DIAS-STP-80-32

On Monopole Systems with Weak Axial Symmetry*

by

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Abstract

Let $(\vec{\Phi},\vec{A})$ be an SO(3) Yang-Mills-Higgs system which is a real-analytic, static, finite-energy solution of the Bogomolny field equations $\vec{B}=\vec{D}\Phi$. We show that the zero-set of the current $\vec{J}=\Phi_\wedge\vec{D}\Phi$ is of dimension at most one. Using this property of $\vec{J}$ we obtain the curious result that if the system is axially symmetric, in the weak sense that all local scalar gauge-invariants are axially symmetric, the topological charges must be located on the axis of symmetry and must be of equal magnitude and alternate sign. In particular, if the charges are of uniform sign they must be concentrated at a single point. The fact that the charges of spherically symmetric monopoles are bounded by unity is obtained as a corollary. It is also shown that a master-potential for the invariant fields that was found earlier to exist for systems with additional symmetry, exists as a direct consequence of weak axial symmetry alone.

* Talk presented at the 1980 Conference on Differential Geometric Methods in Mathematical Physics, Clausthal, Germany.

1. Introduction

Let $(\Phi,\vec{A})$ be a static $(\partial_t=0)$ purely magnetic $(A_0=0)$ finite energy SO(3) Yang-Mills-Higgs system satisfying the first-order Bogomolny[1] field equation in Euclidean 3-space E(3),

$$\vec{B}=\vec{D}\Phi\ , \qquad \text{where} \quad \vec{B}=\vec{\nabla}\times\vec{A}+\tfrac{1}{2}\vec{A}\overset{\times}{\wedge}\vec{A}\ , \quad \vec{D}\Phi=\vec{\nabla}\Phi+\vec{A}_\wedge\Phi\ , \tag{1.1}$$

with boundary conditions

$$\vec{B}\to 0\ , \qquad \Phi^2\to c^2\neq 0\ , \quad \text{as} \quad |\vec{x}|\to\infty \tag{1.2}$$

Here $\times$ and $\wedge$ denote outer-product in space and isospace respectively, c is a constant, and we suppose that $(\vec{A},\Phi)$ are real analytic. The real analyticity is not a strong assumption because it has been shown[2] to hold(in at least one gauge) for solutions of (1.1)(1.2) which satisfy quite mild conditions concerning the Sobolov norms of the fields.

In some previous papers we have shown[3] that if the system $(\vec{A},\Phi)$ is axially symmetric in the strong or conventional[4] sense that there exists a local (scalar) isovector $\omega(x)$ such that for any local(scalar) isovector $\lambda(x)$ we have

$$D_\psi\,\lambda(x)=\omega(x)_\wedge\lambda(x) \qquad \text{where} \qquad D_\psi=xD_y-yD_x\ , \tag{1.3}$$

then the topological charge distribution must be as stated in the abstract. It has also been shown[5] that if the strong axially symmetric system is mirror-symmetric (symmetric with respect to reflexions in planes through the axis of symmetry) then it admits a (scalar-isoscalar) masterpotential $W(x)$ from which invariant fields such as (Φ,Φ) and $(D_\psi\Phi,D_\psi\Phi)$ can be obtained by differentiation.(Bracket denotes inner product in isospace).

The rather surprising nature of the result concerning the charge-distribution raises the question as to whether (1.1) is really the most general definition of axial symmetry and whether the results would still hold under a weaker definition. Accordingly, the purpose of this note is to reconsider the situation under what would seem to be the weakest reasonable definition of axial symmetry, namely, that local gauge-invariants such as the inner-products

$$(\Phi,\Phi),\quad (\Phi,B_\alpha)\quad,\quad (B_\alpha,B_\psi)\quad,\qquad x_\alpha=(z,\rho), \tag{1.4}$$

be independent of the azimuthal angle ψ. (Here the space indices are expressed in cylindrical coordinates to avoid a spurious ψ-dependence).

It turns out that, for real analytic fields satisfying (1.1) and (1.2), weak axial symmetry actually implies strong axial symmetry so that the previous results still hold. Furthermore, it turns out that the result for the charge-distribution can be obtained more or less directly from weak axial symmetry, and that the masterpotential W exists even without the hypothesis of mirror-symmetry. The role played by W also becomes much clearer.

In order to establish these results it is first necessary to establish that the zero-set $\chi(J)$ of the current

$$\vec{J} = \Phi\wedge\vec{D}\Phi \tag{1.5}$$

is located on a manifold which is at most 1-dimensional i.e. consists of at most isolated points and analytic curves. This particular result is independent of axial symmetry, and in the axially symmetric case it implies that $\chi(J)$ can lie on at most the axis of symmetry and symmetrical rings around the axis.

Finally for completeness we derive as a simple corollary the known result that a spherically symmetric monopole must have unit charge and derive also the single equation for the master potential which is sufficient to close the system of field equations in mirror-symmetric case. In the latter derivation we use mirror symmetry only in the weak form

$$(B_\psi,B_\alpha)=0\ . \tag{1.6}$$

2. Zero Sets of the Higgs Field and the Current.

We commence with the result that the zero-set $\chi(J)$ the current $\vec{J}$ is at most 1-dimensional, and it will be convenient to consider also the zero-set of the Higgs field Φ, although from the definition of $\vec{J}$ the zero-set of Φ is contained in $\chi(J)$. Since Φ and $\vec{J}$ are real analytic their zero-sets are analytic submanifolds of $E(3)$, and hence what we have to do is eliminate $E(3)$ itself and 2-dimensional submanifolds. As mentioned before, the results are quite general(independent of axial symmetry) and in the(weak)axially symmetric case they reduce the possibilities for $\chi(J)$ to the axis of symmetry and isolated symmetrical rings around the axis.

In the case of the Higgs field we first note that the boundary condition (1.2) excludes $E(3)$ itself, and requires that any 1- and 2-dimensional sub-manifolds be closed. Next using the Bianchi identity $\vec{D}\cdot\vec{B}=0$ we obtain from (1.1) the usual second-order field equation

$$D^2\Phi = 0, \tag{2.1}$$

for the Higgs field, and from (2.1) we obtain at once the equation

$$\Delta(\Phi,\Phi) = 2(\vec{D}\Phi,\vec{D}\Phi)\geqslant 0 \tag{2.2}$$

for (Φ,Φ). Since (2.2) shows that (Φ,Φ) is a subharmonic function it follows that (Φ,Φ) cannot vanish on a closed 2-dimensional surface without vanishing in the interior, and hence vanishing throughout $E(3)$, in contradiction to the boundary condition. Thus the zero set of Φ consists of at most isolated points and analytic closed curves[6]. In particular, in the axially symmetric case it reduces to at most isolated points on the axis of symmetry and isolated rings around the axis. (The whole symmetry axis is excluded by the boundary condition).

In the case of the current $\vec{J}$ the space $E(3)$ itself is excluded for a different reason, namely, that if $\vec{J}$ is identically zero the gauge and Higgs field completely decouple and it is well-known that there are no non-trivial finite-energy solutions for the decoupled system. Now suppose that $\vec{J}$ vanishes on an analytic 2-surface Σ . The results for Φ show that Σ contains finite elements $\delta\Sigma$ on which $(\Phi,\Phi) \neq 0$. But from the Bianchi identity and the field equations (1.1) we obtain after some algebraic manipulations

$$\vec{D}\cdot\vec{J} = 0 \quad \text{and} \quad \vec{D}\times\vec{J} = -\Phi_\wedge\vec{J} + (\Phi,\Phi)^{-2}\left\{\vec{J}\overset{\times}{\wedge}\vec{J} + 2(\Phi,\vec{B})\times\vec{J}\right\}, \qquad (2.3)$$

and it is easy to see that this equation implies that on $\delta\Sigma$ the normal derivative to $\vec{J}$ can be expressed as a linear combination of $\vec{J}$ and the tangential derivatives to $\vec{J}$, with coefficients which are smooth on $\delta\Sigma$ (and may be functions of $\vec{J}$). Since, by iteration, the same will be true of the normal derivative of any order, and $\vec{J}$ is real analytic, it follows that $\vec{J}$ cannot vanish on $\delta\Sigma$ without vanishing on a finite 3-volume containing $\delta\Sigma$ and hence vanishing throughout $E(3)$. Thus in the non-trivial case the zero-set $\chi(J)$ of $\vec{J}$ can be at most 1-dimensional i.e. can consist only of isolated points and analytic curves(not necessarily closed). In particular, in the (weak) axially symmetric case the zero-set of $\vec{J}$ can consist of at most points on the symmetry-axis and isolated rings around the axis.

3. Orthonormal Triads in the Complement of $\chi(J)$.

The reason that we need to locate the zero-set $\chi(J)$ of $\vec{J}$ is that in the complement $\tilde{E}(3) = E(3) - \chi(J)$ we can construct orthonormal triads of isovectors and use them to implement the weak axial symmetry. In this section, we give the construction. First we note that since $\chi(J)$ is at most 1-dimensional $\tilde{E}(3)$ is connected, though not necessarily simply-connected. Now let P be any point of $\tilde{E}(3)$. Then at P , and by analyticity, in a finite neighbourhood N of P , we have $\vec{J} \neq 0$. But then in N we have $\Phi \neq 0$, and for at least one component, J say, of $\vec{J}$ we have $J \neq 0$. Furthermore from the definition of $\vec{J}$ we have $(\Phi, J) = 0$. It follows that in N the isovectors

$$\omega_1 = \Phi/|\Phi| , \qquad \omega_2 = J/|J| \qquad \text{and} \qquad \omega_3 = \omega_1 \wedge \omega_2 \text{, in } N \qquad (3.1)$$

form an orthonormal triad. The triad(3.1) formed with fixed* component J of $\vec{J}$ may not be extendable to all of $\tilde{E}(3)$, because J might vanish at finite distances from P . But since $\vec{J} \neq 0$ in $\tilde{E}(3)$ and $\tilde{E}(3)$ is connected, it is clear that $\tilde{E}(3)$ can be covered with overlapping neighbourhoods N, each with at least one triad. As we shall see in the next section this result is sufficient to implement real axial symmetry in $\tilde{E}(3)$, and that is all that we shall need. Note that the ω's are real analytic and single-valued in $\tilde{E}(3)$ since they are quotients of functions which are real analytic in $E(3)$.

* Here and throughout the components of J are understood to be expressed in cylindrical coordinates.

4. Implementation of Weak Axial Symmetry on $\tilde{E}(3)$

Using the triad (3.1) we can construct the isovector

$$\omega = -\tfrac{1}{2}\,\varepsilon_{abc}(\omega_a, \mathcal{D}_\varphi \omega_b)\,\omega_c \quad \text{in } N \text{, where } a,b, = 1,2,3\,. \tag{4.1}$$

Now let $\lambda(x)$ be an arbitrary axial-scalar isovector (space-tensor isovector whose space-indices are expressed in cylindrical coordinates). Then $\lambda(x)$ has the expansion

$$\lambda(x) = \big(\omega_a(x), \lambda(x)\big)\, \omega_a(x) \qquad \text{in} \qquad N\,, \tag{4.2}$$

and since by weak axial symmetry

$$\mathcal{D}_\varphi(\omega_a, \lambda) \equiv \nabla_\varphi(\omega_a, \lambda) = 0 \quad \text{in } N\,, \tag{4.3}$$

we obtain at once from (4.1) the relation

$$\mathcal{D}_\varphi \lambda(x) = \omega(x) \wedge \lambda(x) \qquad \text{in } N\,. \tag{4.4}$$

Equation (4.4) shows that the vector $\omega(x)$ implements the covariant derivative $\mathcal{D}_\varphi$ in N. To extend it to $\tilde{E}(3)$ we note that if ω_a' is any alternative basis e.g. for a neighbourhood N', and ω' is the isovector constructed as in (4.1) from ω_a', then from (4.4) we have

$$(\omega' - \omega, \lambda) = 0 \qquad \text{in} \qquad N \cap N'\,. \tag{4.5}$$

Thus we have

$$(\omega' - \omega, \omega_a) = 0 \quad \text{or} \quad \omega = \omega' \quad \text{in} \quad N \cap N'\,. \tag{4.6}$$

Equation (4.6) shows that ω is unique and basis-independent in $N \cap N'$, a result that can also be verified directly from (4.1) using weak axial symmetry. Since $\tilde{E}(3)$ can be covered with overlapping neighbourhoods N, it follows that ω and eq. (4.4) extend to all of $\tilde{E}(3)$ as required. Note that ω will also be real analytic in $\tilde{E}(3)$. In particular ω will be unique or single-valued in $\tilde{E}(3)$.

From (4.4) on $\tilde{E}(3)$ it also follows (for $\lambda = \omega_a$) that

$$F_{i\varphi} \wedge \omega_a = (\mathcal{D}_i \omega) \wedge \omega_a \text{ on } E(3), \text{ where } F_{ij} = [\mathcal{D}_i, \mathcal{D}_j] = \varepsilon_{ijk} B_k\,, \tag{4.7}$$

and since the ω_a are non-degenerate on $\tilde{E}(3)$, we then have

$$\mathcal{D}_i \omega = F_{i\varphi} = \varepsilon_{i\varphi k} B_k \qquad \text{on} \quad \tilde{E}(3)\,. \tag{4.8}$$

We shall refer to equations (4.4) on $\tilde{E}(3)$ and (4.8) as the equations of weak axial symmetry. With these equations in hand we turn to the topological charge distribution.

5. The Topological Charge

The general expression for the topological charge contained in a volume V with smooth surface S on which $\Phi \neq 0$ is well-known[7] to be

$$Q_V = \frac{1}{4\pi}\int_S f_{ij}\,dx^i dx^j \qquad \text{where} \qquad f_{ij} = (\phi, F_{ij}) - (\phi, D_i\phi_\wedge D_j\phi), \qquad \phi = \Phi/|\Phi|, \tag{5.1}$$

is the Maxwell field projected out of F_{ij} by ϕ. In the axially symmetric case f_{ij} is independent of ψ and hence if we choose V to be a volume of revolution we have

$$Q_V = \frac{1}{4\pi}\int f_{i\psi}\,dx^i d\psi = \frac{1}{2}\oint_C f_{i\psi}\,dx^i\,, \tag{5.2}$$

where the line integral is along a curve C in S orthogonal to the azimuthal direction. The precise nature of C depends on the topology of S and will be specified later. Now from (5.1) we have, in particular,

$$f_{i\psi} = (\phi, F_{i\psi}) + (D_i\phi, \phi_\wedge D_\psi\phi)\,. \tag{5.3}$$

Hence using the equations (4.4)(4.8) of weak axial symmetry, we have

$$f_{i\psi} = (\phi, D_i\omega) + (D_i\phi, \phi_\wedge(\omega_\wedge\phi)),$$

$$= (\phi, D_i\omega) + (D_i\phi, \omega) = \nabla_i(\phi,\omega), \qquad \text{on } \tilde{E}(3). \tag{5.4}$$

Suppose now that the curve C lies in $\tilde{E}(3)$ except possibly for the end-points x_1 and x_2. Then (5.4) can be used in the integral (5.2) and we obtain the closed expression

$$Q_V = \frac{1}{2}\left[b(x_1) - b(x_2)\right] \qquad \text{where} \qquad b(x_a) = \lim_{x\to x_a}(\phi(x), \omega(x)), \tag{5.5}$$

for the topological charge Q_V contained in V.

It is well-known that the topological charge as defined in (5.1) must be located at the zeros of Φ. From our results on these zeros, we see that in the axially symmetric case the charges can be located only at isolated points on the axis of symmetry and on rings around the axis. Our first step will be to show that the rings are not possible, so that the charge must be located only on the axis.

6. Elimination of Rings of Topological Charge.

To show that rings of topological charge are not possible we let R be a ring of zeros of Φ. From the results on $\chi(J)$.., R is isolated in $\tilde{E}(3)$ and hence can be surrounded by a torus of revolution whose surface lies entirely in $\tilde{E}(3)$. Letting the volume V of the previous section be such a torus, the curve C must be a circle in S which loops the torus in the direction orthogonal to the toroidal axis. Then, since C is closed and (ω, ϕ) is single-valued in $\tilde{E}(3)$, the expression (5.5) for the charge inside the torus yields zero, as required. Thus the rings are eliminated and the charge is located only on the symmetry axis.

7. Limits of $\omega_\wedge\bar{\Phi}$ and ω^2 on the Axis of Symmetry

In order to determine the charge distribution on the symmetry axis(z-axis, say) we shall need the limits of $\omega_\wedge\bar{\Phi}$ and ω^2 as $\rho\to 0$, and hence we consider these limits in this section. First from (4.4) we have

$$|(\omega_\wedge\bar{\Phi})| = |D_\psi\bar{\Phi}| \leq \rho(|D_x\bar{\Phi}|+|D_y\bar{\Phi}|) \quad \text{in} \quad \tilde{E}(3), \tag{7.1}$$

and since $\vec{A}$ and Φ are real analytic throughout $E(3)$ we then have

$$\underset{\rho=0}{\mathrm{Lt}}\,(\omega_\wedge\bar{\Phi}) = 0. \tag{7.2}$$

Next from (4.8) we have

$$|\nabla_i\omega^2| = |2(\omega,F_{i\psi})| \leq 2\rho|\omega|(|F_{ix}|+|F_{iy}|) \leq \rho(\omega^2+K^2) \quad \text{in } \tilde{E}(3), \tag{7.3}$$

where K^2 is the maximum of $(\vec{B},\vec{B})$ in $E(3)$. Since $\vec{B}$ is non-trivial and is real analytic, we have $0<K<\infty$, and hence on integrating (7.3) along a straight line between any two points x and x_0 in $\tilde{E}(3)$ we have

$$\frac{\omega^2(x)+K^2}{\omega^2(x_0)+K^2} \leq e^{R|\vec{x}-\vec{x}_0|} \quad \text{where} \quad R=\max(\rho,\rho_0). \tag{7.4}$$

Keeping ρ_0 fixed and letting $\rho\to 0$ we see from Cauchy convergence that $\omega^2(x)$ has a finite limit as $\rho\to 0$. Then, by letting $x=(z,\rho)$ and $x_0=(z_0,\rho)$ we see that

$$\underset{\rho=0}{\mathrm{Lt}}\,\omega^2(x) = n^2 \tag{7.5}$$

where the finite value n^2 is <u>independent of z</u>. Equations (7.2) and (7.5) give the required limits.

Incidentally, we note that since $\omega_\wedge\bar{\Phi}$ is real analytic in $\tilde{E}(3)$, it must be periodic in ψ, and hence, since from (4.4) we have

$$\omega^2(\omega_\wedge\bar{\Phi}) = D_\psi^2(\omega_\wedge\bar{\Phi}) \to \nabla_\psi^2(\omega_\wedge\bar{\Phi}), \quad \text{as } \rho\to 0, \tag{7.6}$$

the constant n in (7.5) must be an <u>integer</u>. In the next section n will be identified with the topological charge.

8. Charge Distribution on the Symmetry-Axis.

To determine the charge distribution on the z-axis, we let the volume of the previous section be any volume of revolution which lies inside all rings of zeros of Φ and cuts the z-axis at just two points z_1 and z_2 where $\bar{\Phi}\neq 0$. Then, apart from z_1 and z_2, the surface of V lies entirely in $\tilde{E}(3)$ and the curve C is a curve joining z_1 to z_2 with all its interior points in $\tilde{E}(3)$. From (5.5) we then have for the charge in

$$Q_V = \tfrac{1}{2}[b(z_1)-b(z_2)], \quad \text{where} \quad b(z) = \underset{\rho=0}{\mathrm{Lt}}\,b(z,\rho). \tag{8.1}$$

But from the limits obtained in section 7 we have for $\Phi\neq 0$,

$$b^2(z) = \underset{\rho=0}{\mathrm{Lt}}\,(\omega,\phi)^2 = \underset{\rho=0}{\mathrm{Lt}}\,[\omega^2-(\omega_\wedge\phi)^2] = \underset{\rho=0}{\mathrm{Lt}}\,\omega^2 = n^2 \tag{8.2}$$

where n^2 is independent of z. It follows that

$$Q_V = 0, \pm n. \tag{8.3}$$

But since the volume V may contain any number of charges, and two successive charges of the same sign would yield $Q_V=\pm 2n$, eq. (8.3) implies that the charges must be of alternate sign and of the same magnitude.

In particular, if the charges are required to have the same sign then there can be only a single charge(of arbitrary magnitude).

9. Equivalence of Weak and Strong Axial Symmetry.

The results for the charge-distributions were obtained using only the weak axial symmetry equations (4.4) and (4.8). However, for completeness and for the discussion of the masterpotential, we wish to show that for analytic fields satisfying (1.1) and (1.2) weak axial symmetry actually implies strong axial symmetry i.e. eqs. (4.4) and (4.8) can be extended from $\tilde{E}(3)$ to $E(3)$.

For this purpose we note that eq. (4.8) can be integrated along any curve Γ from x_0 to x in $\tilde{E}(3)$ to yield

$$\omega(x) = \omega_\Gamma(\omega(x_0), \vec{A}) + \int_\Gamma \varepsilon_{i\varphi k} B_k dx^i , \tag{9.1}$$

where ω_Γ denotes the parallel transfer of $\omega(x_0)$ along Γ with respect to the connection $\vec{A}$. The value of $\omega(x)$ is path-independent because the integrability condition for (4.8) is just (4.4) in $\tilde{E}(3)$ in the special case when λ is replaced by the components of $\vec{B}$.

But now since $\vec{A}$ and $\vec{B}$ are real analytic throughout $E(3)$ and the complement $\chi(J) = E(3) - \tilde{E}(3)$ consists only of points and curves (so that $\tilde{E}(3)$ is connected) eq. (9.1) defines an analytic extension of $\omega(x)$ as $x \to \chi(J)$. Furthermore, for any $\lambda(x)$ which is real analytic in $E(3)$ eq. (4.4) then extends analytically to $E(3)$, and this is just the condition of strong axial symmetry.

Note that the result would not necessarily hold if $\chi(J)$ contained a 2-dimensional submanifold Σ because Σ would necessarily disconnect $\tilde{E}(3)$. Then $\omega(x)$ would not necessarily be path-independent and the values of $\omega(x)$ obtained coming from the two sides might not agree. This can perhaps be seen more clearly by considering the infinitesimal version of the above proof. First we note from (7.4) that ω^2 remains uniformly bounded as $x \to \chi(J)$ and that by recycling this result into (4.8) $\omega(x)$ and all its finite derivatives are uniformly bounded as $x \to \chi(J)$. Thus ω has a smooth(C^∞ limit) as $x \to \chi(J)$. But if $\chi(J)$ contained a 2-dimensional Σ, the smooth limits on either side of Σ might not agree. When $\chi(J)$ is at most one-dimensional, however, the values as $x \to \chi(J)$ are independent of the direction of approach and so ω has a limit which is unique as well as smooth when $x \to \chi(J)$. The analyticity of the extension follows by differentiating (4.8) again to obtain the elliptic equation

$$D^2 \omega = J_\varphi . \tag{9.2}$$

Since the coefficients in this elliptic equation are analytic the smooth solutions must be analytic as required.

10. The Existence of the Masterpotential W.

We wish to show that the existence of the masterpotential W found in previous papers[5] is a direct consequence of axial symmetry condition (1.3). Inserting the field equation (1.1) in (1.3) we have

$$D_i \omega = \varepsilon_{i\varphi k} D_k \Phi , \tag{10.1}$$

and taking the inner product of this equation with ω we obtain

$$\nabla_i \omega^2 = 2\varepsilon_{i\varphi k} \nabla_k(\omega, \Phi) - \rho^2 \nabla_i \Phi^2 . \tag{10.2}$$

But eq. (10.2) is just the Cauchy-Riemann-type equation which was used previously to deduce the existence of a masterpotential W such that

$$\frac{\partial W}{\partial z} = (\omega, \Phi), \qquad \rho \frac{\partial W}{\partial \rho} = \omega^2, \qquad \Delta W = \Phi^2, \tag{10.3}$$

and thus the result is established and the role of W clarified. Note that (10.1) is itself a type of covariant Cauchy-Riemann equation and hence might be of some use in seeking explicit solutions of the field equations.

11. Field equation for W in the Mirror-Symmetric Case.

It is known that when the system is mirror-symmetric the field equation for W contains only the fields ω^2, $h=|\Phi|$ and $b=(\omega,\phi)$ which occur in (10.3) and hence this equation and (10.3) form a closed system. We therefore wish to derive the field equation for W directly from our present results. From the second-order field equation (2.1) for Φ we obtain

$$\Delta h = (\vec{D}\phi)^2 h . \tag{11.1}$$

But from the normalization of ϕ and the mirror-symmetry condition (1.6) we have

$$(\phi, D_\alpha \phi) = 0 \qquad \text{and} \qquad (D_\varphi \phi, D_\alpha \phi) = 0 \tag{11.2}$$

respectively, where $x_\alpha = (z, \rho)$. Hence $D_\alpha \phi$ has a component only in the direction of the vector

$$\hat{\omega} = \phi_\wedge D_\varphi \phi = \phi_\wedge(\omega_\wedge \phi) = \omega - (\phi,\omega)\phi , \tag{11.3}$$

and

$$(\vec{D}\phi)^2 = \frac{1}{\rho^2}(D_\varphi\phi)^2 + \frac{(\omega, D_\alpha\phi)^2}{\tilde{\omega}^2} = \frac{k^2}{\rho^2} + \frac{u^2}{k^2} \tag{11.4}$$

where

$$k^2 = \omega^2 - b^2 \qquad \text{and} \qquad u_\alpha = \nabla_\alpha b - \rho\, \varepsilon_{\alpha\beta} \nabla_\beta h . \tag{11.5}$$

Hence from (11.1) we have

$$\Delta h = \left(\frac{k^2}{\rho^2} + \frac{u^2}{k^2}\right) h , \tag{11.6}$$

and this is the required equation for W. Of course, we must use (11.5) and (10.3) to make it explicit in W.

12. Corollary for the Spherically Symmetric Case.

We wish to show here that previous results[(12)], which state that a spherically symmetric charge distribution must be of maximum strength unity can be derived as a simple corollary to our present results. First we note that the results for $\chi(J)$ imply that a spherically symmetric charge must be located at the origin. Then since spherical symmetry simply extends axial symmetry to all three axes, we have from (1.3)

$$\vec{L}\lambda + \vec{\omega}_\wedge \lambda = 0 , \tag{12.1}$$

where $\vec{L}$ is the angular momentum operator, λ is any scalar isovector and $-\omega_z$ is our previous ω. But from (12.1) we have

$$(\vec{L}\times\vec{L})\lambda + 2\,\vec{\omega}_\wedge\lambda - \tfrac{1}{2}(\vec{\omega}\,\overset{\times}{\wedge}\,\vec{\omega})_\wedge\lambda = 0 , \tag{12.2}$$

and hence

$$\tfrac{1}{2}(\vec{\omega}\,\overset{\times}{\wedge}\,\vec{\omega})_\wedge\lambda = \vec{\omega}_\wedge\lambda \qquad \text{or} \qquad \tfrac{1}{2}(\vec{\omega}\,\overset{\times}{\wedge}\,\vec{\omega}) = \vec{\omega} , \tag{12.3}$$

since the λ are non-degenerate in $E(3)-O$. But this means that for each fixed x in $E(3)-O$ the $\vec{\omega}$ form a canonical set of generators for the isospin group. Furthermore since they act only on the 3-dimensional space of the λ's they can generate only the trivial or 3-dimensional representation. Thus

$$(\omega, \omega) \leq 2 \qquad \text{and} \qquad (\omega_z, \omega_z) \leq 1 . \tag{12.4}$$

But then from (5.5) we have for a volume V enclosing the origin

$$Q_V \leq |b(z)| = |(\omega_z, \phi)| \leq |\omega_z| \leq 1 , \tag{12.5}$$

which is the required result.

Acknowledgements

We are indebted to E. Seiler, C. Taubes and S. Deser for some very illuminating discussions.

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