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On Axially-Symmetric Finite-Energy Monopole Configurations

by

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Abstract

Axially symmetric finite energy monopole configurations are investigated for the gauge group SO(3) with the Higgs field in the adjoint representation . To avoid the complications due to gauge freedom gauge invariant fields are introduced and used throughout . From topological and continuity considerations it is argued that the only regular axially symmetric magnetic charge distributions permitted are isolated charges of uniform strength and alternate sign located along the axis of symmetry . In particular , if there is only one sign , the magnetic charge must be located at a single point . For a zero Higgs potential the minimal energy (first order Bogomolny) field equations take a simple form when written in terms of the gauge-invariant fields . In general , there are nine equations for nine (axially symmetric) fields , but these reduce to five equations for five fields if a further symmetry (invariance under reflexions in planes through the axis of symmetry) is imposed . Remarkably , four of the equations are the same whether the reflexion symmetry is imposed or not , and these four equations can be completely solved in terms of a master potential . From these and the remaining equations (just one for the case of mirror symmetry) the asymptotic behaviour of the functions at large distances and in the neighbourhood of the origin (the location of the charge) is obtained and studied in some detail .

1. Introduction

The discovery of the complete set of finite-action solutions [1] to the static 4-dimensional self-dual Yang-Mills equations (instantons) has revived interest in the problem of finding finite energy solutions to the static 3-dimensional Yang-Mills-Higgs equations (monopoles). So far, only the spherically symmetric solution with unit magnetic charge has been found explicitly [2] , but recently multi-monopole solutions have been shown to exist [3] . The existence is shown for the first-order (Bogomolny) field equations which are obtained in the Prasad-Sommerfield limit in which (for the gauge-group SO(3) and the Higgs field in the adjoint representation) the Higgs potential is set equal to zero. So far the proof is valid also only in the case the magnetic charges are isolated and of equal strength unity. (The charges should be expected to have the same sign in any case since otherwise even the long-range forces between them do not vanish [4]).

In the present paper we wish to study axially-symmetric configurations for the gauge-group SO(3) and the Higgs field in the adjoint representation, both from the point of view of the topology and of the field equations. To avoid the problems of gauge freedom we introduce fields which are gauge invariant and use these throughout (such fields will in general have string singularities) .

Our results come under two headings. First we rederive a result concerning the charge distribution which we have already reported elsewhere [5] , using the gauge-invariant fields to strengthen and generalize it. The present result is that any regular axially symmetric charge distribution can be located only at isolated points situated on the axis of symmetry, with equal and opposite values of the charge at alternate points. In particular, if only one sign of the charge is allowed, then all the charge must be concentrated at a single point. This result is suprising since we should expect any colinear charge configurations, in particular the 2-monopole configuration, to be symmetric about the axis of colinearity

Apparently, the axial symmetry of the topological charge does not necessarily imply the axial symmetry of the fields. The result may, perhaps, be understood by considering the standard example of a 2-monopole system [6], where the Higgs field breaks the axial symmetry by rotating about an orthogonal axis as one proceeds from one monopole to the other.

The second set of results concern the field equations. These will now, of course, be equations for single monopoles with charge located at the origin. They are derived from the static Hamiltonian

$$H = \int \{ \tfrac{1}{2}(\underline{B}.\underline{B}) + \tfrac{1}{2}(\underline{D}\Phi, \underline{D}\Phi) + V(\Phi) \} d_3\underline{x} , \tag{1.1}$$

where Φ is the Higgs field, $\underline{B}$ the static magnetic Yang-Mills field

$$\underline{B} = \nabla \times \underline{A} + \underline{A} \times \underline{A} , \tag{1.2}$$

V the potential, and the inner product is in isospace. In the case that G = SO(3), Φ is in the adjoint representation and $V(\Phi)$ is zero (and is replaced by the boundary condition $(\Phi,\Phi) \to c^2$, as $r \to \infty$) the Hamiltonian (1.1) can be re-written [7] as

$$H = \int \tfrac{1}{2}(\underline{B} - \underline{D}\Phi)^2 d_3\underline{x} + 4\pi c Q , \tag{1.3}$$

where Q is the topological charge, and hence the field equations which minimize H are the first-order 'self-dual' equations

$$\underline{B} = \underline{D}\Phi , \tag{1.4}$$

which we shall refer to as the Bogomolny-Prasad-Sommerfield (BPS) equations. This is the system of equations for which the multi-monopole solutions have been shown to exist and which we shall study in the axially-symmetric case.

In general the BPS equations (1.4) constitute nine equations for twelve functions (three functions corresponding to the gauge freedom). By introducing the gauge-invariant fields we reduce (1.4) to nine equations for nine functions. Axial symmetry removes the azimuthal dependence of these nine functions, but, being an abelian group, does not reduce their number. Nevertheless, in terms of the gauge-invariant functions the equations take quite a simple intuitive form and split naturally into two quartets and a single elliptic equation of the form

$$\Delta h = \sigma(h) h , \tag{1.5}$$

for the norm h of the Higgs field , where σ is a positive functional of h . These equations are presented in section 5.

The nine equations for nine unknown functions of section 5 can be reduced to five equations for five unknown functions by imposing a further symmetry, namely, symmetry with respect to reflexions in any plane through the axis of axial symmetry. A remarkable result is that four of the nine equations (one of the quartets mentioned above) remain completely unchanged by the imposition of this reflexion symmetry, and these four can be solved completely in terms of a single function, which we call the master-potential. The solution is given in section 9.

In the reflexion symmetric case there is only one further equation to solve, namely (1.1), and this can be considered as an equation for the master-potential. As it may be difficult to obtain exact solutions to this equation we concentrate for the rest of the paper on the asymptotic form of the fields at spatial infinity and at the origin, where the charge is located. At infinity it is shown that most of the fields fall-off exponentially, and we obtain exact solutions for the remaining fields when these fields are neglected. These exact solutions are singular at the origin but serve as asymptotic values for the fields in question, and they can even be used to improve the estimate for the fields which fall-off exponentially. At the origin the behaviour of all the fields is shown to be determined by a set of Legendre and associated Legendre functions and a number of free parameters are permitted. These parameters probably correspond to those predicted by Weinberg [8] on general grounds (index-theorem).

Finally, in view of the central role (1.5) plays , this equation is derived in section 2 for arbitrary gauge-groups , arbitrary representations of the Higgs field, and arbitrary (not necessarily axially symmetric) field configurations . The functional σ is given , in general , by the equation

$$\sigma = (\underline{D}\phi, \underline{D}\phi) + 2V'(h^2) \quad , \text{ where } \Phi = h\phi , \tag{1.6}$$

and prime denotes differentiation . It is shown for certain classes of potentials, including non-decreasing functions of h^2 and renormalizable potentials, that (1.6) implies the bound

$$h^2(\underline{x}) \leq c^2 \quad , \text{ where } h^2(\underline{x}) \to c^2 \text{ as } r \to \infty . \tag{1.7}$$

In conclusion it should be mentioned that not all of the results presented here are new. The existence of a master-potential was noted in ref. 9, the boundary conditions on the axis of symmetry were discussed in ref. 10, and finally, the boundary conditions we obtain have been derived independently for the mirror-symmetric case by S. Adler and P. Rossi [11] . We are grateful to Prof. Adler for communicating his results to us and for discussions concerning them.

Note added : After this paper was completed an explicit $Q = 2$ monopole solution was constructed by R. Ward [12] using the self-duality aspect of the Bogomolny equations. The solution describes a single monopole of charge 2 which is located at the origin and is axially symmetric.

2. Equation and Bound for the Norm of the Higgs Field.

We begin by obtaining eqns. (1.6) and (1.7) for the norm of the Higgs field Φ for any gauge group and any representation of the Higgs field. Writing

$$\Phi = h\phi \text{ , where } h^2 = (\Phi, \Phi) , \; (\phi, \phi) = 1 , \tag{2.1}$$

the Hamiltonian (1.1) becomes

$$H = \int d_3\underline{x} \left\{ \tfrac{1}{2}(\underline{B}, \underline{B}) + \tfrac{1}{2}(\underline{\nabla} h)^2 + \tfrac{1}{2} h^2 (\underline{D}\phi, \underline{D}\phi) + V(h^2) \right\} . \tag{2.2}$$

Note that the cross-term in $\underline{\nabla} h$ and $\underline{D}\phi$ drops out because of the normalization of ϕ . On varying (2.2) with respect to h we obtain (1.6) as required.

Equation (1.6) implies that

$$\Delta h^2 = 2(\underline{\nabla} h)^2 + 2h^2(\underline{D}\phi, \underline{D}\phi) + 4h^2 V'(h^2) , \tag{2.3}$$

and this equation can be used to obtain the bound (1.7) for certain classes of potentials. Let us consider three classes.

(i) $V(h^2)$ non-decreasing. Then (2.3) shows that h^2 is a subharmonic function, and hence automatically satisfies (1.7). However, unless $V = 0$ this result means that for such potentials the finite energy solutions must be trivial since finite energy requires that $V \to 0$ as $r \to \infty$.

(ii) $V(h^2) = 0$. In this case (2.3) still implies that h^2 is subharmonic and the finite energy imposes no condition. Accordingly h^2 satisfies (1.7) with any boundary value c^2.

(iii) $V(h^2)$ renormalizable and bounded below: Then $V(h^2)$ must be quadratic in h^2, and assuming that it is not monotonically increasing as in (i) it must be of the form

$$V(h^2) = \lambda(h^2 - c^2)^2 \quad , \quad \lambda > 0 . \tag{2.4}$$

This will be recognized as a spontaneously broken potential. The constant c has been chosen so that $h^2 \to c^2$ as $V \to 0$ at $r = \infty$. In this case h^2 is not subharmonic, but, nevertheless, if h^2 is sufficiently smooth, it can be shown that (1.7) still holds. For suppose not. Then the maximum of h^2 would be at $h^2 > c^2$, and since at the maximum Δh is negative, the two sides of (2.3) would have opposite signs at this point. Thus (1.7) holds also for renormalizable, spontaneously broken, potentials. Note that by substituting eq. (2.3) in (2.2) one sees that σh^2 is integrable.

In the case that we shall be concerned with in this paper, namely $G = SO(3)$ and Φ in the adjoint representation we can obtain an insight into the meaning of σ by linking it to the current $\underline{J}$ which appears in the second-order field equations

$$\underline{D}^2\underline{\Phi} = \frac{\partial V}{\partial \Phi} \quad , \quad \underline{D}\times\underline{B} = -\underline{J} \quad , \quad \underline{D}\cdot\underline{B} = 0 \; , \tag{2.5}$$

because in that case $\underline{J}$ is given by

$$\underline{J} = \underline{\Phi}\wedge\underline{D}\underline{\Phi} \equiv h^2\underline{j} \quad , \quad \text{where} \quad \underline{j} = \phi\wedge\underline{D}\phi \; , \tag{2.6}$$

the wedge denoting the usual SO(3) outer product. Hence

$$\underline{j}^2 = (\underline{D}\phi, \underline{D}\phi) \quad \text{and} \quad \sigma = \underline{j}^2 + 2V'(h^2) \, . \tag{2.7}$$

Thus for $G = SO(3)$ and Φ in the adjoint representation σ consists of a potential term and the square of the normalized current.

In the case when SO(3) is spontaneously broken the only long-range fields are the electromagnetic field (belonging to the unbroken SO(2) subgroup) and (if $V = 0$) a neutral Higgs field, and for both these fields there is no current. Since the remaining fields are massive and hence may be expected to fall-off exponentially we should therefore expect $\underline{j}$ and hence σ to fall-off exponentially as $r \to \infty$ and we now give a heuristic proof that such is the case, at least in the Bogomolny limit (1.4). In this limit, (2.6) can be written in the form

$$\underline{J} = \underline{\Phi}\wedge\underline{B} \, , \tag{2.8}$$

and hence from (2.5) and (1.4) we have

$$\underline{D}\times\underline{J} = \underline{B}\overset{\times}{\wedge}\underline{B} - \underline{\Phi}\wedge\underline{J} \, . \tag{2.9}$$

At first sight it might appear that the first term on the right-hand-side of (2.9) should dominate since $\underline{B}$ has a long-range component and we are expecting $\underline{J}$ to decay exponentially. However, the long-range part of $\underline{B}$ is abelian and drops out in the wedge product, as can be seen explicitly from the identity

$$h^2\,\underline{B}\overset{\times}{\wedge}\underline{B} = \underline{J}\overset{\times}{\wedge}\underline{J} + 2(\underline{\Phi},\underline{B})\times\underline{J} \, , \tag{2.10}$$

which follows from (2.8). This identity shows that not only is $\underline{B}\overset{\times}{\wedge}\underline{B}$ not dominant, but it may even be neglected, because the finite-energy conditions $h^2 \to c^2$, $\underline{J} \to 0$ and $(\underline{\Phi},\underline{B}) \to 0$, show that each of the terms in (2.10) is dominated by $\underline{\Phi}\wedge\underline{J}$ (the first because $|\underline{J}|^2 \ll |\underline{J}|$ and the second because $|(\underline{J},\underline{\Phi})| \ll |\underline{\Phi}|^2$). Thus we have the asymptotic equation

$$\underline{D}\times\underline{J} \approx -\underline{\Phi}\wedge\underline{J} \, , \tag{2.11}$$

as $r \to \infty$. Taking the covariant curl of this equation, using the identities

$$(\underline{\Phi},\underline{J}) = 0 \quad , \quad \underline{D}\cdot\underline{J} = 0 \; , \tag{2.12}$$

which follow from the definition of $\underline{J}$ and the Bianchi identity in (2.5) respectively, and again neglecting terms of order $|\underline{B}||\underline{J}|$ compared with $h^2|\underline{B}|$, we obtain the covariant Helmholtz equation

$$\underline{D}^2\underline{J} \approx h^2\underline{J} \approx c^2\underline{J} \, . \tag{2.13}$$

in the asymptotic region. This equation exhibits the exponential decay of $\underset{\sim}{J}$ and hence of $\sigma = h^{-4}(\underset{\sim}{J}, \underset{\sim}{J})$.

In section 12 we verify the exponential decay in more detail for the axially symmetric case.

3. Axially Symmetric Gauge-Invariant Fields for SO(3).

For the rest of this paper we limit ourselves to the case in which the gauge-group is SO(3) and the Higgs field is in the adjoint representation. There are then twelve fields altogether (nine $\underset{\sim}{A}$'s and three $\underset{\sim}{\Phi}$'s) and three of these correspond to the gauge freedom.

We wish to impose axial symmetry (around the z-axis), and to avoid the usual complications due to gauge freedom we shall introduce instead of the twelve fields ($\underset{\sim}{A}$, $\underset{\sim}{\Phi}$) nine gauge-invariant fields ($\mathcal{A}$, h). The proceedure is as follows : We introduce an orthonormal triad ω_a, a = 1,2,3 with $\omega_1 = \phi$, and ω_2, $\omega_3 = \omega_1 \wedge \omega_2$ arbitrary, but related to the covariant fields $\underset{\sim}{J}$. The base vector ω_1 is well defined except at the zeroes of $\underset{\sim}{\Phi}$ and in appendix B it is shown that for any real analytic fields satisfying the field equations these zeroes can be only at isolated points on the z-axis (and possibly isolated rings around the axis). Since ($\underset{\sim}{\Phi}$, $\underset{\sim}{J}$) = 0, the base vectors ω_2, ω_3 are well defined everywhere except at the zeroes of $\underset{\sim}{J}$ and in appendix B it is shown that these zeros can only be the whole z-axis (and possibly isolated rings around the axis). Thus in principle the space can be covered by an overlapping set of such non-degenerate triads (ω_1, ω_2, ω_3) constructed from the fields except on the z-axis (and possibly rings around it). Note that although the region where the triads are well-defined may not be simply connected, the vectors $\omega_1 = \Phi/|\Phi|$, $\omega_2 = J_i/|J_i|$, etc. are single-valued because they are quotients of functions which are single-valued throughout the whole space.

In practice, however, to obtain the field equations it will be convenient to chose the single triad defined by

$$\omega_1 = \phi \,,\ \omega_2 = D_\varphi\phi/|D_\varphi\phi| \,,\ \omega_3 = \omega_1 \wedge \omega_2 \,, \tag{3.1}$$

because of its relationship with the axial symmetry. Although in principle the triad (3.1) could become degenerate on some 2-surface, this does not appear to happen and in any case should not affect the content of the field equations, which are self-consistent even at points where $D_\psi \phi = 0$.

From the orthogonality of the ω_a-basis in (3.1) we have

$$(\omega_a, D_i\omega_b) + (\omega_b, D_i\omega_a) = \nabla_i(\omega_a, \omega_b) = 0\,, \qquad (3.2)$$

from which we see that the $D_i\omega_a$ have the expansions

$$D_i\omega_a = \epsilon_{abc} A_i^b \omega_c\,, \text{ where } A_i^a(\underline{x}) = \tfrac{1}{2}\epsilon_{abc}(\omega_b, D_i\omega_c)\,. \qquad (3.3)$$

There are actually only eight $\underset{\sim}{A}$ -fields because from (3.1) we have

$$A_\psi^2 = 0\,, \qquad (3.4)$$

where the subscript ψ denoted azimuthal component . The eight $\underset{\sim}{A}$ -fields are gauge- invariant by construction , and are single valued wherever they are defined because they are quotients of single-valued functions . Together with h these eight fields constitute the required nine gauge-invariant fields ($\underset{\sim}{A}$, h) .

Because of their gauge invariance , axial symmetry is trivially implemented on ($\underset{\sim}{A}$, h) as

$$A^a_{i,\psi} = h_{,\psi} = 0\,, \qquad (3.5)$$

where ψ is the azimuthal angle . Thus axial symmetry is simply the statement that $\underset{\sim}{A}$ and h are independent of ψ .

4. Topological Charge Density, Hamiltonian Density and BPS Equations for the Fields ($\underset{\sim}{A}$, h).

In order to express the topological charge and Hamiltonian derivatives in terms of ($\underset{\sim}{A}$, h) it is convenient to introduce the formal curvature tensor

$$\mathcal{F}_i^a = -\epsilon_{ijk}(\nabla^j A^{ak} + \tfrac{1}{2}\epsilon_{abc} A^{bj} A^{ck})\,. \qquad (4.1)$$

If the topological charge density $Q(\underline{x})$ is then defined in the usual way as

$$Q(\underline{x}) = \underline{\nabla}.\underline{f}(\underline{x})\,, \quad \text{where } \underline{f}(\underline{x}) = (\phi, \underline{B}) - \tfrac{1}{2}(\phi, \underline{D}\phi \times \underline{D}\phi)\,, \qquad (4.2)$$

is the Maxwellian magnetic field, a short calculation (appendix A) shows that $\underline{f}(\underline{x})$ is simply

$$f_i(\underline{x}) = \epsilon_{ijk}\nabla^j A^{1k}\,. \qquad (4.3)$$

Thus the Maxwell field can be written as the curl of an SO(3)-invariant. Of course, the fact that $\underline{f}$ is a curl means that, just as in the Dirac monopole theory, we can have a monopole only if $\underset{\sim}{A}$ (and hence the triad ω_a)has a string singularity.

From (2.2) one finds that the Hamiltonian density takes the form

$$2\mathcal{H}(\underline{x}) = (\underline{\nabla}h)^2 + h^2\sum_{b\neq 1}(\underset{\sim}{A}^b)^2 + \sum_a(\underline{\mathcal{F}}^a)^2 + V(h^2)\,. \qquad (4.4)$$

This Hamiltonian exhibits the Higgs mechanism explicitly, with 'mass' h. (The apparent 'gauge-freedom' corresponding to rotations around the ϕ -axis is forbidden by (3.4)) . The second-order field equations can be obtained from (4.4) in the usual way, but we shall not consider them here (especially as they correspond only to extrema which are constrained by axial symmetry). Instead we consider the limit V = 0, when the absolute minimum can be obtained by the Bogomolny trick. For (4.4) the trick consists of writing it in the equivalent form

$$2\mathcal{H}(\underline{x}) = (\underline{\nabla}h + \underline{\mathcal{F}}^1)^2 + \sum_{b\neq 1}(\underline{\mathcal{F}}^b - h\epsilon^{1bc}\underset{\sim}{A}^c)^2 - 2(\underline{\nabla}h.\underline{\mathcal{F}}^1 - h\epsilon^{1bc}\underline{\mathcal{F}}^b.\underset{\sim}{A}^c)\,. \qquad (4.5)$$

Using the Bianchi identity

$$\underline{\nabla}.\underline{\mathfrak{F}}^{a} + \epsilon^{abc}\underline{\mathfrak{F}}^{b}.\underline{\mathcal{A}}^{c} = 0 \qquad (4.6)$$

the last term in (4.5) is seen to be a pure divergence, and using (4.3) its integral is identified with the topological charge. Thus the energy is minimal when the squared-terms in (4.5) are zero, and so we obtain the BPS equations

$$\underline{\mathfrak{F}}^{1} = -\underline{\nabla} h, \quad \underline{\mathfrak{F}}^{b} = h\,\epsilon^{1bc}\underline{\mathcal{A}}^{c}, \; b \neq 1. \qquad (4.7)$$

Clearly (4.7) contains nine equations for the nine unknown fields ($\underline{\mathcal{A}}$,h). Finite Energy Bounds on the seperate fields are obtained by noting that (4.4) is a sum of positive terms .

5. <u>Separation of the Azimuthal Components. Explicit Field Equations.</u>

For practical purposes it will be convenient to separate the azimuthal and non-azimuthal components of the $\underline{\mathcal{A}}$-fields,by setting

$$\mathcal{A}^{a}_{\psi} = (-b, 0, -k), \quad \mathcal{A}^{a}_{\alpha} = (-t_{\alpha}, u_{\alpha}, -v_{\alpha}), \qquad (5.1)$$

where $x_{\alpha} = (z, \rho)$, $\alpha = 1, 2$. From the axially symmetric condition (3.5) the formal curvature tensor $\underline{\mathfrak{F}}^{a}$ separates correspondingly into

$$\mathfrak{F}^{a}_{\psi} = \left(\epsilon_{\alpha\beta}(t_{\alpha,\beta} + u_{\alpha}v_{\beta}),\ \epsilon_{\alpha\beta}(u_{\beta,\alpha} + t_{\alpha}v_{\beta}),\ \epsilon_{\alpha\beta}(v_{\alpha,\beta} + t_{\alpha}u_{\beta})\right)$$
$$\mathfrak{F}^{a}_{\alpha} = \left(\epsilon_{\alpha\beta}(-b_{,\beta} + k u_{\beta}),\ \epsilon_{\alpha\beta}(k t_{\beta} - b v_{\beta}),\ \epsilon_{\alpha\beta}(-k_{,\beta} - b u_{\beta})\right) \qquad (5.2)$$

and the Maxwell field $\underline{f}(\underline{x})$ into

$$f_{\alpha} = \epsilon_{\alpha\beta} b_{,\beta}, \quad f_{\psi} = \epsilon_{\alpha\beta} t_{\beta,\alpha}. \qquad (5.3)$$

The Hamiltonian density takes the form

$$2\mathcal{H}(\underline{x}) = (\underline{\nabla} h)^2 + h^2\Big(\sum_{\alpha}(u_{\alpha}^2 + v_{\alpha}^2) + \frac{k^2}{\rho^2}\Big) + \sum_{a}\Big(\sum_{\alpha}(\mathfrak{F}^{a}_{\alpha})^2 + \frac{1}{\rho^2}(\mathfrak{F}^{a}_{\psi})^2\Big), \qquad (5.4)$$

where the $\underline{\mathfrak{F}}$'s are given by (5.2), and the nine BPS field equations can be written as

$$b_{,\alpha} + \rho\,\epsilon_{\alpha\beta} h_{,\beta} = k u_{\alpha},$$
$$b u_{\alpha} + \rho h\,\epsilon_{\alpha\beta} u_{\beta} = -k_{,\alpha}, \qquad (5.5)$$

$$\rho\,\epsilon_{\alpha\beta}(u_{\alpha,\beta} + v_{\alpha}t_{\beta}) = hk, \qquad (5.6)$$

and

$$\epsilon_{\alpha\beta}(t_{\alpha,\beta} + u_{\alpha}v_{\beta}) = 0,$$
$$\epsilon_{\alpha\beta}(v_{\alpha,\beta} + t_{\alpha}u_{\beta}) = 0, \qquad (5.7)$$
$$b v_{\alpha} + \rho h\,\epsilon_{\alpha\beta} v_{\beta} - k t_{\alpha} = 0.$$

The nine equations have been separated into the sets (5.5)(5.6)(5.7) for the following reason: The first quartet of equations (5.5) are independent of the four fields t_α, v_α in (5.1) and the last quartet (5.7) are linear in these fields. Hence if we set $t_\alpha = v_\alpha = 0$ we obtain a consistent subset of five eqns. (5.5)(5.6) for the five variables (h, b, k, u_α) and the first four of these equations will remain unchanged. Thus any solution of the BPS equations will be an extension of the simpler set (5.5)(5.6) with $t_\alpha = v_\alpha = 0$. This circumstance is not accidental, but due to the fact that t_α and v_α can be eliminated by imposing a further symmetry, namely the reflexion symmetry discussed in the next section. What is remarkable, however, is that the quartet (5.5) remains unchanged, even when the reflexion-symmetry is not imposed. In the mirror-symmetric case the full set of equations for the five functions (h, b, k, u_α) are just (5.5) and

$$hk = \rho\, \epsilon_{\alpha\beta}\, u_{\alpha,\beta} \,. \tag{5.8}$$

and the Hamiltonian density (5.4) reduces to

$$2\mathcal{H}(\underline{x}) = (\nabla h)^2 + h^2\Big(\sum_\alpha u_\alpha^2 + \frac{k}{\rho^2}\Big) + (\epsilon_{\alpha\beta} u_{\alpha,\beta})^2 + \frac{1}{\rho^2}\sum_\alpha\big[(b_\alpha - k u_\alpha)^2$$

$$+ (k_{,\alpha} + b u_\alpha)^2\big]. \tag{5.9}$$

The fact that the quartet of eqns. (5.5) are common to both the mirror-symmetric and non-reflexion-symmetric cases is fortunate because, as we shall see, this quartet can be solved explicitly.

6. Geometrical Interpretation of (k,b) and Relationship with Conventional Axial Symmetry.

A geometrical interpretation of the azimuthal components $A_\psi^a = (-b, 0, -k)$ may be obtained as follows: First, from the ψ-invariance of the inner-product (Φ, Φ) we see that Φ and $D_\psi \Phi$ are orthogonal and then from the definition of ω_2 we have

$$D_\psi \Phi = hk\omega_2 \,. \tag{6.1}$$

Similarly from the ψ-invariance of the inner-product $(D_\psi\Phi, D_\psi\Phi)$ we see that $D_\psi\Phi$ and $D_\psi^2\Phi$ are orthogonal. Hence $D_\psi^2\Phi$ lies in the (ω_1, ω_3)-plane and it is easy to check that the expansion coefficients are

$$D_\psi^2 \Phi = hk(b\omega_3 - k\omega_1) \,. \tag{6.2}$$

Furthermore, from the definition of $k\omega_3$ as $\phi \wedge D_\psi \phi$ we find that

$$D_\psi^2(D_\psi\Phi) = -(b^2+k^2)(D_\psi\Phi), \quad h \neq 0 \,. \tag{6.3}$$

Thus the 2-space spanned by $D_\psi\Phi$ and $D_\psi^2\Phi$ is an eigenspace of D_ψ^2 with eigenvalue $-(b^2+k^2)$. Since for each $\underline{x}$, $(\omega_a, D_\psi\omega_b)$ is a real antisymmetric matrix, it follows that D_ψ^2, and hence D_ψ, must be zero on the orthogonal 1-space i.e. we must also have

$$D_\psi w = 0 \,, \quad \text{where} \quad w = b\omega_1 + k\omega_3 \,, \tag{6.4}$$

and this is easily verified directly. Thus finally we have that the vectors w and $D_\psi^2\Phi$ both lie in the (ω_1, ω_3)- plane (perpendicular to ω_2) and are orthogonal to each other. Note that w is related to k and b by

$$(w, w) = b^2 + k^2, \quad (w, \phi) = b \,. \tag{6.5}$$

To connect these results with the standard formulation of axial symmetry we first note from (6.1)-(6.4) that

$$D_\psi \omega_a = \omega_\wedge \omega_a . \tag{6.6}$$

In other words, ω implements the covariant derivative D_ψ. On the other hand the conventional definition of axial symmetry is that

$$\omega_a(z,\rho,\psi+\epsilon) = R(z,\rho,\psi,\epsilon).\omega_a(z,\rho,\psi) , \tag{6.7}$$

where $R(\underline{x},\epsilon)$ is an a-independent orthogonal matrix in isospace , which is a smooth deformation of a constant matrix $R(\epsilon)$. Hence on differentiation , we have

$$\nabla_\psi \omega_a = \lambda(\underline{x})_\wedge \omega_a , \text{ where } \lambda = (\nabla_\epsilon R.R^{-1})_{\epsilon=0} , \tag{6.8}$$

and is a smooth deformation of a constant vector . Comparing (6.8) with (6.6) we see that

$$\omega = \lambda + A\psi , \tag{6.9}$$

which is the required relationship . For future reference we note that , from equation (6.6) , we have

$$\begin{aligned} \rho B_{i\wedge}\omega_a &= \epsilon_{ij\psi}[D_j, D_\psi]\,\omega_a = \epsilon_{ij\psi}(D_j(\omega_\wedge\omega_a) - \omega_\wedge D_j\omega_a) , \\ &= \epsilon_{ij\psi} D_j \omega_\wedge \omega_a , \end{aligned} \tag{6.10}$$

and hence

$$\rho B_i = \epsilon_{ij\psi} D_j \omega , \tag{6.11}$$

which includes (6.4) as a special case .

In view of the importance of the vector ω , it is worth remarking that ω can be defined directly using any non-degenerate basis by writing

$$\omega = -\tfrac{1}{2}\epsilon_{abc}(\omega_a, D_\psi\omega_b)\,\omega_c . \tag{6.12}$$

It is easy to verify that ω is independent of the basis chosen and has the required property

$$D_\psi \cdot = \omega_\wedge \cdot , \tag{6.13}$$

if all inner products of covariant quantities are ψ-independent. Since ω is independent of the basis it is well defined everywhere that a basis exists which we have seen to be the compliment of the z-axis (and possibly some rings around it). We shall see in the next section that the limit as we approach the z-axis presents no real problem.

7. Topological Charge Distributions.

It has been shown by Taubes that in at least one gauge solution to the BPS field equations (1.4) are real analytic and hence that the zeros of the norm h can be located only at isolated points or on isolated curves and surfaces. Furthermore, since $h \to c$ as $r \to \infty$, the curves and surfaces must be closed, and then, since h is subharmonic, the surfaces are ruled out. Thus the zeros of h, and hence the possible locations of the topological charge, are at isolated points or on isolated closed curves.

In the axially symmetric case such distributions would include a priori horizontal rings centred on the z-axis and isolated charges of arbitrary strength on the z-axis itself. However, it seems that not all such solutions are allowed. Indeed the only permissible ones seem to be isolated charges of uniform strength and alternating sign on the z-axis. In particular, if the charge has only one sign (as required by the BPS equations) then it would seem to be concentrated at a single point, corresponding to superimposed monopoles at that point.

These results follow essentially from the expression (5.3) for the Maxwell current. For let V be smooth volume of revolution either not intersecting the z-axis (torus) or intersecting the z-axis at two points (z_1, z_2), let S be the surface of V, L the projection of S on the (xz)-plane and t any smooth parameter for L. Then from (5.3) we have for the charge ΔQ contained in V

$$\Delta Q = \frac{1}{4\pi}\int_V d_3x(\nabla\cdot\underline{f}) = \frac{1}{4\pi}\int_S d_2s(\underline{n}\cdot\underline{f}) = \frac{1}{2}\oint_L dt\,\frac{\partial x^\alpha}{\partial t}(b)_{,\alpha} = \frac{1}{2}[b]_L, \tag{7.1}$$

where [b] denotes the values of b at the end-points of L. Now suppose first that there is a ring of charges. Since the triad ω_α is assumed to be well-defined except on the z-axis and on isolated rings we can surround the ring of charges with a torus on whose surface ω_α, and hence b, is well-defined. Letting the volume in (7.1) be such a torus the curve L becomes a closed ring, and hence [b] is zero provided that b is single-valued. But we recall from section 3 that b is single-valued wherever it is defined because it is the quotient of single-valued functions. Thus the term $[b]_L$ in (7.1) is indeed zero and we conclude that there cannot be a ring of charges.

Consider next the points on the z-axis ($\rho = 0$). For regular $\underline{A}$ and h we have

$$A_\psi = xA_y - yA_x \to 0, \quad D_\psi\Phi \to \nabla_\psi\Phi \to 0, \text{ for } \rho \to 0. \tag{7.2}$$

Hence by taking the limit of (6.1), as $\rho \to 0$ we obtain

$$hk = 0, \text{ for } \rho = 0. \tag{7.3}$$

Furthermore, since Φ must be periodic in ψ and the eigenvalues of ∇_ψ on periodic functions are $\pm in$ where n is an integer, we see from (7.2) and the limit of (6.3) that

$$b^2 + k^2 = n^2, \text{ for } \rho = 0. \tag{7.4}$$

The crucial point now is that n^2 is the same integer, for all z, because, as shown in appendix C, equation (6.11) implies that in the limit $\rho \to 0$ $b^2 + k^2$ is finite and independent of z. If we now let the smooth volume of revolution V above be simply connected and intersect the z-axis at two points z_1 and z_2 (where $h \neq 0$) the line L is no longer closed, but is a curve joining z_1 and z_2 and so for the topological charge inside such a volume we have from (7.1)

$$\Delta Q = \frac{1}{2}\left(b(z_1) - b(z_2)\right). \tag{7.5}$$

Since $h(z_a) \neq 0$, $a = 1,2$, we see from (7.3) and (7.4) that

$$k(z_a) = 0\,,\ b^2(z_a) = n^2\,, \tag{7.6}$$

and hence that

$$\Delta Q = 0\,, \pm n\,. \tag{7.7}$$

Since the volume V may contain any number of isolated charges, it follows at once that these charges must all have the same magnitude n and must alternate in sign. In particular, if only one sign is allowed, the total charge n must be located at a single point. Thus the BPS field equations (4.7) must actually describe a system with the charge located at the origin, and for the rest of this paper we shall work under this assumption.

Alternative derivations of the result of this section, using somewhat different assumptions have been given elsewhere. In reference 5 the result is proved assuming the existance of an ω which is regular everywhere, and satisfies (6.6), and in reference 13 the existance of a regular ω is established using the Bogomolny field equations and the fact that the fields are real analytic.

8. Reflexion and Vertical Symmetry.

The reflexion symmetry referred to in sect. 5 is symmetry with respect to the group of reflexions in the planes through the z-axis, and the reduction in the number of fields occurs because the combination of this group with the axial symmetry group is non-abelian (it is a semi-direct product). In practice, on account of the axial symmetry, it suffices to consider reflexions in any single-plane (the xz-plane say) and the reduction in the number of $\mathcal{A}$-fields occurs because four of them are odd under the reflexions R_y in this plane.

To determine which of the $\mathcal{A}$-fields are odd, we need to determine first the R_y-parities of the base-vectors ω_a. From the original BPS equation (1.4) we see that Φ is a psuedo-scalar with respect to any reflexion. Since D_ψ is odd with respect to R_y it follows that the R_y-parities of $(\omega_1, \omega_2, \omega_3)$ are (-1, 1, -1) respectively, and since the D_α are even with respect to R_y, it then follows from (3.3) that the R_y-parities of the $\mathcal{A}$-fields are as follows

$$(\mathcal{A}^1_\psi, \mathcal{A}^2_\alpha, \mathcal{A}^3_\psi)\ \text{even}\ ;\ (\mathcal{A}^1_\alpha, \mathcal{A}^2_\psi = 0, \mathcal{A}^3_\alpha)\ \text{odd}\,. \tag{8.1}$$

Thus reflexion symmetry eliminates four fields by imposing the condition

$$\mathcal{A}^1_\alpha = \mathcal{A}^3_\alpha = 0\,, \tag{8.2}$$

and in terms of the separated fields this condition is just

$$t_\alpha = v_\alpha = 0\,, \text{or}\quad \mathcal{A}^a_\psi = (-b, 0, -k),\ \mathcal{A}^a_\alpha = (0, u_\alpha, 0)\,, \tag{8.3}$$

as stated in section 5.

A second symmetry which is admitted by the BPS equations of section 5 is the 'vertical' symmetry $z \to -z$, with the parity assignments

$$(h, k, u_z) \text{ even}, \quad (b, u_\rho) \text{ odd}; \quad (t_\rho, V_z) \text{ opposite to } (t_z, V_\rho). \tag{8.4}$$

The vertical symmetry completely commutes with axial and reflexion symmetry and fixes the origin on the z-axis (Note that without vertical symmetry the BPS equations are invariant with respect to translations in the z-direction).

Finally, to make contact with standard results we remark that in the spherically symmetric case the fields (8.3) and the norm h of the Higgs fields become

$$b = -\frac{z}{r}, \quad u_\alpha = -\frac{K(r)}{r}\epsilon_{\alpha\beta}\frac{x_\beta}{r}, \quad k = \frac{\rho}{r}K(r), \quad h = \frac{H(r)}{r}, \tag{8.5}$$

where r is the polar radius and $K(r)$ and $H(r)$ the original two independent spherical symmetric fields [1,2].

9. Solutions of the First Quartet. Existence of a Master-Potential.

As pointed out in section 5, the quartet of equations (5.5) is common to both the axial and axial + reflexion symmetric cases, and we now wish to solve these equations explicitly. First, we note that (5.5) contains u_α linearly and without derivatives, and hence can be solved explicitly for u_α to give

$$u_\alpha = \frac{-bk_{,\alpha} + \rho h \epsilon_{\alpha\beta} k_{,\beta}}{b^2 + \rho^2 h^2} = \frac{kb_{,\alpha} - bk_{,\alpha} + \rho\epsilon_{\alpha\beta}(hk)_{,\beta}}{b^2 + k^2 + \rho^2 h^2}. \tag{9.1}$$

Reinserting this result for u_α in the remaining two equations we obtain

$$(b^2+k^2)_{,\alpha} + 2\rho\,\epsilon_{\alpha\beta}(hb)_{,\beta} = \rho^2(h^2)_{,\alpha}. \tag{9.2}$$

To solve equations (9.2) we note that they are almost of Cauchy-Riemann type and it is then easy to see that they imply the existence of a scalar function W, unique up to a constant, such that

$$\frac{\partial W}{\partial z} = hb, \quad 2\rho\frac{\partial W}{\partial \rho} = \rho^2 h^2 - b^2 - k^2, \quad \Delta W = h^2. \tag{9.3}$$

Thus W acts as a master potential from which the functions h, b, k (and u_α) are obtained by differentiation. The existence of W with the properties (9.3) is the full content of the quartet (5.5).

In practice it is sometimes convenient to replace W by the related family

$$W_m = W - \tfrac{1}{4}c^2\rho^2 + m^2 \log\rho, \tag{9.4}$$

where m is an integer, and then the analogue of (9.3) is

$$\frac{\partial W_m}{\partial z} = hb, \quad 2\rho\frac{\partial W_m}{\partial \rho} = (m^2 - b^2 - k^2) + \rho^2(h^2 - c^2), \quad \Delta W_m = (h^2 - c^2). \tag{9.5}$$

From (9.5) and the bound (1.7) we see that the W_m are superharmonic functions, and that W_0 is non-increasing in ρ. On the other hand, we shall see that W_n where n is the topological charge, has the best behaviour on the z-axis.

In the spherically symmetric case, the known solution [2] is generated by

$$W_1 = \log K(r) = \log\left(\frac{rc}{\sinh rc}\right), \tag{9.6}$$

and (9.5) implies the well-known algebraic relation

$$(H(r)+1)^2 = K(r)^2 + c^2r^2, \tag{9.7}$$

between the two spherically symmetric functions $H(r)$ and $K(r)$.[2]

The existance of the master potential W, and the fact that it is independent of reflexion symmetry can be seen from the following alternative and more direct derivation of equation (9.2). In the condition of axial symmetry (6.11) of section 6 we insert the BPS-equations (1.4), giving

$$D_i \omega = -\rho\, \epsilon_{ij\psi} D_j \Phi . \tag{9.8}$$

Hence we can write

$$\begin{aligned} \nabla_\alpha(\omega^2) &= 2(\omega, D_\alpha \omega) = -2\rho\,\epsilon_{\alpha\beta}(\omega, D_\beta \tilde{\Phi}), \\ \nabla_\alpha(\omega, \Phi) &= (D_\alpha \omega, \Phi) + (\omega, D_\alpha \Phi), \\ &= -\tfrac{1}{2}\rho\,\epsilon_{\alpha\beta}\nabla_\beta(\tilde{\Phi}, \Phi) + (\omega, D_\alpha \tilde{\Phi}). \end{aligned} \tag{9.9}$$

Eliminating the factor $(\omega, D_\alpha \tilde{\Phi})$ by combining these expressions we obtain

$$\nabla_\alpha(\omega,\omega) + 2\rho\,\epsilon_{\alpha\beta}\nabla_\beta(\omega, \tilde{\Phi}) = \rho^2 \nabla_\alpha(\tilde{\Phi}, \Phi), \tag{9.10}$$

which, from (6.5), is the required result.

10. The Second Quartet

It might be asked whether the second quartet of equations (5.7) for the four non-reflexion-symmetric functions t_α, v_α could be solved in a similar manner. We have not studied this question in detail, but one sees by inspection that the variables t_α can be eliminated algebraically to give

$$t_\alpha = (b v_\alpha + \rho h\, \epsilon_{\alpha\beta} v_\beta)/k . \tag{10.1}$$

Inserting (10.1) in the remaining two equations and using the first quartet we then obtain the following two equations for the v_α.

$$\epsilon_{\alpha\beta}((k v_\alpha)_{,\beta} + 2b v_\alpha u_\beta) = 0, \tag{10.2}$$

$$h(\rho^{v_\alpha/k})_{,\alpha} + 2\rho(h_{,\alpha} - h b\, u_\alpha/k)(v_\alpha/k) = 0 . \tag{10.3}$$

These equations will be useful later for studying the behaviour of t_α and v_α in the asymptotic regions $r \to \infty$ and $r \to 0$.

11. Elliptic Equations.

If we compute $u_{\alpha,\beta}$, and $\underline{u}^2$ from (9.1) and insert the result in the remaining equation of section 5 namely (5.6) we obtain the elliptic equation

$$\Delta h = \left(\frac{k^2}{\rho^2} + \underline{u}^2 + \underline{v}^2\right)h, \tag{11.1}$$

where

$$\underline{u}^2 = \frac{(\nabla k)^2}{d}, \quad \underline{v}^2 = \frac{k^2 \underline{t}^2}{d}, \quad d = b^2 + \rho^2 h^2, \tag{11.2}$$

for the norm of the Higgs field h. Eq. (11.1) is easily identified as the explicit form of the eqn. for h discussed in section 2. In the reflexion symmetric case $t_\alpha = v_\alpha = 0$ and (11.1)(11.2) and the master-potential equations (9.3) form a complete set of equations for the three unknown functions (h,b,k). Without reflexion symmetry the four extra equations of section 10 must be included.

It will be convenient, especially in considering boundary conditions, to obtain similar elliptic equations for hb, hk and k. From (9.3) we see by inspection that

$$\Delta(hb) = (h^2)_{,z}, \tag{11.3}$$

and from (11.1) and (5.5)(5.7) we obtain after some computation

$$\Delta k = \left(\frac{\rho h d_{,\alpha} - b\,\epsilon_{\alpha\beta} d_{,\beta}}{\rho h d}\right)k_{,\alpha} + \left(\frac{d}{\rho^2} + \underline{t}^2 - \underline{u}^2\right)k. \tag{11.4}$$

and

$$\Delta(hk) = \frac{2X_\alpha}{h^2(d+k^2)}(hk)_{,\alpha} + \left(\frac{d+k^2}{\rho^2} + \underline{t}^2 + \underline{v}^2 + \frac{2Y}{h^2(d+k^2)}\right)hk, \tag{11.5}$$

where

$$X_\alpha = \rho h^2(\rho h^2)_{,\alpha} - hb\,\epsilon_{\alpha\beta}(\rho h^2)_{,\beta}\ ; \quad Y = \epsilon_{\alpha\beta}(hb)_{,\alpha}(\rho h^2)_{,\beta}. \tag{11.6}$$

It should be stressed, however, that equations (11.3)-(11.6) are derived from previous ones and contain no new information. In the mirror-symmetric case they simplify slightly on setting $t_\alpha = v_\alpha = 0$.

12. Exponential Fall-off of σ at Large Distances.

In section 2 we saw that the coefficient σ in eq.(1.5) may be expected to decay exponentially as $r \to \infty$, on account of the Higgs mechanism. In our case σ is given by (11.1) and hence we shall have the predicted asymptotic behaviour, provided that k, u_α and v_α fall-off exponentially.

In this section we wish to show that k, u_α and v_α do indeed decay exponentially as $r \to \infty$. Since u_α and v_α are given in terms of k (and v_α) by (9.1) and (10.1) we need only show that k has the above asymptotic behaviour (and that $t_\alpha \to 0$). For this purpose we consider the elliptic equation (11.4) for k. For simplicity, and because the term $\underline{t}^2$ on the right-hand side of (11.4) can at most <u>decrease</u> k as $r \to \infty$, we shall consider in this section only the reflexion-symmetric case $\underline{t}^2 = 0$, leaving the case $\underline{t}^2 \neq 0$ to section 15. From the boundary conditions and finite-energy (see 5.4) we have

$$h \to c - \frac{n}{r}, \quad b \to -n\frac{z}{r}, \quad \underline{u}^2 \to O\left(\frac{1}{r^3}\right), \quad k \to 0, \tag{12.1}$$

as $r \to \infty$ and inserting these values in (11.4) we obtain

$$\Delta k = \frac{2nc}{r^3}\left(\frac{\rho^2 + r^2}{n^2 + \rho^2 c^2}\right) z\,k_{,z} + \left(\frac{2c^2\rho}{n^2 + \rho^2 c^2}\right)k_{,\rho} + \left(c^2 + \frac{n^2}{\rho^2}\right)k. \tag{12.2}$$

Although (12.2) is a linear equation for k, it is not easily solved in general, and so it is convenient to consider the two overlapping sectors (a) ρ finite, $|z| \to \infty$ (b) $\rho = r\sin\theta,\ z = r\cos\theta,\ r \to \infty,\ \sin\theta \neq 0$, fixed. In the first sector (12.2) reduces to

$$\Delta k = \left(\frac{2nc}{n^2 + \rho^2 c^2}\right)\frac{z}{|z|}k_{,z} + \left(\frac{2c^2\rho}{n^2 + \rho^2 c^2}\right)k_{,\rho} + \left(c^2 + \frac{n^2}{\rho^2}\right)k, \tag{12.3}$$

which has the exact solution

$$k = \text{const.}\,\rho^n e^{-c|z|}, \tag{12.4}$$

and hence has the required exponential fall-off. In sector (b) eq. (12.2) reduces to

$$\Delta k = \frac{2n}{c}\,\frac{z(r^2+\rho^2)}{r^3\rho^2}\,k_{,z} + \frac{2}{\rho}\,k_{,\rho} + c^2 k. \tag{12.5}$$

On making the substitution

$$k = \rho\,\exp\left(\frac{n}{c}\,\frac{z^2}{r\rho^2}\right)\bar{k}\,, \tag{12.6}$$

eq. (12.5) reduces to the Helmholtz equation

$$\Delta\bar{k} = c^2\bar{k}\,, \tag{12.7}$$

which again has the required exponential fall-off ($\exp(-cr)$).

13. Asymptotic Behaviour of the Fields as $r \to \infty$.

Once k (and u_α, v_α) are known to fall-off exponentially more precise information can be obtained concerning the asymptotic behaviour of all the fields. For simplicity we again consider only the reflexion symmetric case, leaving the general axially symmetric case to section 15.

First, from (5.5) we see that in the 'exterior' region where k and u_α are neglected completely, the remaining two variables h and b satisfy just the equations

$$b_{,\alpha} + \rho\,\epsilon_{\alpha\beta}\,h_{,\beta} = 0\,. \tag{13.1}$$

It is easy to see that (13.1) implies that h is a harmonic function, and indeed that there exists a harmonic function U such that

$$\Delta U = 0\,, \quad h = U_{,z}\,, \quad \text{and} \quad b = -\rho\,U_{,\rho}\,. \tag{13.2}$$

This is the full solution of the field equations when the exponential terms are neglected. Since the corrections to (13.2) are of order k^2 and $\underline{u}^2$ one may therefore write for k and b the multipole expansions

$$\begin{aligned} h &= c - \frac{n}{r} + \frac{\mu z}{r^3} + \ldots + O(e^{-2cr})\,, \\ b &= -\frac{nz}{r} - \frac{\mu\rho^2}{r^3} + \ldots + O(e^{-2cr})\,. \end{aligned} \tag{13.3}$$

Note that the dipole moment $\mu = 0$ and all the other terms which are odd in z vanish if we have vertical symmetry $z \to -z$.

Using (13.3) in turn we can improve the estimate for k obtained in the previous section. In the sector (a) it suffices to verify that (13.3) makes no difference to the leading-order terms in (12.3) and hence that the estimate (12.4) is already the best one to this order.

In the sector (b) on the other hand, if expansions (13.3) are inserted in (11.4) they modify (12.5) to

$$\Delta k = 2\left[\frac{1}{r}+\frac{n}{cr^2\sin^2\theta}\right]\frac{\partial k}{\partial r}+2\frac{\cos\theta}{r^2\sin\theta}\frac{\partial k}{\partial\theta}+\left[c^2-\frac{2nc}{r}+\frac{n^2}{r^2\sin^2\theta}+\frac{2\mu c\cos\theta}{r^2}\right]k, \quad (13.4)$$

and if we now make the substitutions,

$$k = r^{\alpha}e^{-cr}f(\theta) \quad , \quad \text{and} \quad f(\theta) = \sin^n\theta . g(y) , \quad (13.5)$$

where $y = \cos\theta$ and $g(y)$ is regular for all θ, $0 \leq \theta \leq \pi$,

$$\alpha = n, \text{ and } (1-y^2)g''(y) - 2nyg'(y) = (2\mu c y)g . \quad (13.6)$$

Thus in sector (b) the asymptotic behaviour of k is given by

$$k = \rho^n e^{-cr} g(y), \quad (13.7)$$

where g satisfies (13.6). Note that when the dipole moment μ vanishes (as in the case of vertical symmetry) the only regular solution of (13.6) is g = const. Then (13.7) and (12.4) coincide and (12.4) gives the asymptotic behaviour at all angles.

14. Behaviour of the Fields at the Origin.

In section 7 we saw that any axially-symmetric charge distribution must be concentrated at a single point, which we take to be the origin. Furthermore, we have the conditions

$$b^2+k^2 = n^2 \quad \text{(all z)} \quad \text{and} \quad k = 0 \quad (z \neq 0) \quad (14.1)$$

on the z-axis ($\rho = 0$).

We now wish to determine the behaviour of the fields in the neighbourhood of the origin r = 0. For simplicity, we consider first only the five reflexion-symmetric functions(h, b, k, u_α), leaving the other four functions (t_α, v_α) to the next section (where we shall see that they do not alter the results). We shall also use the results of Taubes[3] to assume that the functions are regular at r = 0. From (5.9) we see that the finite-energy condition at $\rho = 0$ implies that

$$bu_\alpha \to -k_{,\alpha}, \; ku_\alpha \to b_{,\alpha}, \; hk \to 0, \; \text{and} \; \rho^{3/2}\underline{\nabla} h \to 0, \quad (14.2)$$

and using these conditions and (14.1) we find that the leading terms in the elliptic equations (11.3) and (11.5) for hb and hk may be written as

$$\Delta(hk) \approx \frac{n^2}{\rho^2}(hk), \quad (14.3)$$

$$\Delta(hb) = (h^2)_{,z} \approx \frac{1}{n^2}((hb)^2+(hk)^2)_{,z} . \quad (14.4)$$

Since eq. (14.3) is just the associated Legendre equation it shows that the leading behaviour of hk at r = 0 is just

$$hk \sim r^m P_m^n(\cos\theta) \; , \; m \geq n . \quad (14.5)$$

where P_m^n is an associated Legendre function of the first kind. From eq. (14.4) we see that the leading behaviour of hb is determined either by a solution of the homogeneous equation $\Delta(hb) = 0$, or by a special solution of the form $y \sim r^{\alpha}$ to one of the equations

$$\Delta y = \frac{1}{n^2} y^2 , \quad \Delta y = \left(\frac{r^{2m}}{n^2} (P_m^n)^2 \right)_{,z} , \tag{14.6}$$

whichever of these three is regular and dominant . But any power-like solution of the first equation in (14.6) , which is not harmonic , is of the form $y \sim r^{-2}$ and hence is not regular . Similarly , any non-homogeneous solution to the second equation is of the form $y \sim r^{2m+1}$ for $m \geq n$, which implies that $hb < hk$, and hence $b < k$, as $r \to 0$, in contradiction to (14.1) . It follows that the leading term in hb has to be a solution of the homogeneous equation i.e.

$$hb \sim r^s P_s(\cos\theta) , \quad 1 \leq s \leq m , \tag{14.7}$$

where P_s is a Legendre polynomial. The condition $s \leq m$ is imposed so that $b \not< k$, in contradiction to (14.1).

Collecting these results together and using (14.1) we find that the leading behaviour at the origin is

$$\begin{aligned} hb &\to nE_s r^s P_s(\cos\theta), & 1 \leq s \leq m , \\ hk &\to nE\, r^m P_m^n(\cos\theta), & n \leq m , \\ h^2 &\to (E_s r^s P_s)^2 , & s < m , \\ h^2 &\to r^{2m}\left((E_s P_m)^2 + (E\, P_m^n)^2\right), & s = m , \end{aligned} \tag{14.8}$$

where E_s and E are constants and n is the topological charge. Note that vertical symmetry (section 8) would require s to be odd and $(m-n)$ to be even. Note also that in the special case $m = n$:

$$hk \to nE \rho^n . \tag{14.9}$$

and that the special cases $s = m = n$ and $s = 1$ $m = n$ would appear to be the most 'natural'. However, in the case $s = m = n$ vertical symmetry would require that the topological charge n to be odd.

Using (14.8) the estimate (14.1) for b^2+k^2 can be improved to

$$b^2 + k^2 = n^2 + \frac{2nE_s}{s+1} r^{s+1} \sin^2\theta \, P_s'(\cos\theta) , \tag{14.10}$$

and whenever $s < m$, the estimate (14.8) can itself be improved as follows: since from (14.8) $k \sim r^{m-s}$ as $r \to 0$ we obtain from the field equations (5.5) the estimate

$$\left((b^2+k^2)^{\frac{1}{2}}\right)_{,\alpha} + \rho\, \epsilon_{\alpha\beta} \left(hb (b^2+k^2)^{-\frac{1}{2}} \right)_{,\beta} = O(hk^4) = O(r^{4m-3s}) . \tag{14.11}$$

These equations show that up to the order shown, there exists a harmonic function U such that

$$(b^2+k^2)^{\frac{1}{2}} = -\rho\, U_{,\rho} , \quad hb(b^2+k^2)^{-\frac{1}{2}} = U_{,z} , \tag{14.12}$$

and hence that

$$(b^2+k^2)^{\frac{1}{2}} = n + \sum_{\ell=s}^{4m-3s-1} \frac{E_\ell}{\ell+1} r^{\ell+1} \sin^2\theta \, P_\ell'(\cos\theta) + O(hk^4) . \tag{14.13}$$

$$hb(b^2+k^2)^{-\frac{1}{2}} = \sum_{\ell=s}^{4m-3s-1} E_\ell r^\ell P_\ell(\cos\theta) + O(hk^4) , \tag{14.14}$$

where the E_ℓ are constants. Note that the odd E_ℓ vanish in the case of vertical symmetry. These equations give the leading behaviour of h and b in terms of harmonic function and k. In particular, using the leading behaviour for k from (14.8) we obtain

$$b^2 \to \left\{ n + \sum_{\ell=s}^{2m-2s-1} \frac{E_\ell}{\ell+1} r^{\ell+1} \sin^2\theta \, P_\ell'(\cos\theta) \right\}^2 - \left(\frac{nE}{E_s}\right)^2 \left(\frac{P_m^n(\cos\theta)}{P_s(\cos\theta)} \right)^2 r^{2m-2s} , \tag{14.15}$$

and

$$h \to \left| \sum_{\ell=s}^{2m-s} E_\ell r^\ell P_\ell(\cos\theta) + \frac{nE^2}{2E_s} \frac{(P_m^n(\cos\theta))^2}{P_s(\cos\theta)} r^{2m-s} \right| . \tag{14.16}$$

15. Boundary Behaviour Without Reflexion Symmetry.

In the case that reflexion-symmetry is not assumed we must consider the equations of section 10 for the four non-reflexion symmetric functions t_α and v_α. From the finite-energy condition both at $r \to \infty$ and $\rho \to 0$(see(5.4)) we have

$$bu_\alpha \to -k_{,\alpha} \,, \text{ and } \quad kt_\alpha \to bv_\alpha \,. \tag{15.1}$$

From the first of these equations we see that eqns. (10.2) and (10.3) for reduce in these limits to

$$\epsilon_{\alpha\beta}(v_\alpha/k)_{,\beta} = 0, \text{ and } \quad (\rho^2 h^2 k^2\, v_\alpha/k)_{,\alpha} = 0\,, \tag{15.2}$$

respectively. The first of these equations implies the existence of a scalar function Ω of which v_α/k is the gradient. Hence for $r \to 0, \infty$, we have

$$v_\alpha \to k\,\Omega_{,\alpha}\,, \text{ and } \quad t_\alpha \to b\,\Omega_{,\alpha}\,, \tag{15.3}$$

where

$$(\rho^2 h^2 k^2 \Omega_{,\alpha})_{,\alpha} = 0\,. \tag{15.4}$$

First, let us consider the asymptotic behaviour as $r \to \infty$. From (15.4) and (11.5) we find that hk and Ωhk satsify the same asymptotic behaviour, namely,

$$\Delta(hk)/(hk) = \Delta(\Omega hk)/(\Omega hk) = c^2 + b^2(\nabla\Omega)^2. \tag{15.5}$$

These equations suggest that both hk and Ωhk fall-off like exp(-cr) as $r \to \infty$, and that Ω behaves at most like a polynomial. Following these suggestions we make the Ansätze

$$\Omega(r,\theta) \to r^\alpha\,\Omega(\theta)\,, \quad hk \to r^\beta e^{-cr} f(\theta)\,. \tag{15.6}$$

From (15.5) we then find that $\alpha = 0$, and inserting this result in(15.4) we find that

$$\sin\theta . f^2(\theta) . \frac{d\Omega(\theta)}{d\theta} = \text{Const.} \tag{15.7}$$

But since $f(\theta)$ and $\Omega'(\theta)$ are regular, for all θ, including $\sin\theta = 0$, the constant must be zero and hence $\Omega'(\theta)$ must be zero for all θ. Thus, subject to the ansatze (15.6) we have

$$\Omega \to \text{constant}\,, \quad v_\alpha, t_\alpha \to 0\,, \text{ as } r \to \infty\,, \tag{15.8}$$

and so the four non-reflexion-symmetric functions drop out asymptotically. Note that the term $\underline{v}^2$ in $|\underline{D}\phi|^2$ decays at least like exp(-2cr), as $r \to \infty$.

Next let us consider the behaviour at the origin. The only difference that the four functions v_α, t_α make to the reflexion-symmetric discussion of section 14 is that equation (14.3) is replaced by

$$\Delta(hk) \approx n^2\left\{\frac{1}{\rho^2} + (\nabla\Omega)^2\right\}(hk)\,, \tag{15.9}$$

where Ω is the function defined above. By regularity we have

$$\Omega(r,\theta) \approx r^\gamma\,\Omega(\theta)\,, \tag{15.10}$$

where $\gamma \geq 0$, and if $\gamma > 0$, the term $(\nabla\Omega)^2$ in (15.9) is negligible compared to $1/\rho^2$. If $\gamma = 0$ on the other hand, the same arguement that led from (15.4) to (15.7) and (15.8) leads again to $\Omega(\theta) =$ constant, and hence for $\gamma = 0$ also $(\nabla\Omega)^2$ can be neglected. Thus for all $\gamma \geq 0$.

$$\Omega \to \text{constant}\,, \quad t_\alpha, v_\alpha \to 0\,, \text{ as } r \to 0\,, \tag{15.11}$$

and the four non-reflexion-symmetric functions drop out. In particular, the behaviour (14.8) for hb, hk, and h^2 remains unaffected.

In actual fact, we can determine the next to leading behaviour of Ω, as $r \to 0$ by assuming $\Omega = \text{Constant} + r^\Gamma g(\theta)$ near the origin. From (15.9), (15.4) and the fact that $\Omega_{,\alpha} \to 0$, as $r \to 0$ we get the following equation

$$\Delta(\Omega hk) = n^2/\rho^2(\Omega hk)\,, \text{ as } r \to 0\,. \tag{15.12}$$

Thus Ω satisfies

$$\Omega = \text{Constant} + F r^{p} P^{n}_{m+p}(\cos\theta)/P^{n}_{m}(\cos\theta), \text{ as } r \to 0, \qquad (15.13)$$

where p is a positive non zero integer and F is a constant (the non uniform dependence on θ of the coefficient of r^p in Ω we expect to be due to the fact that $\Omega(r,\theta)$ relates only to ratios of well behaved quantities).

The fact that the four functions V_κ, t_κ play no role in the equations for hb , hk , and h^2 as $r \to 0$ and $r \to \infty$ suggests that the only axially symmetric solutions may be the reflexion symmetric ones .

Appendix A Maxwell Field as the Curl of an SO(3)-Invariant

Let $(\omega_1, \omega_2, \omega_3)$ be any triad of isovectors with $\omega_1 = \phi$. The choice of ω_2 is not relevant and includes, of course, the possibility $\omega_2 = D_\mu\phi/|D_\mu\phi|$ of (3.1). Using $\phi = \omega_2 \wedge \omega_3$ and the anti-symmetry of the wedge-product we can write the standard formula (4.2) for the Maxwell field $\underline{f}(\underline{x})$ in the form

$$f_i(\underline{x}) = \tfrac{1}{2}\epsilon_{ijk}\{(\omega_2\wedge\omega_3, F_{jk}) + (\omega_3\wedge\omega_2, D_j\omega_1\wedge D_k\omega_1)\}. \qquad (A.1)$$

Using the usual vector identities

$$(a, b\wedge c) = (c, a\wedge b), \quad \text{and} \quad (a\wedge b, c\wedge d) = (a,c)(b,d) - (a,d)(b,c), \qquad (A.2)$$

we then have

$$f_i(\underline{x}) = \epsilon_{ijk}\{\tfrac{1}{2}(\omega_3, F_{jk}\wedge\omega_2) + (\omega_3, D_j\omega_1)(\omega_2, D_k\omega_1)\}. \qquad (A.3)$$

But from the orthogonality of the ω_a -basis we have

$$(\omega_a, D_i\omega_b) + (\omega_b, D_i\omega_a) = \nabla_i(\omega_a, \omega_b) = \nabla_i\delta_{ab} = 0. \qquad (A.4)$$

Hence the position of the D_i in the inner product can be switched and we have

$$f_i(\underline{x}) = \epsilon_{ijk}\{\tfrac{1}{2}(\omega_3, F_{jk}\wedge\omega_2) + (\omega_1, D_j\omega_3)(\omega_1, D_k\omega_2)\}. \qquad (A.5)$$

From the definition of F_{jk} as the commutator of D_j and D_k and the anti-symmetry of the Levi-Civita symbol we then have

$$f_i(\underline{x}) = \epsilon_{ijk}\{(\omega_3, D_jD_k\omega_2) + (\omega_1, D_j\omega_3)(\omega_1, D_k\omega_2)\}. \qquad (A.6)$$

But now, since ω_2 and ω_3 are normalized, they are orthogonal to $D_k\omega_2$ and $D_j\omega_3$ respectively. Hence

$$(D_j\omega_3, D_k\omega_2) = (D_j\omega_3, \omega_a)(\omega_a, D_k\omega_2) = (D_j\omega_3, \omega_1)(\omega_1, D_k\omega_2) \qquad (A.7)$$

and thus from (A.6) we then have

$$j_i(\underline{x}) = \epsilon_{ijk}\{(\omega_3, D_j D_k \omega_2) + (D_j\omega_3, D_k\omega_2)\},$$

$$= \epsilon_{ijk}\{\nabla_j(\omega_3 D_k \omega_2)\},$$

$$= -\epsilon_{ijk}\nabla_j \mathcal{A}^1_k , \qquad (A.8)$$

as required.

Appendix B The Zero Set of $\underline{\Phi}$ and $\underline{J}$.

Here, we give the arguements which show that the zero set of $\underline{\Phi}$ and $\underline{J}$, $\Gamma(\underline{\Phi})$ and $\Gamma(\underline{J})$ respectively can be at worst isolated points along the z-axis, or the whole z-axis, or possibly isolated rings around the z-axis. In addition, in the case of $\Gamma(\underline{\Phi})$ the zeros along the z-axis must be isolated.

Quite generally, it is known that $\underline{\Phi}$, and $\underline{J}$ are real analytic functions[3] (at least in some gauge). Thus $\Gamma(\underline{\Phi})$ and $\Gamma(\underline{J})$ are real analytic varieties and can consist of isolated points, smooth curves and smooth 2-surfaces. It is shown below that the 2-surfaces can be eliminated. Then, $\Gamma(\underline{\Phi})$ and $\Gamma(\underline{J})$ can be at most isolated points and lines, which for axial symmetry is the required result. In addition, for $\Gamma(\underline{\Phi})$ we have the boundary condition $(\underline{\Phi}, \underline{\Phi}) \to c^2$, as $r \to \infty$, which only allows isolated points and closed loops.

First, to eliminate 2-surfaces for $\Gamma(\underline{\Phi})$ we note that due to the boundary condition, as $r \to \infty$ any surface S on which $\underline{\Phi} = 0$ must be closed. But, since $(\underline{\Phi}, \underline{\Phi})$ is a subharmonic function, $\underline{\Phi}$ would then vanish everywhere inside S and hence by analyticity, $\underline{\Phi}$ would vanish everywhere.

Next, suppose $\Gamma(\underline{J})$ contains a 2-surface. Such a 2-surface will always contain local neighbourhoods for which $h \neq 0$. But from equations (2.9), (2.10) and (2.12) we see that for any such neighbourhood the normal derivative of $\underline{J}$ is linear in $\underline{J}$ and its tangential derivatives, with smooth coefficients. By iteration, the same is true for the normal derivatives, to any order. It then follows from real analyticity that $\underline{J}$ cannot vanish on a neighbourhood of 2-surface (with $h \neq 0$) without vanishing in a finite 3-volume containing it. In that case, the real analyticity will force $\underline{J}$ to vanish everywhere.

Appendix C The Behaviour of (ω, ω) on the z-axis.

In this appendix we show that the limit of $(\omega, \omega) = b^2 + k^2$, as $\rho \to 0$ is bounded and constant all along the z-axis. The necessity of proving this arises from the fact that we have imposed only a ' weak ' form of axial symmetry, namely that all inner products of covariant quantities are independent of the azimuthal angle. Hitherto [6], we assumed a strong axial symmetry condition, i.e. that there existed a continuous ω such that

$$D_\varphi \cdot = \omega_\wedge \cdot , \tag{C.1}$$

Both the boundedness and constancy of (ω, ω), along the z-axis follow from equation (6.11), i.e.

$$\rho B_\alpha = \epsilon_{\alpha\beta} D_\beta \omega . \tag{C.2}$$

Since we know that the following bound is valid [3]

$$0 < K^2 \equiv \sup_{\mathbb{R}^3} |(\underline{B}, \underline{B})| < \infty , \tag{C.3}$$

we have the following sequence

$$\begin{aligned} |\nabla_\alpha |\omega|^2| &= 2 |(\omega, D_\alpha \omega)| = 2\rho |-\epsilon_{\alpha\beta}(\omega, B_\beta)| \\ &\leq 2\rho |\omega| K \leq \rho (|\omega|^2 + K^2) . \end{aligned} \tag{C.4}$$

From this it follows since $0 < K < \infty$ that

$$|\nabla_\alpha \log(|\omega|^2 + K^2)| \leq \rho , \quad \text{for all } \rho > 0. \tag{C.5}$$

The boundedness and constancy (and hence continuity) of $\log(|\omega|^2 + K^2)$, and hence $|\omega|^2$ along the z-axis are an immediate consequence of equation (C.5).

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