . \tag{6.8}$$

It is approximately constant for all frequencies up to the cut-off frequency ω_c and then drops off to zero exponentially. The other components of the Riemann tensor in (5.6) contain at least one factor $\sin\vartheta(t)$ and are, therefore, much smaller. In the Fourier transform of the Riemann tensor components which contain a factor of $\sin\vartheta$ all frequencies except those close to ω_c are suppressed and furthermore the amplitudes are down by one power of γ compared to (6.8), while the amplitudes of those containing $\sin^2\vartheta$ are down by two powers of γ but have a relatively flatter spectrum. We note that in practice the Fourier transform of the dominant pulse may be approximated by a step function which terminates at ω_c while the others, being negligibly small, can be ignored altogether.

The terms with coefficient $\cos\vartheta$ and $\cos^2\vartheta$ in (4.9) and (4.10) are appropriate complementary electric and magnetic parts for waves in the 2-direction. Hence where the time dependence of ϑ is slow compared to that of σ as we have in this case, the largest pulses appear to be plane waves. The other components of the Riemann tensor are the analogues of the 'small pulse' which arises in the same problem in electromagnetism.

The frequency spectrum can now be written down according to (5.6) and we find

$$\frac{d^2E}{d\omega dA} = \frac{4}{\pi} M^2 \gamma^{-2} \omega^2 K_2{}^2\left(\frac{\omega}{\omega_c}\right) \qquad (6.9)$$

where we have neglected all except the contribution from the strong pulse. In the high-frequency limit we have

$$\frac{d^2E}{d\omega dA} \sim 2\left(\frac{M}{b}\right)^2 \frac{\omega}{\omega_c}\, e^{-2\omega/\omega_c} \qquad \omega \gg \omega_c \qquad (6.10)$$

where the frequency dependence exhibits the behavior to be found for all massless fields. For low frequencies the frequency spectrum

$$\frac{d^2E}{d\omega dA} \sim \frac{1}{\pi}\left(4\,\frac{M}{b}\,\frac{\omega_c}{\omega}\right)^2 \qquad \frac{1}{b} < \omega \ll \omega_c \qquad (6.11)$$

has the characteristic infrared divergence that frequently appears in problems involving gravitational radiation.

This low frequency behavior may be compared to the expression for $d^2E/d\omega dA$ of the equivalent plane wave in the electromagnetic case, Eq. (2.4). The transition from electromagnetism to gravity can be viewed as replacing the electromagnetic charge Q by its gravitational counterpart (γM) and inserting the factor $(b\omega)^{-2}$. This latter factor, which strongly suppresses the high-frequency part of gravitational radiation, is typical of such radiation, as it arises from the tensor nature of the field. Technically, this is because the energy flux calculated depends on the integral of the physical object (the Riemann tensor) for the gravitational case. This amounts to dividing by the frequency ω, i.e. emphasizing the low frequency components. In the electromagnetic case one calculates the flux by squaring the Maxwell tensor directly. Since the Maxwell and Riemann tensors transform similarly (though not of course identically), the ω^{-2} difference in energy fluxes persists and is responsible for the fact that the bulk of the total radiation is produced in low-frequency low-angular momentum waves, even though there is very substantial peaking at high frequencies for large γ .

We are also left with the difficulty of interpreting this result, since we need to use it in applying the virtual-quantum approach to gravitational radiation. Clearly the divergence is spurious, and clearly it is associated with the non-localizability of gravitational energy density. We have said that we need to average the gravitational waves over several wavelengths. But it makes no sense to find a "localized" effective energy density if we must average over lengths which are comparable to the system undergoing the interaction. The minimum size of the system is b, which is the approximate distance of closest approach. Hence it definitely makes no sense to consider wavelengths (seen in the frame of the small mass) which are longer than b. Hence we insert a cut-off lower frequency,

$$\omega_{min} = b^{-1} \quad , \tag{6.12}$$

and arbitrarily truncate the low-frequency part of the spectrum at this frequency. The low frequency part of the produced radiation will thus be uncertain because of our inability to handle the long wavelength part of the equivalent pulse. We hope, however that we have made a reasonable <u>physical</u> choice in this cut-off. A suggestion of Smarr[13] for the low frequency behavior based on the zero-frequency-limit also supports this cut-off. We should emphasize, however that this is the first brush with a cutoff made necessary by the non-localizability of the gravitational energy, i.e. by the equivalence principle.

VII. An Estimate of Gravitational Radiation Produced in a Distant Bremsstrahlung Encounter.

Sections III - VI have been devoted to obtaining just one of the three steps described in Section II as necessary to carry out the virtual quantum approximation for the gravitational case. We still must find the gravitational scattering cross section, and carry out the transformation of the scattered radiation into the frame of the large mass. In order to provide a glimpse of the direction of this work, and to allow the exercise of some physical insight (i.e. guesses) we will estimate the total gravitational radiation in a bremsstrahlung encounter, given the flux via (6.9) and (6.12), and estimating the cross section and the transformation to the frame of the large particle.

The total flux per unit area in the equivalent pulse is (for large γ)

$$\frac{dE}{dA} \cong \int_b^{\omega_c} \frac{d^2E}{d\omega dA} \, d\omega \tag{7.1a}$$

$$\cong \frac{M^2}{b^2} \frac{\omega_c^{\ 2}}{(1/b)} = \frac{\gamma^2 M^2}{b^3} \tag{7.1b}$$

The cross section for purely gravitational scattering can be expected to be of the order of the square of the Schwarzschild radius of the scatterer, i.e. $\sigma = m^2$. Hence

$$\left.\frac{dE}{d\Omega}\right|_m \cong \gamma^2 \frac{m^2 M^2}{b^3} \tag{7.2}$$

Finally, the energy scattered into the hemisphere centered on the motion of the small mass is directed so that it is blue shifted by a factor $\sim\gamma$ on going to the large mass frame. One thus estimates

$$E_{tot}\{\text{Grav.Rad.;Grav.Defl.}\} \tilde{=} \gamma^3 \frac{m^2M^2}{b^3} \quad . \tag{7.3}$$

Expression (7.3) is the grail we seek. Let us return to a more rigorous path of computation of the cross section and the energy transformation. The path will lead back close to (7.3). The final result via the virtual-quantum method is given by Eq (9.14).

VIII Cross Sections for Scattering Gravitational Radiation.

It is important to note that there exist two distinct types of gravitational radiation which a system such as we describe will give rise to. The Riemann tensor on which we have based our computation finds its expression in the physical phenomenon of tidal forces. Hence if we assign the test body an extension, the relative accelerations will generate radiation. Such a tidally induced gravitational radiation can alternatively be described by the scattering of the equivalent pulse of gravitational waves. In order to model such tidal effects, we might consider a cloud of particles undergoing tidal distortions in reaction to the incident pulse of plane waves and reradiating as a consequence of the changing quadropole moment which is thus induced. Calculated in linearized theory, this gives a cross-section

$$\sigma \propto m^2(R\omega)^4 \quad , \tag{8.1}$$

where m is the mass of the scatterer, R is its radius, and ω is the frequency of the incident radiation. Extrapolating to the case where the scattering sphere is a black hole, dimensional analysis suggests replacing R by Schwarzschild radius 2m. In this case we see that the resultant cross-section is proportional to m^6. Notice this is a scattering local to the particle, analogous to the Thomson scattering of the electromagnetic example.

For non-structured particles we cannot expect this tidal mechanism to operate. But there is another type of radiation which is independent of the structure of the test body, except for its mass.

This type of radiation is produced because there is an essentially Newtonian scattering, different from the tidally mediated scattering discussed above, which deflects the virtual quanta. The scattering mass m deflects massless radiation according to its Newtonian gravitational attraction and the equivalence principle (as, for instance, expressed by the gravitational deflexion of light). The Riemann tensor, which is defined at a point, has no direct bearing on this Newtonian scattering. We use it solely as an indicator of the presence of a plane wave over an extended region about the scattering centre which is the test particle. To discuss the scattering problem we need the cross-section for the scattering of strong gravitational waves on a point particle as well as the complementary problem of calculation of the scattering by a large Schwarzschild mass, when the incident waves are treated as a perturbation. In the very low-frequency limit, this scattering of weak waves has been computed exactly by Matzner and Ryan[14] (as a limit $\omega \to 0$).

The result, which unfortunately does not have a particularly simple analytic expression, is given by squaring the modulus of the amplitude $\lim_{\omega \to 0} f(\omega,\theta)$, where

$$f(\omega,\theta) = me^{2i\eta_o} \frac{y^{2im\omega}}{y} + \frac{2ye^{2i\eta_o}}{2i\omega}\left(y^{2im\omega}\,\frac{1-4im\omega}{1+2im\omega} - 1\right)\Big/\cos^4(\theta/2)$$
$$+ \frac{4e^{2i\eta_o}}{2i\omega}\left(y^{2im\omega} - \frac{1-im\omega}{1+2im\omega}\right)\Big/\cos^4(\theta/2) \quad , \qquad (8.2)$$

η_o is an irrelevant phase, and

$$y = \sin^2\theta/2 \quad . \tag{8.3}$$

The cross section determined by this scattering amplitude has some interesting features. Contrary to first impression, it does not have a backward divergence (where $\cos^2 \theta/2 \to o$) but because of a cancellation of terms vanishes in the backward direction and is quite small in the entire backward hemisphere. The forward direction is dominated by a forward divergence $\sigma \sim m^2\sin^{-4} \theta/2$ just as in the case of the Rutherford cross section. Because $y^{2im\omega}$ is not a factor of this expression, there are interference phenomena between the different terms. But for small $m\omega$, these are confined to the region $\theta \tilde{<} \exp(-1/4m\omega)$. Finally, the cross section so derived is summed over final polarizations; it is independent of the polarization state of the incident wave and is independent of axial angle.

This low frequency limit is appropriate to our calculation because the relevant smallness parameter is $m\omega$, and we are taking the small mass m as a test mass. By making m small enough, we can guarantee that all of the wavelengths in the equivalent pulse exceed the Schwarzschild radius of the small particle, m. An alternative statement of this condition is that the frequency spectrum cuts off at a frequency less than $\omega \sim 1/m$ and this will be shown to require $b/\gamma > m$ in the bremsstrahlung calculation where b is the impact parameter in that problem.
The cross-section is not exactly of this form (there is some absorption for instance) for non-zero frequencies. For those cases one can in principle use the numerically computed cross sections. It is then possible to relax the requirements of the smallness of m to a simpler condition such as $m \ll M$. The requirement $m < b/\gamma$ is thus not essential to the method of virtual quanta.

The Rutherford-like cross section calculated from (8.2) obviously dominates the cross-section for tidally induced scattering (8.1) for small m and we shall henceforth totally ignore the tidal effects. In contrast to the cross-section (8.1) the Rutherford-like cross-section is inherently long range. Hence at least some of the virtual gravitons scattered by this field are scattered far from the small mass and with small angles of deflexion. The equivalent pulse consists of waves which are locally plane waves but deviate from planeless over lengths of the order of the impact parameter b, the characteristic length in the problem. One might consider that the smallest deflexion angle is this $\theta_{min} \sim 4m/b$ (i.e. there is no scattering of virtual qravitons with impact parameter greater than b), and this is the procedure which was followed in Ref [1]. However, a point not considered in [1] is that this deflexion angle is smaller even than the deflection of the small mass past M. Virtual quanta which are ~b from m are $\lesssim$ 2b from M, and, at least after initial disturbance from m, are scattered through ~4M/b by M. Hence in this work we take $\theta_{min} = 4M/b$, and $\omega_{min} \sim 1/b$. The infinities in the energy spectrum of the equivalent pulse and in the cross section are thus cut off.

Again the non-localizability of gravitational interaction leads to ambiguity. And again physical arguments have allowed a reasonable choice of cut-off; the ambiguity is, as before, concentrated in the long wavelength part of the radiation spectrum.

For analytical calculational purposes we will use the following very simplified approximation to Eq (8.2):

$$\tilde{f}(\theta) = m(\sin^{-2}\theta/2 - 1) \tag{8.4}$$

The cross section calculated by squaring (8.4) closely approximates that from (8.2) everywhere but is much easier to handle analytically. Eq (8.4) corresponds to the limiting backward behavior of the cross section, and obviously preserves the forward peak. We should emphasize that although explicit ω dependences appear in (8.2); the result for $m \ll 1$ is in fact independent of ω, except for very small forward angles, for purposes of analysis, we may treat the cross section as independent of ω.

IX. Gravitational Bremsstrahlung

We now multiply the energy spectrum of the incident equivalent pulse by the cross section to obtain the energy per unit solid angle and frequency scattered by the small particle m from the equivalent pulse of plane waves:

$$\left.\frac{d^2E}{d\Omega d\omega}\right|_m = \frac{d^2E}{d\omega dA}\,\frac{d\sigma}{d\Omega} \qquad \begin{array}{l}\theta > \theta_{min} \cong 4M/b \\ \omega > b^{-1}\end{array}$$

$$= 0 \qquad \text{otherwise} \tag{9.1}$$

where we have introduced the small angular and small frequency cutoffs described above. The expression $\theta_{min} \approx 4M/b$ is appropriate for large impact parameters which we consider.

Direct calculation of $(d^2E/d\Omega d\omega)_m$ then inserts expressions (6.9), and the square of (8.2) into (9.1). Because the approximate expression (8.4) is independent of ω , the resultant spectrum at any angle in the frame of the small particle has the approximate behavior

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_m \approx \frac{1}{\pi}\left(\frac{4M\gamma m}{b^2\omega_m}\right)^2 (\sin^{-2}\theta/2-1) \qquad \left\{\theta > \theta_{min}, \frac{1}{b} < \omega < \omega_c = \frac{\gamma}{b}\right\} \tag{9.2}$$

$$= 0 \text{ otherwise,}$$

where as before we indicate with the subscript m the various quantities measured in the frame of the small mass. To obtain the spectrum of radiation at infinity we need to transform to the frame in which the large mass M is at rest. We will use Eqs (2.7) - (2.9) since gravitational wave energy transforms just like electromagnetic wave energy.

Because of the sharp cut-off at the low-frequency end of the spectrum $\omega_m = 1/b$ and because of the existence of the minimum angle of deflexion $(\theta_{min})_m$ the lowest frequency in the spectrum at infinity is given by

$$\omega_{\infty\min} = (\gamma/b)(1+\delta_m v), \tag{9.3}$$

where

$$\delta_m = -\cos(\theta_{\min})_m$$

$$\approx -1+8M^2/b^2$$

The spectrum at infinity has frequencies approximately up to the frequency corresponding to the maximum blue shift applied to ω_c:

$$(\omega_c)_\infty = \frac{1}{b}\frac{1}{1-v} \simeq 2\gamma^2/b.$$

Knowledge of the frequency spectrum at infinity obtained by the quantity $\left.\frac{d^2E}{d\omega d\Omega}\right|_\infty$ over solid angle allows comparison with other discussions of the process or radiation emitted by rapidly moving particles, even though this quantity itself is not of direct physical interest. The integration around the angle Φ is of course trivial and the integration in $dn_{3\infty}$ has the following limits;

$$n_{3\infty} \text{ upper limit} = \min\left\{\begin{matrix} 1, \\ \frac{1}{v}\left(1-\frac{1}{\gamma\omega_\infty b}\right) \equiv \tilde{A} \end{matrix}\right\}, \tag{9.5}$$

$$n_{3\infty} \text{ lower limit} = \max\left\{\begin{matrix} \frac{1}{v}\left(1-\frac{1}{\omega_\infty b}\right) \equiv \tilde{B}, \\ \delta_\infty, \end{matrix}\right\}. \tag{9.6}$$

The limits 1 and δ_∞ in (9.5) and (9.6) are simply the limits defined by the geometry of the problem. The expression $\tilde{A}$ in (9.5) however, gives the most forward angle (greatest blue shift) such that the ω_∞ under consideration corresponds to any blue shifted radiation in the approximation spectrum (9.2). That is, any higher blue shift would shift the lowest frequency in (9.2) above the frequency ω_∞. Similarly, the expression $\tilde{B}$ in (9.6) gives the largest red shift permitted, the amount which red shifts $(\omega_c)_m$ below the ω_∞ under consideration. We may thus use the simple analytic form of (9.2) with the appropriate transformations to express the variables in terms of ω_∞, $n_{3\infty}$, and the limits on the integral given by (9.5) and(9.6)take care of the cut-off defined in (9.2).

If we wish to give a frequency spectrum we must integrate over all angles for each value of ω_∞. This has been carried out, for the approximations of Eq (9.2).

The spectrum starts from zero at

$$(\omega_\infty)_{min} = (\gamma/b)(1+\delta_m v), \tag{9.7}$$

and rises at first linearly, reaching a maximum value at

$$(2\omega_\infty)_{min} = (2\gamma/b)(1+\delta_m v). \tag{9.8}$$

This abrupt initial rise is entirely due to the cutoffs imposed at low frequency and small angle . Thereafter, the spectrum falls approximately like ω_∞^{-1} up to the frequency $\omega_\infty \simeq 1/\gamma b(1-v)$. Then ω_∞^{-2} behaviour begins. As the frequency approaches the upper boundary $((\omega_c)_\infty = 1/b(1-v) \simeq 2\gamma^2/b)$ the simple model shows a linear decrease in $dE/d\omega_\infty$, down to zero flux for $\omega_\infty > (\omega_c)_\infty$. An exact treatment would show instead an exponential decrease in flux above the frequency (ω_c) and this is the only significant difference between our simple analytical model and an exact calculation.

In order to calculate the total energy radiated we can perform an approximate calculation based on the cross section from (8.4), which is $m^2(1-n_m)^2/(1+n_m)^2$. We integrate:

$$E_{TOT} = 2\pi \int dn_\infty \int \frac{d\omega_\infty}{\gamma^2(1-vn_\infty)^2} \left[\frac{4}{\pi} M^2\gamma^{-2} \omega_m^2 K_2^2 \left(\frac{\omega_m b}{\gamma}\right)\right] \frac{d\sigma}{d\Omega} \tag{9.9}$$

$$= \frac{8M^2}{\gamma^2 b^2} \int dn_\infty \int d\omega_\infty \left(\frac{\omega_m b}{\gamma}\right)^2 K_2^2 \left(\frac{\omega_m b}{\gamma}\right) \frac{d\sigma}{d\Omega} \frac{1}{(1-vn_\infty)^2}$$

Now, using (2.8) we have

$$E_{TOT} = \frac{8M^2}{\gamma^2 b^2} \int dn_\infty \int \frac{d\omega_\infty b(1-vn_\infty)}{b(1-vn_\infty)^3} (\omega_\infty b(1-vn_\infty))^2 K_2^2(\omega_\infty b(1-vn_\infty)) \frac{d\sigma}{d\Omega} \tag{9.10}$$

$$= \frac{8M^2}{\gamma^2 b^3} \int dn_\infty \frac{d\sigma}{d\Omega} \frac{1}{(1-vn_\infty)^3} \int_{x_o}^{\infty} x^2 K_2^2(x)\,dx$$

The lower limit of the x integral equals the least value of $\frac{\omega_m b}{\gamma}$ in the frame of m. Thus $x_o = \gamma^{-1}$. We cannot quote here a tabulated value for the Bessel function integral (as we did for the electromagnetic case c.f. Eq (2.10).) However $K_2 \sim 2/x^2$ as $x \to 0$, and (6.5) shows that the effective upper limit of this integral is $x_c \approx 1$, so

$$\int_{x_o}^{\infty} x^2 K_2^2(x)\,dx = 4\gamma[1+0(\gamma^{-1})] \tag{9.11}$$

Hence

$$E_{TOT} \cong \frac{32M^2 m^2}{\gamma b^3} \int_{(n_\infty)_{min}}^{1} dn_\infty \left(\frac{1-n_m}{1+n_m}\right)^2 \frac{1}{(1-vn_\infty)^3} \tag{9.12}$$

It is clear that the contributions to this remaining angular integral come principally from the poles. An analytic form can be found for the integral, but it is not particularly enlightening. The principal terms are obtained by adding terms at the two poles. Note that according to (2.9),

$$\left(\frac{1-n_m}{1+n_m}\right)^2 = \gamma^4(1+v)^4 \left(\frac{1-n_\infty}{1+n_\infty}\right)^2 \tag{9.13}$$

Hence:

$$E_{TOT}\{\text{Grav.Rad; Grav. Defl.}\} =$$

$$\frac{(16Mm)^2}{b^3}\gamma^3\left\{\ell n\ \gamma + H(\gamma)\right\} \tag{9.14}$$

Here $H(\gamma)$ is the contribution from the small angle cutoff part of the integral. $H(\gamma) \sim (b/M)^2\gamma^{-2}$ so long as $(M\gamma/b) < 1$, and $H(\gamma) \sim O(1)$ for $\gamma > b/M$. Hence this term contributes negligibly to the $\gamma \to \infty$ limit of the total energy. As we mentioned earlier, the largest part of the energy is emitted in the low frequency modes, and result (9.14) is uncertain up to numerics owing to the ambiguity in the cut-off. However the γ^3 power is not uncertain although the logarithm appears to be a spurious feature of the total energy result, arising from the angular integration cutoffs (see the discussion in Section VII).

We note that Smarr's[13] study of the zero frequency limit of the gravitational radiation produced in a bremmstrahlung encounter suggests the produced spectrum flattens off to eliminate the divergent low frequency behavior starting at the intermediate frequency $\omega_\infty \sim \gamma/b$. (The spectrum at ∞ ranges over $\omega_\infty \varepsilon < 1/(2b\gamma),\ 2\gamma^2/b >$.)

In Reference [1] the frequency spectrum and total energy were calculated using cross section derived in the Born approximation. The cross section based on Eq (8.2) had not been discovered when the earlier work was performed. The essential difference is that the cross section from (8.2) vanishes in the backward direction, while the Born cross section has the value m^2 in the backward direction. The is the direction which leads to the greatest blue shift and might have been expected to affect the high energy end of the spectrum somewhat. In fact the change is slight. The greatest change appears in the total energy calculated in Eq (9.14) above. When this approximate calculation is compared to the work of Ref [1] the value of E_{TOT} here is exactly (1/2) that of the previous calculation.

We should at this point mention a certain invariance of structure of these calculations. In particular, the effective gravitational wave pulse impinging on the small particle is independent of the source particle; contributions from higher multipoles fall off more rapidly with distance than does the mass contribution. Thus only the mass and charge of the large particle contribute to the large b fields. The scattering cross sections also are largely independent of the structure of the scatterer. Eq (8.2) was derived on the basis of a Schwarzschild black hole. However, both electromagnetic and gravitational waves have non-zero spin. This means that the angular quantum number $\ell \gtrsim 1$, and centrifugal terms in the wave equation ensure that the waves are non-zero only outside the turning points $r_{TP} \sim \ell/\omega >> m$. The scattering amplitude (8.2) is thus in fact calculated for any mass of finite radius, in the limit $m\omega \to 0$. (Again, we expect multipole moments other than the mass to have no effect on the $\omega \to 0$ limit of the cross section; see the more detailed arguments at the end of the following section.)

X. The Method of Virtual Quanta as a Probe of the Equivalence Principle.

We now use the results obtained in the previous sections to apply the virtual quantum technique as a probe of our understanding of the equivalence principle.[15] Besides the classical electromagnetic bremsstrahlung and the gravitational bremsstrahlung worked out above, our discussion requires two intermediate cases, both of which are calculated by the method of virtual quanta (see Appendix):

Case 1:

The electromagnetic radiation produced when a charged test mass (m,e) moves in a hyperbolic orbit in the field of a large uncharged mass M.

Case 2:

The gravitational radiation produced during a bremsstrahlung encounter of two charged particles (m,e), (M,Q).

Although it is clear that both these cases can be treated by the technique of Green's functions, [16,17] an alternative formulation is always desirable. A crucial point enables us to calculate the radiation emitted in the cases above using the method of virtual quanta. This is the fact that in the first order perturbation theory of the Reissner-Nordstrøm [18,19] geometry describing a charged black hole, there is a coupling between electromagnetic and gravitational modes which means there is a cross section for interconversion between them [20,21] . Case 1 above may be calculated in terms of virtual gravitons undergoing conversion scattering on a charged black hole; the conversion cross section gives the electromagnetic radiation produced. Similarly, case 2 may be calculated by utilizing the conversion scattering of incident virtual photons which give an outgoing gravitational wave flux.

The cross sections for the conversion process have been calculated by Matzner [22] (the $\ell=2$ case only) and by Fabbri [23]. In using the cross sections we are modelling our test particle (m,e) as a small black hole. However we argue at the end of this section that the conversion cross sections used here are universal for spinless test particles.

The two dimensionless numbers characterizing the scattering of gravitational waves of frequency ω from the small particle can be formed by multiplying ω with m or e. Both will be small in the test particle limit so that only the long wavelength conversion cross section is relevant to our purposes.

The $\ell = 2$ cross section for conversion between electromagnetic and gravitational radiation on a charged black hole is, in the limit $\omega \to 0$:

$$\sigma_{\substack{\text{conv}\\ \ell=2}} \sim e^2 \tag{10.1}$$

Unfortunately, contrary to statements in ref [22], the $\ell > 2$ terms contribute significantly to the cross section; in fact $\sigma_{\substack{\text{conv}\\ \ell}} \propto \ell^{-1}$, so that

$$\sigma_{\substack{\text{tot}\\ \text{conv}}} = \sum_{\ell=2}^{\infty} \sigma_{\substack{\text{conv}\\ \ell}} \tag{10.2}$$

$$\sim \sigma_{\substack{\text{conv}\\ \ell=2}} \ln \ell_{max} \tag{10.3}$$

$$\sim C^2 e^2 \ln \ell_{max}$$

where C is a constant of order unity, and where ℓ_{max} is some maximum permissible value of the angular momentum, determined by the parameters of the problem. We are interested, in this paper, in distant bremsstrahlung encounters. In such a case, $\ell_{max} \sim \omega_{max} b$, where as before $\omega_{max} \sim \gamma/b$. Hence $\ell_{max} \sim \gamma$.

According to [22] the conversion cross section arises from (differences in) potentials of order r^{-3}. For low frequencies the wave functions are nonzero only outside the turning points $r_{TP} \sim \ell/\omega >> m$. In analogy with our previous discussion (Section IX), we argue that the conversion process occurs very far from the black hole and we thus conclude that the structure of the black hole, even whether it is a black hole, and what the ratio $|e/m|$ is, are all irrelevant. We conclude that any charged mass will have the low-frequency conversion cross section (10.1)-(10.3) regardless of e/m. (The cross sections in (10.1)-(10.3) were calculated assuming $|e/m|<1$.) Hence we conclude that our model of the test charge as a charged black hole is no specialization at all, but merely an intermediate step in arriving at the universal equations (10.1) (10.3) above.

XI. The Equivalence Principle; Does a Falling Charge Radiate?

The equivalence principle states that if gradients of the gravitational field can be ignored, Lorentz physics holds in a freely falling frame. A charged particle possesses a Coulomb field which reaches to infinity, and so does interact with the gradients of the gravitational field. Thus there is radiation expected from a freely-falling charge, but, as we shall calculate, there is substantial suppression of radiation in the free-fall case compared to possible motions under the action of non-gravitational forces. The freely falling charge is trying very hard not to radiate, so the equivalence principle arguments have some relevance. In this section we discuss results of this type and their interpretation in terms of the equivalence principle.

We have already given the large γ expressions for the angle of deflection due to electromagnetic forces (Eq (2.2)) and the angle of deflection due to gravitational forces (Eq (3.1)). Comparing (2.2) with (3.1), we expect that in an encounter between two charged particles one of two different deflection regimes to hold, depending on whether (with the other parameters fixed), Q exceeds the value needed to give $\theta_{em} = \theta_{grav}$. The critical value of Q is

$$Q = 2m(\gamma M)/e \tag{11.1}$$

Naively one might suppose that (11.1) is also the criterion that the electromagnetic radiation produced by electromagnetic deflection equal that produced by gravitational deflection. However we have already mentioned that the equivalence principle suggests the radiation in the gravitational (free-fall) case should somehow be suppressed. A very naive misapplication of the equivalence principle might predict no radiation at all in the free-fall case. Figures (1)-(4) are presented to help explain the physical result, which is intermediate between the two naive extremes. First we recall the argument of Thomson [24], using the Fig. (1) from Reference [26]. An accelerated charge radiates electromagnetically because the sudden acceleration of the particle kinks the field lines near the particle and the kinks then propagate outward along the field lines. This sudden kinking of the field lines near the particle requires an acceleration, but does not require relativistic velocities for the particle. On the other hand, we here concentrate on a virtual quantum picture which does require relativistic velocities (large γ). In Fig. (2) we consider a moving charge (m,e), with $\gamma >> 1$, viewed from the frame of the large charge (M,Q). In this frame the Coulomb field of e is the field that resembles the plane wave, with the field lines schematically represented in Fig. (2a). In Fig. (2b) the field lines are replaced by virtual quanta. Consider now the case of electromagnetic deflection. The virtual photons of Fig. (2b) are uncharged, so when the test charge e is accelerated by the electric field due to Q, the net deflection yanks the charge away froms its attendant cloud of virtual quanta, as in Fig. (3). Some of the quanta (no longer virtual) escape to infinity as the produced bremsstrahlung radiation.

Fig. (4) shows the situation for gravitational deflection. Recall that gravitational fields deflect even light. Now since $\gamma >> 1$, the test particle and the photons undergo the same deflection. The virtual quanta are not separated from their charges, so we expect the electromagnetic radiation to be suppressed in this case.* The freely falling particle is trying very hard not to radiate.

That there is any radiation at all arises from the long-range nature of the electromagnetic and gravitational fields which enables the test particle to sample the non-stationary aspects of its situation. From the virtual quantum picture of Figs. (2)-(4), one would say that the radiation produced is due to gradients in the field of (M,Q); the "photons nearer the mass M are deflected more." This leads to some separation of the virtual photons from their charge, producing radiation, though a substantially smaller amount than in the case of electromagnetic deflection.

Obviously the comments in the above paragraphs also apply to virtual gravitons and the production of gravitational radiation. In Figures (2)-(4) simply interpret the wiggly lines as gravitons instead of photons.

The total amount of bremsstrahlung radiation produced for gravitational and for electromagnetic encounters has been calculated as:

$$E_{tot}\,\{\text{Grav. Rad.; Grav. Defl.}\} = 16\left(\frac{(4\gamma Mm)}{b^3}\right)^2 \ell n(\gamma) \ . \qquad (11.2a)$$

* This shows that the acceleration in Fig (1) should be non-gravitational; otherwise the field lines locally fall with the charge and no strong kinking occurs.

$$E_{tot}\ \{\text{E.M. Rad.; E.M. Defl.}\} = \frac{\pi}{4}\frac{\gamma^2 Q^2 e^2}{b^3}\left(\frac{e}{m}\right)^2 \qquad (11.2b)$$

$$E_{tot}\ \{\text{E.M. Rad.; Grav. Defl.}\} = A^2\ \frac{\gamma^3 e^2 M^2}{b^3}\ \ell n\gamma \qquad (11.2c)$$

$$E_{tot}\ \{\text{Grav. Rad.; E.M. Defl.}\} = \frac{\gamma^2 Q^2 e^2}{b^3}\ B^2\ \ell n\gamma \qquad (11.2d)$$

where A^2 and B^2 are constants of order unity. Equations (11.2a) and (11.2b) are calculated above in Sections IX and II respectively. The other two are worked out in the Appendix.

The calculated amounts of radiation allow us to make comparisons between them. For instance, let us compare the electromagnetic radiation energy produced in a brehmsstrahlung encounter for electromagnetic deflection [Eq (11.2b)] to that for gravitational deflection [Eq (11.2c)]. We find that the two contributions will be equal when Q is adjusted (γ,b,e,m,M fixed) so that (neglecting logarithms)

$$\frac{\gamma^2 Q^2 e^2}{b^3}\left(\frac{e}{m}\right)^2 \sim \frac{\gamma^3 e^2 M^2}{b^3}\ , \qquad (11.3)$$

i.e. when

$$eQ \sim \gamma^{\frac{1}{2}}\ mM. \qquad (11.4)$$

Similarly, we may compare the gravitational radiation produced in an electromagnetic deflection [Eq (11.2d)] with that produced in a gravitational deflection [Eq (11.2a)]. We find, neglecting logarithms, equal contributions when Q is adjusted so that

$$\frac{\gamma^2 Q^2 e^2}{b^3} \sim \frac{\gamma^3 M^2 m^2}{b^3} \tag{11.5}$$

which again leads to (11.4).

It will be noted that (11.4) is not the same condition as the condition (11.1) that angles of deflection in the electromagnetic and gravitational cases be equal. Eq (11.4) says that for $\gamma >> 1$ the total energy produced (whether gravitational or electromagnetic) in an electromagnetic encounter is much greater than that produced in a gravitational encounter with the same deflection. In other words, for the same total power to be radiated, the angle in the case of electromagnetic deflection is a factor $\gamma^{\frac{1}{2}}$ smaller than the angle in the gravitational deflection case.

The higher power of γ (i.e. γ^3 vs γ^2) in formulae for radiation due to gravitational deflection appears because the gravitational deflection angle θ_{grav} is finite as $\gamma \to \infty$, i.e. much larger than θ_{em} which goes to zero. There are simple relationships between the total radiation produced and the angle of deflection (recalling Eqs (2.2) and (3.1)):

$$E_{tot}\ \{\text{Grav. Rad; Grav. Defl.}\} \sim \frac{\gamma^3 m^2}{b}\, \theta^2_{grav} \quad , \tag{11.6a}$$

$$E_{tot}\ \{\text{E.M. Rad; E.M. Defl.}\} \sim \frac{\gamma^4 e^2}{b}\, \theta^2_{em} \quad , \tag{11.6b}$$

$$E_{tot}\ \{\text{E.M. Rad; Grav. Defl.}\} \sim \frac{\gamma^3 e^2}{b}\ \theta^2_{grav} \quad , \tag{11.6c}$$

$$E_{tot}\ \{\text{Grav. Rad; E.M. Defl.}\} \sim \frac{\gamma^4 m^2}{b}\ \theta^2_{em} \quad , \tag{11.6d}$$

which make explicit the smaller radiation for the same angle of deflection in the free-fall case.

An interesting aspect of the equivalence principle applies here. The orbit of the charged test particle in free fall could perhaps be considered an accelerated orbit in some sort of coordinate system in flat space. Case 1 of Section X would thus be viewed as a classic bremsstrahlung with gravitational deflection. Whatever coordinatization is used must agree with (3.1) on the net deflection since this is determined by measurements in the asymptotic region of the spacetime. Taking this viewpoint seriously leads to an incorrect estimate based on the electromagnetic deflection formula (11.6b) but with θ_{grav} incorrectly inserted instead of θ_{em}. The incorrect result $\sim\gamma^4$ is larger by a factor γ than the correct (11.6c).

Another amusing result concerns the situation where the condition

$$eQ = 2\gamma Mm \tag{11.7}$$

is exactly satisfied (e and Q have the same sign). There is then no net deflection and a naive application of flat space electrodynamics predicts no radiation while the actual energy radiated is given by Eqs. (11.2a) - (11.2d) and is dominated by the terms (11.2b),(11.2d) arising from the electromagnetic acceleration.* Figure 5 describes this

*There is a complication here that if M possesses a charge Q also, its Riemann tensor must be calculated on that basis; the Riemann tensor of a charged black hole contains terms depending on the charge. However as noted in Section IX the dominant terms in the Riemann tensor in distant encounters are those that depend only on M.

situation from the virtual quantum viewpoint. The process may be viewed as arising from the fact that the uncharged virtual quanta are gravitationally deflected away from the test particle. More accurately, the quanta are following a straight line (geodesic) and it is the test particle which is deflected away from them.

XII. The Accuracy of Virtual Quantum Calculations

It is well known that for purely electromagnetic calculations the virtual quantum technique is accurate for $\gamma\to\infty$ so Eq(11.2b) is accurate. For the purely gravitational case, the total radiation has been computed via at least three techniques other than the virtual quantum result reported in Eq (11.2a) the Green's function techniques of Peters[16] the "Zero Frequency Limit" which was recently used by Smarr[13] ; and a very detailed calculation by Kovacs and Thorne[26] . The purely gravitational calculation of [1] requires the imposition of a geometrically reasonable low frequency cutoff for the virtual quantum spectrum. The ambiguity of the low frequency cutoff introduces an ambiguity in only a very slow function of the energy produced. For instance, neither Peters' [16] nor Kovacs and Thorne's result [26] have the $\ell n\gamma$ factor of Eq (11.2a) while Smarr does find such a factor.

Kovacs and Thorne have given a very detailed analysis of the radiation process. It appears from their analysis that the method of virtual quanta, together with the zero frequency limit results, could have produced an accurate prediction of the radiation, by simply observing where the two estimates cross, and taking the one which predicts the lower energy. Kovacs and Thorne state that this would change the cut-off behavior in the angular integrals (9.12) eliminating the logarithmic term in Eq (9.14). Aside from the logarithm, when account is taken of the different regimes of applicability all the techniques are in reasonable agreement on Eq (11.2a). The calculation of Eq (11.2c) in the Appendix also requires a low frequency cutoff. However, Eq (11.2c) may be compared to calculations due to Peters [16] who based his result on a weak field Green's function technique. He estimates, based on this technique

$$E_{tot}\,\{\text{E.M. Rad.; Grav. Defl.}\} \sim \frac{64\pi(eM)^2\gamma^3}{b^3}$$

Peters [27] has also given a calculation of E_{tot} {Grav.Rad.; E.M. Defl.} (compare Eq (11.2d), which is

$$E_{tot}\,\{\text{Grav.Rad.; E.M. Defl.}\} = \frac{\pi\gamma^2 Q^2 e^2}{4b^2}$$

Considering the difference in the techniques and the regions of validity of the approximation, these are also in satisfactory agreement.

Appendix

a) Electromagnetic Radiation from Free-Fall Bremsstrahlung

Using the cross sections for conversion (10.1)-(10.3) we may calculate, via the virtual quantum technique, the electromagnetic radiation produced when a charged particle in a hyperbolic orbit in a gravitational field is deflected. (This is case 1 of Section X). We begin with the equivalent spectrum of gravitational radiation given by Eq (6.9). Because the conversion cross sections (10.1)-(10.3) are independent of ω , no natural cutoff presents itself so we retain the lower frequency limit of b^{-1} in the equivalent pulse.

The produced electromagnetic energy in the frame of the small particle, m, is the amount of scattered "conversion" electromagnetic radiation:

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_m = \left.\frac{d^2E}{d\omega dA}\,\frac{d\sigma}{d\Omega}\right|_{G\to E} . \tag{A.1}$$

To obtain the energy spectrum and angular distribution in the frame of the heavy particle, we apply the standard transformation (2.7). The total energy of the produced electromagnetic radiation is gotten by integration equation (A.1) over angle and frequency in the large mass frame. Let $d\sigma/d\Omega = (e^2/\pi)f(n_{3m},\varphi)$. Then we may write

$$\int d\Omega_\infty d\omega_\infty \left.\frac{d^2E}{d\omega d\Omega}\right|_\infty = \int d\Omega_\infty \left(\frac{2Me}{\pi}\right)^2 \frac{\gamma^{-2}}{b^3}\, \frac{f(n_{3m},\varphi)}{(1-vn_{3\infty})^3} \int_{x_0=(1-vn_{3\infty})\omega_{\infty 0}b}^{\infty} x^2K_2^{\,2}(x)dx. \tag{A.2}$$

The Bessel function integral is carried out as in Section IX, and the angular integral is estimated using Eq (10.3); we obtain:

$$E_{tot}\ \{\text{E.M. Rad.; Grav. Defl.}\} = A^2\ \gamma \frac{^3 e^2 M^2}{b^3}\ \ell n\gamma \tag{A.3}$$

with A^2 a constant of order unity.

Again this result, to logarithms, can be estimated in the way described in Section VII.

b) Bremsstrahlung Radiation of Gravitational Waves in Electromagnetic Deflection

The case of the production of gravitational radiation due to the deflection from electromagnetic forces in an electromagnetic bremsstrahlung encounter (case 2 of Section X) can also be treated by the techniques developed here. In this case, there is an electromagnetic plane wave incident on the test mass which gives conversion scattering into gravitational waves, via (10.3).

The incident virtual quantum flux $d^2E/d\omega dA$ is thus that given by (2.3) or (2.4). No cutoff is required. In the lab frame we obtain the result

$$\left.\frac{d^2E}{d\omega d\Omega}\right|_{\infty} = \gamma^{-2}(1 - vn_{3\infty})^{-2}\ \left.\frac{d^2E}{d\omega dA}\ \frac{d\sigma}{d\Omega}\right|_{E\to G} \tag{A.3}$$

in analogy to equation (2.7).

The total gravitational radiation energy produced in this case is found to be

$$E_{tot}\ \{\text{Grav. Rad.; E.M. Defl.}\} = B^2\ \frac{\gamma^2 Q^2 e^2}{b^3}\ \ell n\gamma \tag{A.4}$$

where B^2 is another constant of order unity.
This result is $\sim(m^2/e^2)$ smaller than the electromagnetic radiation produced in such an encounter, and it should be noted that this goes as γ^2, as opposed to the γ^3 quoted in Equation (11.2a) above, for gravitational radiation produced by gravitational deflection.

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F I G U R E S

Fig. 1: An explanation in terms of electric field lines (due to J. J. Thompson[24]) of the radiation induced when a charge undergoes an acceleration. Outside a sphere of size ct, the field lines are centered on the position which would have been occupied by the charge, had it continued to move to the left at speed $|v| << c$. Since the particle reversed direction at $t = 0$, inside the ct sphere the field lines are essentially those of a Coulomb field moving to the right at speed $|v|$. The resulting kinks of the field lines move outward with speed c, and constitute the transverse radiation field. This figure adapted from Ref [25], fig 4.6.

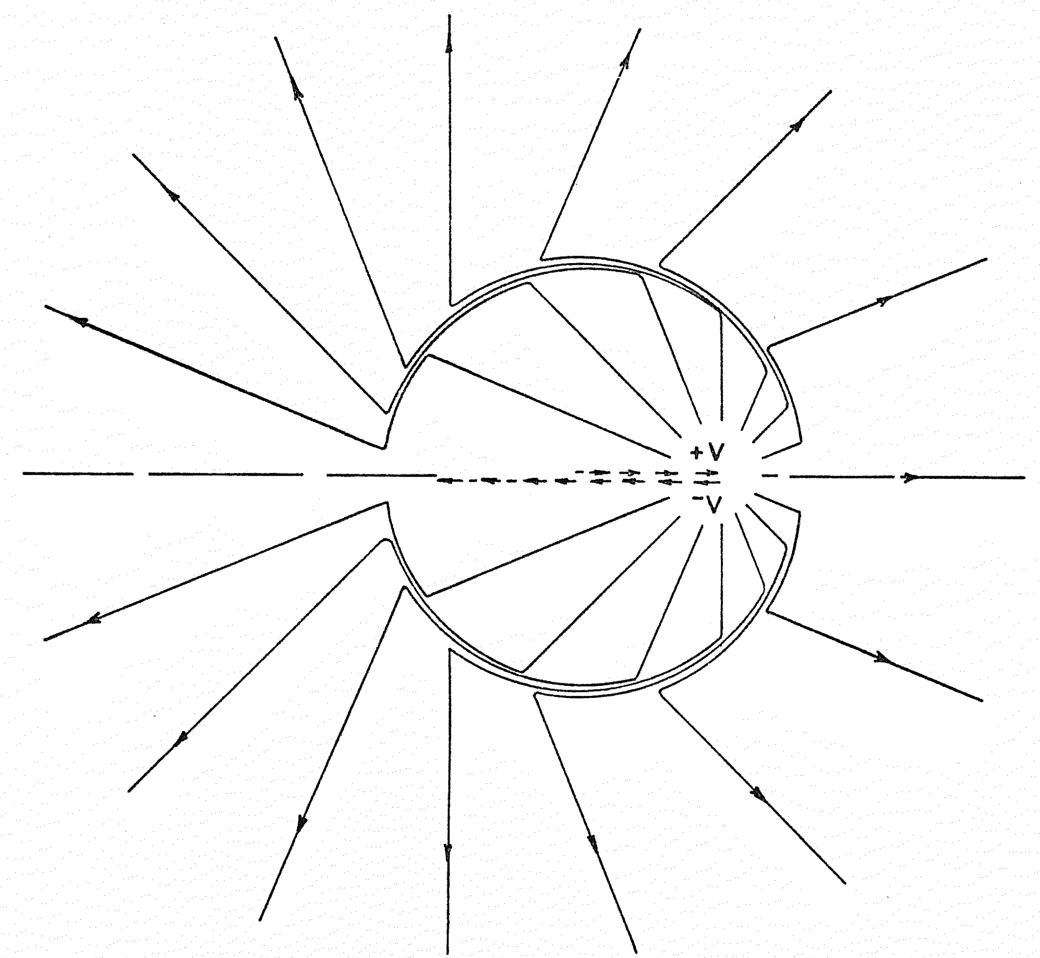

FIG. 1

Fig. 2 a: A charged test particle of mass m, charge e and energy $m\gamma$ undergoes an encounter with impact parameter b on a much more massive particle of mass M, charge Q. In the frame of (M,Q) the Lorentz transformed Coulomb field of (m,e) takes on characteristics closely similar to those of a plane wave [2], [3].

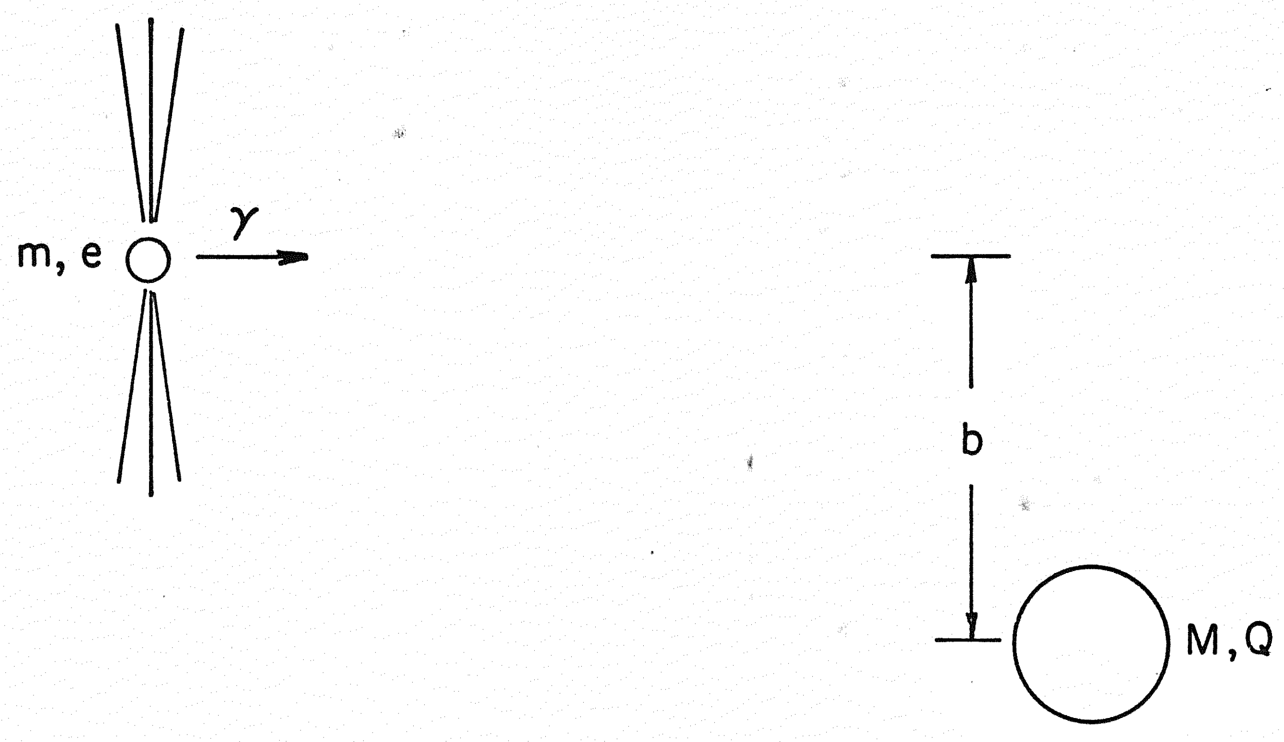

FIG. 2a

Fig. 2 b: The effective plane wave has been replaced by an equivalent pulse of virtual photons. It has been shown[1] that the spherically symmetric gravitational field of (m,e) when viewed from the frame of (M,Q), takes on the character of a plane pulse of gravitational radiation as in Fig 2.a, and can be replaced by an equivalent pulse of virtual gravitons, as in Fig 2.b.

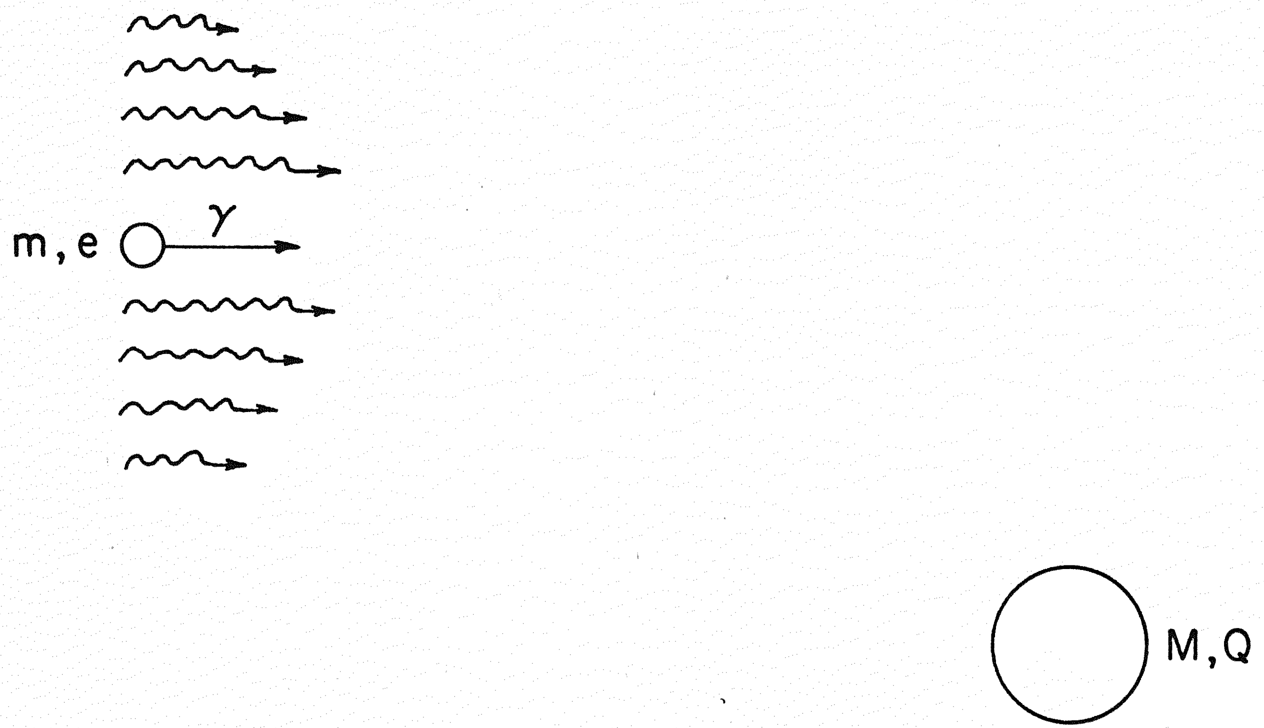

FIG. 2b

Fig 3: An equivalence principle "explanation" of electromagnetic bremsstrahlung. The condition $eQ >> 2\gamma mM$ guarantees that gravitational deflection is negligible. The electromagnetic acceleration (here repulsive) deflects the charged particle (m,e) away from part of its virtual photon cloud. The uncharged virtual photons, separated from the charge, become the produced bremsstrahlung radiation reaching infinity. The total electromagnetic radiation produced is $\sim\gamma^4 e^2 \theta_{em}^2/b$ [Eq 11.6b]. The test particle also possesses a cloud of (uncharged) virtual gravitons, and they behave similarly to the virtual photons just described. Hence we expect gravitational radiation to be produced also, and in fact the total gravitational radiation produced is $\sim\gamma^4 m^2 \theta_{em}^2/b$ [Eq 11.6d].

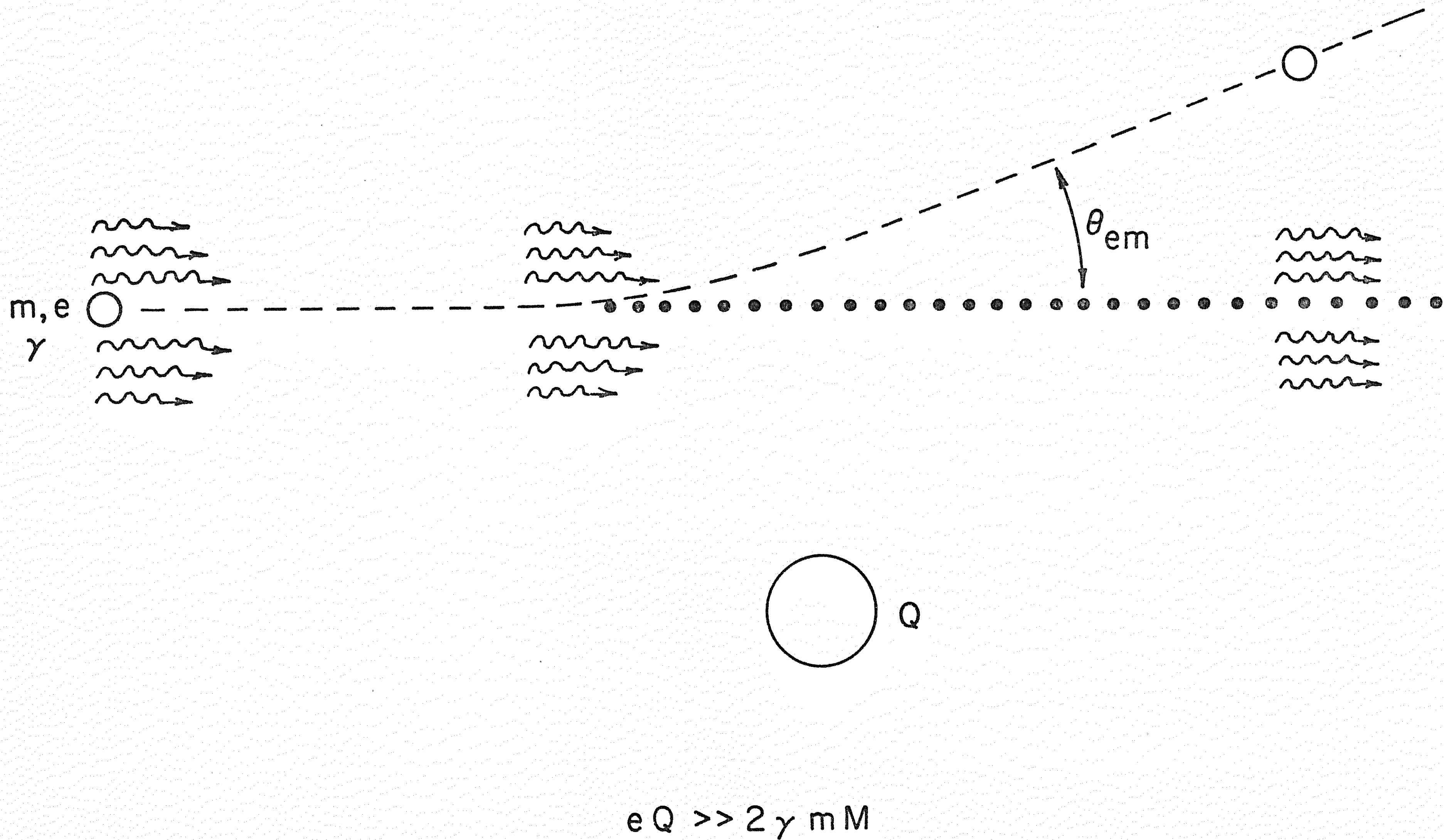

FIG. 3

Fig 4: Gravitational bremsstrahlung, electromagnetic deflection assumed negligible. Here the uncharged virtual gravitons (or photons, if the test particle is charged) fall with the particle (since $\gamma >> 1$) according to the equivalence principle. Some radiation is still produced, but by non-local effects, and Fig. 4 gives a qualitative explanation of the suppression of the radiation (by a factor γ) compared to the case of the same deflection order electromagnetic forces shown in Fig 3. The total gravitational radiation produced is $\sim\gamma^3 m^2 \theta_{grav}^2/b$ [Eq 11.6a]; if the test particle is charged, the total electromagnetic radiation produced is $\sim\gamma^3 e^2 \theta_{grav}^2/b$. [Eq 11.6c]

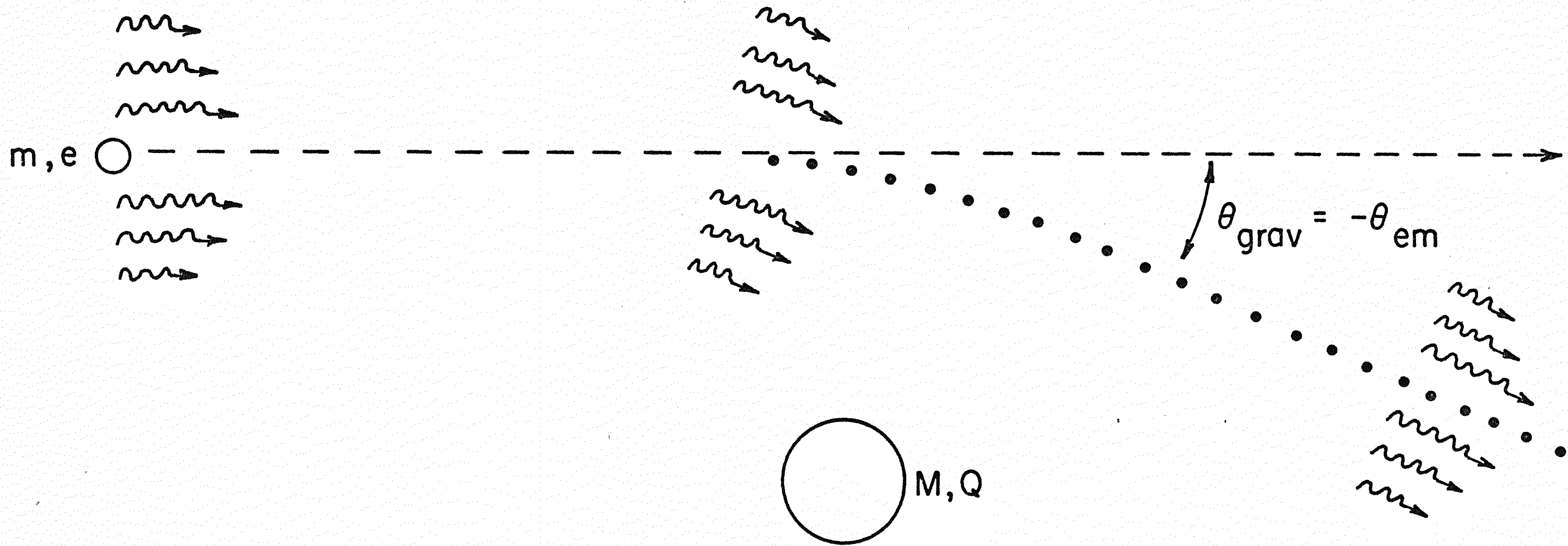

FIG. 5