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# Non-Commutative Gaussian Processes

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## 0. INTRODUCTION

We follow Accardi [1,2] and define a stochastic process to be a family $\{j_t : t \in R\}$ of $^*$-isomorphisms of one algebra $\mathcal{B}$ into another $\mathcal{A}$, together with a state $\omega$ on $\mathcal{A}$; in this paper we work in the category of W*-algebras and normal morphisms, so that classical stochastic processes are included as a special case (when both $\mathcal{B}$ and $\mathcal{A}$ are commutative). In [3] it was shown that a stochastic process is determined up to equivalence by its family of correlation kernels $\omega(j_{t_1}(a_1)^* \ldots j_{t_n}(a_n)^* j_{t_n}(b_n) \ldots j_{t_1}(b_1))$, and that a process can be reconstructed from an inductive family of correlation kernels; Markov processes and their associated semigroups were studied, and examples (related to the Hepp-Lieb models [4]) were constructed using the Clifford algebra. In this paper we study non-commutative Gaussian processes; classical Gaussian processes are a special case.

Gaussian processes are generated by Hilbert space processes (Theorem 1.1), so in §2 we review results about Hilbert space processes. A Gaussian process is determined up to equivalence by its covariance function (Lemma 3.1); it is Gaussian Markov if and only if its covariance function satisfies an evolution condition (Theorem 3.4). There is a one-to-one correspondence between stationary Gaussian Markov processes and semigroups of quasi-free completely-positive identity-preserving maps on the CCR algebra leaving invariant a quasi-free state (Theorem 3.5). A stationary Gaussian Markov process is regular if and only if it satisfies a Langevin equation (Theorem 3.7). The Ford-Kac-Mazur process in quantum theory [5] is Gaussian; it is classical Markov at infinite temperature but non-Markov at finite temperatures.

## 1. Preliminaries and Definitions

We summarize a few results from the theory of cyclic representations of the canonical commutation relations over symplectic spaces [6,7] with possibly degenerate symplectic forms [8,9].

Let H be a Hilbert space, Q a skew-adjoint contraction on H, and let $\sigma$ be the symplectic form on H given by

$$\sigma(h,k) = \langle Qh,k\rangle \qquad \text{for all } h,k \text{ in H.} \tag{1.1}$$

Then there exists a complex Hilbert space $\mathcal{H}$, with a distinguished normalized vector $\Omega$, and a one-to-one map W of H into $\mathcal{B}(\mathcal{H})$ such that

$$W(h)W(k) = e^{-i\sigma(h,k)}W(h+k) \qquad \text{for all } h,k \text{ in H,} \tag{1.2}$$

$h \mapsto W(h)$ is continuous in the ultraweak topology,

the set $\{W(h)\Omega : h\in H\}$ is dense in $\mathcal{H}$,

$$\langle\Omega, W(h)\Omega\rangle = \exp(-\tfrac{1}{2}\|h\|^2) \qquad \text{for all } h \text{ in H.} \tag{1.3}$$

Let $\mathcal{A}$ be the ultraweak closure of the set $\{W(h) : h\in H\}$ ; then $\mathcal{A}$ is a von Neumann algebra. We denote it by W(H).

Definition: If a von Neumann algebra $\mathcal{A}$ is of the form W(H) for some real Hilbert space H, we say that H is a Gaussian space for $\mathcal{A}$. Let $\omega$ be the state on $\mathcal{A}$ defined by $\omega(a) = \langle\Omega, a\Omega\rangle$ for all a in $\mathcal{A}$.

Example 1 Let $Q = 0$ ; then $\mathcal{A} = W(H)$ is abelian, and isomorphic as a W*-algebra to some $L^\infty(\Sigma,\mathcal{F},\mu)$ ; then H may be identified with a closed subspace of $L^2_{\mathbb{R}}(\Sigma,\mathcal{F},\mu)$, and it is then a Gaussian space in the usual sense [10]. The state $\omega$ is faithful since a cyclic vector for $\mathcal{A}$ is also separating for $\mathcal{A}$ if $\mathcal{A}$ is abelian.



Example 2 Let $Q^*Q = 1$ ; then $\sigma$ is non-degenerate, $\omega$ is a Fock state on $\mathcal{A}$ and W is an irreducible representation of the canonical commutation relations (CCR) over $(H,\sigma)$.

Example 3 Let Q be such that $0 < Q^*Q < 1$ ; then $\sigma$ is non-degenerate, $\omega$ is a faithful primary quasi-free state on the algebra of the CCR over $(H,\sigma)$.

Remark: The above examples are universal: one can always decompose H as $H_1\oplus H_2\oplus H_3$ where $H_1 = \ker Q$, $H_2 = \ker(1-Q^*Q)$ and $H_3 = H\ominus(H_1\oplus H_2)$, each $H_n$, $n=1,2,3$, is stable under Q and $W(H) = W(H_1)\otimes W(H_2)\otimes W(H_3)$ : then $W(H_n)$ corresponds to the situation of Example $n$ for $n = 1,2,3$. Note also that $\omega(W(h_1\oplus h_2\oplus h_3)) = \omega(W(h_1))\omega(W(h_2))\omega(W(h_3))$ for $h_n\in H_n$, $n=1,2,3$, and that the centre of W(H) is $W(H_1)\otimes 1\otimes 1$. [8].

In applications one starts from a real vector space V with a symplectic form $\sigma$ which is either identically zero (classical case) or is nondegenerate (boson case). Then one defines a bilinear form $s(\cdot,\cdot)$ on V satisfying

$$\sigma(h,k)^2 \leqslant s(h,h)s(k,k) \qquad \text{for all } h,k \text{ in V} \tag{1.4}$$

(or, equivalently, $\sum \bar{c}_i c_j[s(h_i,h_j)+i\,\sigma(h_i,h_j)] \geqslant 0$ for all finite sequences $\{c_i\}$ in $\mathbb{C}$, $\{h_i\}$ in V), and uses s to define the inner product. Let H be the completion of V with respect to the norm got from the inner product, then $\sigma$ can be extended to H by continuity, and it can be degenerate, in general. For example, for the grand-canonical state of the free boson gas in the thermodynamic limit, $H_1 = \{0\}$ above the critical temperature and $H_1 = \mathbb{R}^2$ below the critical temperature [11].

Definition (Accardi, Frigerio, Lewis [3]): A W*-stochastic process over $\mathcal{B}$ evolving in $\mathcal{A}$ is a pair $(\{j_t : t\in\mathbb{R}\},\omega)$, where $\mathcal{A},\mathcal{B}$ are W*-algebras, $\{j_t : t\in\mathbb{R}\}$ is a family of normal *-isomorphisms of $\mathcal{B}$ into $\mathcal{A}$ with $j_t(1_{\mathcal{B}}) = 1_{\mathcal{A}}$, such that $\mathcal{A}$ is the W*-algebra generated by $\{j_t(b) : t \in \mathbb{R}, b \in \mathcal{B}\}$,

$\omega$ is a normal state on $\mathcal{A}$, such that $\mathcal{A}$ can be identified with the von Neumann algebra of operators $\pi_\omega(\mathcal{A})$ acting on the GNS space $\mathcal{H}$ of $\omega$.

Two W*-processes $(\{j_t\},\omega),(\{\tilde{j}_t\},\tilde{\omega})$ over $\mathcal{B}$ evolving in $\mathcal{A}$, $\tilde{\mathcal{A}}$ respectively are equivalent if there is a normal *-isomorphism $u$ of $\mathcal{A}$ onto $\tilde{\mathcal{A}}$ such that $\tilde{j}_t = u \cdot j_t$ for all $t$ in $\mathbb{R}$ and $\tilde{\omega} \cdot u = \omega$.

Here we construct W*-stochastic processes determined by families of isometries in real Hilbert spaces.

Definition Let M, H be real Hilbert spaces. A Hilbert space process (or H-process, for the sake of brevity in reference) over M evolving in H is a family $\{X_t : t \in \mathbb{R}\}$ of isometries $X_t : M \to H$ such that $H = \bigvee\{X_t m : t \in \mathbb{R}, m \in M\}$, $t \mapsto X_t m$ is continuous for all m in M. To construct a stochastic process in our sense from an H-process $\{X_t\}$ over M evolving in H, all we require is to be given a pair $(Q_M, Q)$ of skew-adjoint contractions, on M and H respectively, which is compatible with $\{X_t\}$ in the sense that

$$Q_M = X_t^* Q X_t \qquad \text{for all } t \text{ in R.} \tag{1.6}$$

The main result of this section is the following

Theorem 1.1 Let $\{X_t\}$ be an H-process over M evolving in H, and let $(Q_M, Q)$ be a pair of skew-adjoint contractions compatible with $\{X_t\}$. Let $\mathcal{B}$ be the W*-algebra W(M), with symplectic form $\sigma_M$ given by $Q_M$, and let $\mathcal{A}$ be the W*-algebra W(H), with symplectic form $\sigma$ given by Q. Then there exists a W*-stochastic process $(\{j_t\},\omega)$ over $\mathcal{B}$ evolving in $\mathcal{A}$ for which

$$j_t W(m) = W(X_t m) \qquad \text{for all m in M, t in } \mathbb{R}, \tag{1.7}$$

$$\omega(W(h)) = \exp(-\tfrac{1}{2}\|h\|^2) \qquad \text{for all h in H.} \tag{1.3}$$

Occasionally we shall write

$$j_t = W(X_t). \tag{1.8}$$

Proof Let $\mathcal{B}^0$, $\mathcal{A}_t^0 (t \in \mathbb{R})$ and $\mathcal{A}^0$ be the C*-algebras generated by $\{W(m) : m \in M\}$, $\{W(X_t m) : m \in M\}$ and $\{W(h) : h \in H\}$ respectively. By condition (1.6), we have, for all m,m' in M,

$$W(m)W(m') = e^{-i\langle Q_M m, m'\rangle} W(m+m'), \tag{1.9}$$

$$\begin{aligned} W(X_t m)W(X_t m') &= e^{-i\langle QX_t m, X_t m'\rangle} W(X_t(m+m')) \\ &= e^{-i\langle Q_M m, m'\rangle} W(X_t(m+m')), \end{aligned} \tag{1.9'}$$

so that, by Slawny's theorem [21], there is a C*-algebra isomorphism $j_t^0$ of $\mathcal{B}^0$ onto $\mathcal{A}_t^0$ such that $j_t^0(W(m)) = W(X_t m)$ for all m in M. In particular, we have $j_t^0(1_{\mathcal{B}^0}) = 1_{\mathcal{A}^0}$. Let $\omega_M^0$ and $\omega^0$ be the states on $\mathcal{B}^0$ and on $\mathcal{A}^0$ respectively defined by

$$\omega_M^0(W(m)) = e^{-\frac{1}{2}\|m\|^2}, \quad \omega^0(W(h)) = e^{-\frac{1}{2}\|h\|^2}.$$

Upon identifying $\mathcal{B}^0$ and $\mathcal{A}^0$ with their respective GNS representations determined by $\omega_M^0$ and $\omega^0$, we have $\mathcal{B} = (\mathcal{B}^0)''$, $\mathcal{A} = (\mathcal{A}^0)''$. The normal extensions of $\omega_M^0$ and of $\omega^0$ to $\mathcal{B}$ and to $\mathcal{A}$ will be denoted by $\omega_M$ and by $\omega$ respectively. Since $X_t$ is an isometry, we have

$$\omega^0 \circ j_t^0 = \omega_M^0. \tag{1.10}$$

It remains to prove that $j_t^0$ can be extended to a normal *-isomorphism of $\mathcal{B}$ into $\mathcal{A}$. By the Remark made after Examples 1—3, it suffices to construct the normal extension in the cases

(i) $\mathrm{Ker}(1_M - Q_M^+ Q_M) = \{0\}$ ($\omega_M$ faithful)

(ii) $Q_M^+ Q_M = 1_M$ ($\omega_M$ Fock).

Case (i): We adapt the argument of [14], Theorem 4.2. Let $\mathcal{S}$ be the set of states on $\mathcal{B}^0$ which are majorized by a scalar multiple of $\omega_M^0$: any state $\varphi^0$ in $\mathcal{S}$ has a normal extension $\varphi$ to $\mathcal{B}$, and the set of such normal extensions spans a norm-dense subset in the predual space of $\mathcal{B}$, since $\omega_M$ is faithful. If $\varphi^0$ is in $\mathcal{S}$, then, by (1.10), $\varphi^0 \circ j_t^0$ is majorized by a scalar multiple of $\omega$, hence it has a normal extension to $\hat{\mathcal{A}}$. By density, the predual space of $\hat{\mathcal{B}}$ is mapped into the predual space of $\hat{\mathcal{A}}$ by the transpose of $j_t^0$, so that $j_t^0$ has a normal extension to $\mathcal{B}$.

Case (ii): If $\omega_M$ is Fock, we have

$$1_M = Q_M^+ Q_M = X_t^* Q^+ X_t X_t^* Q X_t \leq X_t^* Q^+ Q X_t \leq X_t^* 1_H X_t = 1_M,$$

hence $(1 - X_t X_t^*) Q X_t = 0$ and $Q$ maps the subspace $H_t = X_t M$ of $H$ into itself. It follows that $W(X_t m)$ commutes with $W(k)$ for all $m$ in $M$ and for all $k$ in the orthogonal complement $K_t$ of $H_t$ in $H$, and we have the natural identification

$$W(H) = W(H_t) \otimes W(K_t) \; : \; H_t = X_t M, \; K_t = H \ominus H_t .$$

Let $\mathcal{H}_t$ be the Hilbert space $\overline{W(H_t)\Omega}$, where $\Omega$ is the cyclic vector for $W(H)$ satisfying (1.3); let also $\mathcal{H}_M$ denote the space on which $W(M)$ acts with cyclic vector $\Omega_M$ satisfying the analogue of (1.3). By (1.9) and (1.10), $(\mathcal{H}_M, W(m), \Omega_M)$ and $(\mathcal{H}_t, W(X_t m)\restriction_{\mathcal{H}_t}, \Omega)$ are two cyclic representations of the CCR over $(M, \sigma_M)$ with the same generating functional, hence there exists a unitary operator $U_t$ of $\mathcal{H}_M$ onto $\mathcal{H}_t$ such that

$$W(X_t m)\restriction_{\mathcal{H}_t} = U_t W(m) U_t^* \quad \text{for all } m \text{ in } M.$$

Then we define $j_t : \mathcal{B} \to \hat{\mathcal{A}}$ by $\quad j_t(b) = U_t \, b \, U_t^* \times W(0_{K_t}); \quad b \in \mathcal{B}$,

where $0_{K_t}$ is the zero vector in $K_t$ and $j_t$ is a normal *-isomorphism satisfying (1.7) and $j_t(1_{\mathcal{B}}) = 1_{\mathcal{A}}$.

Definition A process as in Theorem 1.1 is a Gaussian process. It is clear that classical Gaussian processes (see [10]) are covered by the above definition. We recall a few definitions from [3]:

A stochastic process $(j_t, \omega)$ over $\mathcal{B}$ evolving in $\hat{\mathcal{A}}$ is said to be stationary if there is a group $\{\mathcal{U}_t : t \in \mathbb{R}\}$ of *-automorphisms of $\hat{\mathcal{A}}$ such that $\mathcal{U}_t \circ j_s = j_{s+t}$ for all $t, s$ in $\mathbb{R}$, $\omega \circ \mathcal{U}_t = \omega$ for all $t$ in $\mathbb{R}$. Define the following W*-subalgebras of relative to the time $t$:

(past and present): $\mathcal{A}_{t]} = \vee\{j_s(b) : s \leq t, \, b \in \mathcal{B}\}$,

(present): $\mathcal{A}_t = \vee\{j_t(b) : b \in \mathcal{B}\}$,

(present and future): $\mathcal{A}_{[t} = \vee\{j_u(b) : t \leq u, \, b \in \mathcal{B}\}$,

and assume that the functions $t \mapsto j_t(b)$ are continuous in the weak* topology, so that the W*-subalgebra $\mathcal{A}_{t)} = \vee\{j_s(b) : s < t, \, b \in \mathcal{B}\}$ coincides with $\mathcal{A}_{t]}$ for each $t$ in $\mathbb{R}$. Then the W*-stochastic process is said to be Markov if there exists a family of normal conditional expectations $E_{t]}$ of $\mathcal{A}$ onto $\mathcal{A}_{t]}$, $t \in \mathbb{R}$, which is compatible with $\omega$ in the sense that

$$\omega = \omega\restriction_{\mathcal{A}_{t]}} \circ E_{t]} \qquad \text{for each } t \text{ in } \mathbb{R},$$

and satisfies the Markov property

$$E_{t]} \mathcal{A}_{[t} = \mathcal{A}_t \qquad \text{for each } t \text{ in } \mathbb{R}.$$

For a Gaussian process, it will be possible to relate the above properties to the corresponding properties, (defined in §2), of the underlying H-process. To do so, we introduce maps on $W(H)$ which are determined by operators on $H$. The following theorem is a slight modification of a result of Evans and Lewis [12], Demoen, Vanheuverzwijn and Verbeure [13]:

Theorem 1.2 A map $Z$ defined on the $W(h)$, $h \in H$, by

$$Z[W(h)] = W(Th) \exp\left(\tfrac{1}{2}\|Th\|^2 - \tfrac{1}{2}\|h\|^2\right), \tag{1.11}$$

where $T$ is a linear operator on $H$, extends by linearity and continuity to a completely positive identity preserving normal map of $W(H)$ into itself if and only if

$$\sum_{i,j} \bar{c}_i c_j \left[\langle h_i, h_j\rangle - \langle Th_i, Th_j\rangle + i\langle Qh_i, h_j\rangle - i\langle QTh_i, Th_j\rangle\right] \geq 0 \tag{1.12}$$

for all finite sequences $\{c_i\}$ in $\mathbb{C}$ and $\{h_i\}$ in $H$.

**Remark** Eq. (1.12) implies that T is a contraction on H, and any contraction T on H commuting with Q satisfies (1.12).

**Proof** (Sketch). It has been shown in [12,13] that (1.12) holds if and only if Z extends to a completely positive identity preserving map $\bar{Z}$ of the C*-algebra generated by $\{W(h): h \in H\}$ into itself, so we only have to prove that $\bar{Z}$ can be further extended to a normal map of W(H) into itself if (1.12) holds. Assume first that T is a contraction commuting with Q, and let $H = H_1 \oplus H_2 \oplus H_3$ be the decomposition introduced after Examples 1,2,3. Then also Z decomposes as

$$ZW(h_1 \oplus h_2 \oplus h_3) = Z_1 W(h_1) \otimes Z_2 W(h_2) \otimes Z_3 W(h_3)$$

and it suffices to extend $Z_1, Z_2, Z_3$ to normal maps on the von Neumann algebra. For $Z_1$ and $Z_3$ this is possible since they leave invariant a separating state (then use [14] Theorem 4.2), for $Z_2$ this follows as a special case from [15] Theorem 4.4. We shall complete the proof after Theorem 3.5.

The extension of Z to W(H) will be denoted by W(T). The set S of the contractions on H which satisfy (1.12) is a semigroup, and $T \mapsto W(T)$ is a homomorphism of S into the set of completely positive identity preserving normal maps on W(H). If O is the zero operator on H, then $W(O) = \omega(\cdot)\mathbb{1}$. All maps W(T), T in S, leave $\omega$ invariant. W(T) is a *-automorphism of W(H) if and only if T is a unitary operator commuting with Q, and it is a conditional expectation if and only if T is an orthogonal projection commuting with Q. Notice that in the classical case $Q = 0$, so that S is the set of all contractions on H; then each unitary on H determines a *-automorphism of W(H) and each orthogonal projection determines a conditional expectation.

## 2. H-PROCESSES

Since the structure of a Gaussian process will be determined by the underlying H-process, we give a short and systematic account of those properties of H-processes which we shall need. H-processes have been studied in connection with classical Gaussian or weakly stationary stochastic processes, see e.g. [16-20,10]. The basic object of a H-process is the <u>covariance function</u> $K(\cdot,\cdot)$ on $\mathbb{R}\times\mathbb{R}$ with values in $\mathcal{B}(M)$ defined by

$$\langle m, K(s,t)m'\rangle = \langle X_s m, X_t m'\rangle \quad \text{for all m, m' in M.} \tag{2.1}$$

It satisfies the following properties:

K1: $K(\cdot,\cdot)$ is a positive definite kernel,

K2: $K(t,t) = \mathbb{1}$ for all t in $\mathbb{R}$,

K3: $s,t \mapsto \langle m, K(s,t)m'\rangle$ is continuous for all m, m' in M.

By the theorem on minimal Kolmogorov decompositions of positive definite kernels [21], an H-process is determined uniquely up to equivalence by its covariance function, and any function $K: \mathbb{R}\times\mathbb{R} \to \mathcal{B}(M)$ satisfying K1-K3 is the covariance function of an H-process. Consequently, any property of an H-process can in principle be stated either in terms of isometries $\{X_t\}$ or in terms of the covariance function $K(\cdot,\cdot)$.

An H-process is <u>stationary</u> if $K(t,t+s)$ is independent of t for all s. By the uniqueness up to equivalence of the minimal Kolmogorov decomposition of a positive definite kernel [21], this holds if and only if there is a group $\{T_t : t \in \mathbb{R}\}$ of unitary operators on H such that $T_t X_s = X_{t+s}$ for all s, t in $\mathbb{R}$.

For each t in $\mathbb{R}$ define the following subspaces of H relative to the time t:

(past and present) $H_{t]} = \vee\{X_s m : s \le t, m \in M\}$,

(present) $H_t = \vee\{X_t m : m \in M\}$,

(present and future) $H_{[t} = \vee\{X_u m : t \le u, m \in M\}$.

Denote by $P_{\#}$ the orthogonal projection onto $H_{\#}$, where $\#$ stands for $t], t$, or $[t$. Because of the assumed continuity, the space $H_{t)} = \vee\{X_s m : s < t, m \in M\}$ coincides with $H_{t]}$. Define also the subspaces

$$D_t^- = H_{t]} \ominus H_t \qquad , D_t^+ = H_{[t} \ominus H_t \quad .$$

Definition An H-process is said to be

deterministic if $H_{t]} = H$ for all t;

regular (or completely nondeterministic) if $\wedge\{H_{t]} : t \in \mathbb{R}\} = \{0\}$;

Markov if $D_t^-$ is orthogonal to $D_t^+$ for all t in $\mathbb{R}$.

In Section 3 we shall see that the following result leads to a generalization of Doob's theorem [17]:

Lemma 2.1 Let $\{X_t\}$ be an H-process over M, with covariance function $K(\cdot,\cdot)$. The following are equivalent:

(i) the H-process is Markov;

(ii) the covariance function satisfies the evolution equation

$$K(s,t)K(t,u) = K(s,u) \qquad \text{for all } s \le t \le u \text{ in } \mathbb{R}\ ; \qquad (2.2)$$

$$\text{(iii)} \quad P_{t]} H_{[t} = H_t \qquad \text{for all t in } \mathbb{R}\ . \qquad (2.3)$$

Proof: (i)$\Leftrightarrow$(ii) For all m, m' in M, $s \le t \le u$ in $\mathbb{R}$ we have

$$\langle m, [K(s,u) - K(s,t)K(t,u)]m' \rangle = \langle X_s m, [1 - X_t X_t^*] X_u m' \rangle \ .$$



The operator $X_t X_t^*$ is the projection $P_t$ onto $H_t$, and $\{X_s m : s \le t, m \in M\}$ $\{X_u m' : t \le u, m' \in M\}$ generate $H_{t]}$, $H_{[t}$ respectively. The result then follows. (i)$\Leftrightarrow$(iii) $D_t^-$ and $D_t^+$ are orthogonal if and only if $H = D_t^- \oplus H_t \oplus D_t^+$. If this is the case $P_{t]} H_{[t} = H_t$. Conversely, if $P_{t]} H_{[t} = H_t$, then $P_{t]} D_t^+ = 0$ and $D_t^+$ is orthogonal to $D_t^-$.

Corollary An H-process is stationary and Markov if and only if there is a strongly continuous semigroup $\{S_t : t \ge 0\}$ of contractions on M such that

$$K(s,s+t) = S_t \quad , K(s,s-t) = S_t^* \text{ for all } t \ge 0 \text{ and s in } \mathbb{R}\ . \qquad (2.4)$$

Hence, given a strongly continuous semigroup $\{S_t\}$ of contractions on a Hilbert space M, let $\{T_t : t \in \mathbb{R}\}$ be a minimal unitary dilation of $\{S_t\}$, acting on a Hilbert space H, with isometric embedding $X_0$ of M in H such that $S_t = X_0^* T_t X_0$. Then $\{X_t = T_t \cdot X_0 : t \in \mathbb{R}\}$ defines a stationary Markov H-process; and each stationary Markov H-process arises in this way.

Definition Let $\{X_t\}$ be an H-process over M; then an H-process $\{\bar{X}_t\}$ over $\bar{M}$ (evolving in the same space H) is said to be an extension of $\{X_t\}$ if there exists an isometry W from M into $\bar{M}$ such that $X_t = \bar{X}_t \cdot W$ for all t in $\mathbb{R}$. An extension $\{\bar{X}_t\}$ is said to be non-anticipating if $\vee\{\bar{X}_s W M : s \le t\} = \vee\{\bar{X}_s \bar{M} : s \le t\}$ for all t, and minimal if $\vee\{\bar{X}_0^* \bar{X}_t W M : t \ge 0\} = \bar{M}$. A non-anticipating extension of a stationary H-process is clearly stationary, with the same group $\{T_t\}$ of unitaries as the original H-process.

Lemma 2.2 (Rozanov [22]). A stationary H-process has a unique minimal non-anticipating Markov extension, given by

$$\bar{M} = P_{0]} H_{[0}\ , \bar{X}_t = T_t \bar{X}_0 \ , \qquad \bar{X}_0 \text{ being the natural embedding of } P_{0]} H_{[0} \text{ into } H\ . \qquad (2.5)$$

Proof From stationarity, it follows $P_{t]} = T_t P_{0]} T_{-t}$ , hence $\{P_{0]} T_t : t \geq 0\}$ is a semigroup of contractions on H. $P_{0]} H_{[0}$ is the smallest subspace of H which is stable under $\{P_{0]} T_t : t \geq 0\}$ : let $\bar{S}_t$ be the restriction of $P_{0]} T_t$ to $P_{0]} H_{[0}$ , then $\{\bar{S}_t : t \geq 0\}$ is a semigroup of contractions. Define $\bar{M}, \bar{X}_t$ as in (2.5). It is easy to see that $\{\bar{X}_t\}$ is a minimal non-anticipating extension of $\{X_t\}$, and it is Markov since $\bar{X}_s^* \bar{X}_{s+t} = \bar{S}_t$ , $\bar{X}_s^* \bar{X}_{s-t} = \bar{S}_t^*$ , $\{\bar{S}_t\}$ being a semigroup.

Let $\bar{\bar{X}}_t : \bar{\bar{M}} \to H$ be another extension of $\{X_t\}$ , and $\bar{\bar{P}}_{t]}$ the associated family of projections onto $\bar{\bar{H}}_{t]} = V\{\bar{\bar{X}}_s \bar{\bar{M}}, s \leq t\}$ . If $\{\bar{\bar{X}}_t\}$ is non-anticipating, $\bar{\bar{P}}_{t]} = P_{t]}$ for all t, if it is Markov $\bar{\bar{X}}_0 \bar{\bar{M}}$ is mapped into itself by $P_{0]} T_t$ ; if it is minimal, $\bar{\bar{X}}_0 \bar{\bar{M}} = \bar{M}$ . Then $\bar{M}$ and $\bar{\bar{M}}$ can be identified, and so can $\bar{X}_t$ and $\bar{\bar{X}}_t$ .

Lemma 2.3 A stationary Markov H-process is regular if and only if the associated contraction semigroup $\{S_t\}$ contracts strongly to zero as $t \to \infty$ , i.e.

$$\lim_{t\to\infty} \| S_t m \| = 0 \qquad \text{for all m in M.} \qquad (2.6)$$

Proof For all $t \geq 0$ , $s \geq -t$ we have

$$\| S_{t+s} m \| = \| X_0 S_{t+s} m \| = \| P_0 X_{t+s} m \|$$

$$= \| P_{0]} X_{t+s} m \| \qquad \text{(by the Markov property)}$$

$$= \| T_{-t} P_{0]} T_t X_s m \| = \| P_{-t]} X_s m \| \qquad \text{(by stationarity),}$$

so that (2.6) holds if and only if $\lim_{t\to\infty} P_{-t]} = 0$, which is equivalent to regularity.

Let G denote the infinitesimal generator of $\{S_t\}$ .

Lemma 2.4 [23] A stationary Markov H-process is regular if and only if there is a family $\xi_t^G$ of linear operators from $\mathcal{D}(G)$ into H such that

$$\langle \xi_t^G m, \xi_{t'}^G m' \rangle = (t \wedge t') [- \langle Gm, m' \rangle - \langle m, Gm' \rangle]$$

$$\text{for all t, t' in } \mathbb{R} \text{ , m, m' in } \mathcal{D}(G). \qquad (2.7)$$

$$H = V\{\xi_t^G m : t \in \mathbb{R}, m \in \mathcal{D}(G)\},$$

and $\{X_t\}$ satisfies the following Langevin equation:

$$X_t m - X_s m = \int_s^t X_u G m \, du + (\xi_t^G - \xi_s^G) m$$

$$\text{for all } s, t \text{ in } \mathbb{R} \text{ and m in } \mathcal{D}(G). \qquad (2.8)$$

Proof: See [23] Theorem 4.2 and [21] Theorem 3.15. Notice that if $\mathcal{D}(G) = \mathcal{D}(G^*)$ then $\xi_t^G = \xi_t^{(-1)} \left\{ -\frac{(G+G^*)}{2} \right\}^{1/2}$.

In their abstract study of scattering theory [24] Lax and Phillips introduced an interesting Hilbert space structure, which we now define:

Definition Let H be a Hilbert space, and let $\{T_t : t \in \mathbb{R}\}$ be a group of unitaries on H. Suppose that there is an orthogonal decomposition

$$H = D^- \oplus K \oplus D^+$$

such that

$$H = V\{T_t D^- : t \in \mathbb{R}\} = V\{T_t K : t \in \mathbb{R}\} = V\{T_t D^+ : t \in \mathbb{R}\},$$

$$T_t D^- \subseteq D^- \quad \text{for all } t \leq 0 \quad , \quad T_t D^+ \subseteq D^+ \quad \text{for all } t \geq 0 .$$

We shall refer to the above structure as a cyclic LP-structure. The following result follows from a careful inspection of the proof of Theorem 4.4 of [23]. However the statement of the Theorem there is incorrect.

Proposition A cyclic LP structure defines a stationary Markov H-process over K such that both $S_t$ and $S_t^*$ contract strongly to zero as $t \to \infty$ .

*Conversely, a stationary Markov process with $S_t$ and $S_t^*$ contracting strongly to zero defines a cyclic LP-structure with $K=H_0$, $D^-=D_0^-$, $D^+=D_0^+$.*

Proof(Sketch) Given a cyclic LP-structure, let $X_0$ be the natural embedding of K into H, and put $X_t = T_t X_0$ : Then $\{X_t\}$ is a stationary H-process. Lax and Phillips show in [24] that $S_t = X_0^* X_t$ is a contraction semigroup on K (so that the H-process is Markov by Lemma 2.1) and that both $S_t$ and $S_t^*$ contract strongly to zero.

Conversely, given a stationary Markov H-process, we have

$$D_t^- = T_t D_0^- = H_{[t}^{\perp} \qquad \text{and} \qquad D_t^+ = T_t D_0^+ = H_{t]}^{\perp} .$$

Since $H = \bigvee\{T_t H_0 : t \in \mathbb{R}\}$, we only have to prove that $\bigwedge\{H_{[t} : t\in\mathbb{R}\} = \{0\}$. By the same reasoning as in Lemma 2.3, this is equivalent to the condition that both $S_t$ and $S_t^*$ contract strongly to zero as $t \to \infty$. The associated scattering operators of such processes have been studied by Okabe [25].



## 3. Properties of Gaussian processes

Lemma 3.1 A Gaussian process over W(M) is determined up to equivalence by its covariance function

$$\omega(j_t(W(m))\, j_{t'}(W(m')))/\omega(j_t(W(m)))\,\omega(j_{t'}(W(m')))$$

$$= \exp[-\langle m, K(t,t')m'\rangle - i\langle m, K^Q(t,t')m'\rangle] , \qquad m,\ m' \text{ in } M, \qquad (3.3)$$

where $K(t,t')$ and $K^Q(t,t')$ in $\mathcal{B}(M)$ satisfy

$$K(t,t) = 1 \quad \text{for all } t, \quad K(t,t')^* = K(t',t) \quad \text{for all } t,\ t', \qquad (3.4a)$$

$$K^Q(t,t) = -Q_M \quad \text{for all } t, \quad K^Q(t,t')^* = -K^Q(t',t) \quad \text{for all } t,\ t', \qquad (3.4b)$$

$$\sum \bar{c}_i c_j [\langle m_i, K(t_i,t_j)m_j\rangle + i\langle m_i, K^Q(t_i,t_j)m_j\rangle] \geqslant 0 \qquad (3.4c)$$

for all finite sequences $\{c_i\}$ in $\mathbb{C}$, $\{m_i\}$ in M and $\{t_i\}$ in $\mathbb{R}$, (3.4d) the functions $t,t' \mapsto \langle m, K(t,t')m'\rangle$, $t,t' \mapsto \langle m, K^Q(t,t')m'\rangle$ are continuous for all m, m' in M.

Proof (cf. AFL [3] and Lindblad [26]) Given a Gaussian process $(\{W(X_t)\}, \omega)$ we have

$$K(t,t') = X_t^* X_{t'} \quad , \quad t,\ t' \in \mathbb{R} \quad , \qquad (3.5a)$$

$$K^Q(t,t') = -X_t^* Q X_{t'} \quad , \quad t,\ t' \in \mathbb{R} \quad ; \qquad (3.5b)$$

then eqs. (3.3) and (3.4a-d) hold. Conversely, given a pair of operator-valued functions $K(\cdot,\cdot), K^Q(\cdot,\cdot)$ on $\mathbb{R}\times\mathbb{R}$ satisfying (3.4a-b), let $(H, X_t)$ be the minimal Kolmogorov decomposition of $K(\cdot,\cdot)$ : then $\{X_t\}$ is an H-process over M evolving in H (see Section 2). Define a skew-symmetric form $\sigma$ on H by

$$\sigma(\sum c_i X_{t_i} m_i, \sum c_j' X_{t_j'} m_j') = \sum c_i c_j \langle m_i, K^Q(t_i, t_j) m_j'\rangle \qquad (3.6)$$

for all finite sequences $\{c_i\},\{c_i'\}$ in $\mathbb{C}$, $\{t_i\},\{t_i'\}$ in $\mathbb{R}$, $\{m_i\},\{m_i'\}$ in M. By eq. (3.4c) there is a skew-adjoint contraction Q on H such that $\sigma(h,k) = \langle Qh,k\rangle$ for all h, k in H; moreover $X_t^*QX_t = K^Q(t,t)$ is a skew-adjoint contraction $Q_M$ independent of t. Thus $\{X_t\}$ is compatible with the pair $(Q_M,Q)$, a Gaussian process can be constructed as in Theorem 1.1 and eq(3.3) holds for it. H, $\{X_t\}$ and Q are determined up to unitary equivalence, hence also the process is unique up to equivalence. We refer to Lindblad [26] for a different kind of elaboration on Gaussian processes on the CCR algebra and their covariance function.

Theorem 3.1 A Gaussian process is stationary if and only if the underlying H-process is stationary and the unitaries $T_t$ on H commute with Q: then $u_t = W(T_t)$.

Proof If $(\{j_t\},\omega)$ is stationary,

$$\omega(j_s(W(m))j_t(W(m'))) = \exp\{-i\langle QX_sm,X_tm'\rangle - \langle m,K(s,t)m'\rangle - \tfrac{1}{2}\|m\|^2 - \tfrac{1}{2}\|m'\|^2\}$$

is a function of $t-s$ alone, for all m, m' in M. Then $K(s,t)$ is a function of $t-s$, so that $\{X_t\}$ is stationary, and

$\langle X_s m, [Q - T_t^{-1}QT_t]X_u m'\rangle = 0$ for all m,m' in M, s, t, u in $\mathbb{R}$, so that $T_t$ commutes with Q for all t. Then $W(T_t)$ is well defined. We have

$$u_t W(X_s m) = W(X_{t+s}m) = W(T_tX_sm) \quad \text{for all m in M, } s,t \text{ in } \mathbb{R},$$

so that indeed $u_t = W(T_t)$.

Conversely, if $T_t$ is a unitary operator commuting with Q, $W(T_t)$ leaves $\omega$ invariant and is a *-automorphism of W(H). If $T_sX_t = X_{s+t}$ for all t,s, then $W(T_s)W(X_t) = W(X_{s+t})$, so the process is stationary with $u_s = W(T_s)$.



By the assumed continuity properties, we have

$$\mathcal{A}_{t)} = W(H_{t)}) = W(H_{t]}) = \mathcal{A}_{t]},$$

$$\mathcal{A}_t = W(H_t), \quad \mathcal{A}_{[t} = W(H_{[t}),$$

for all t in $\mathbb{R}$. Hence a Gaussian process is Markov if and only if there is a family of conditional expectations $E_{t]}$ of W(H) onto $W(H_{t]})$, compatible with $\omega$ and satisfying

$$E_{t]}W(H_{[t}) = W(H_t).$$

Theorem 3.2 A Gaussian process is Markov if the underlying H-process is Markov and the projections $P_{t]}$ of H onto $H_{t]}$ commute with Q; then $E_{t]} = W(P_{t]})$. The converse also holds when $\omega$ is faithful.

Proof If $[P_{t]},Q] = 0$, then $W(P_{t]})$ is well-defined and is a conditional expectation of $\mathcal{A}$ onto $\mathcal{A}_{t]}$, compatible with $\omega$, and $W(P_{t]})W(h)$ is proportional to $W(P_{t]}h)$ for all h in H; then $E_{t]}\mathcal{A}_{[t} = \mathcal{A}_t$ if and only if $P_{t]}H_{[t} = H_t$. When $\omega$ is faithful (equivalently, $Q^*Q < 1$), we can use Takesaki's theorem [27] and the explicit form of the modular automorphism group in terms of Q [28] to prove that a conditional expectation onto $\mathcal{A}_{t]}$ compatible with $\omega$ exists if and only if $[P_{t]}, Q] = 0$, and if it exists it is unique: then it is $W(P_{t]})$.

In order to simplify the discussion we make the following

Definition A Gaussian process is said to be Gaussian Markov if it is Markov and $E_{t]} = W(P_{t]})$ for each t in $\mathbb{R}$. Then the statement of Theorem 3.2 can be rephrased as follows: A Gaussian process is Gaussian Markov if and only if the underlying H-process is Markov, the projections $P_{t]}$ commute with Q and

$E_{t]} = W(P_{t]})$ for all t; if $\omega$ is faithful a Gaussian process is Gaussian Markov if and only if it is Markov. The following Theorem is a consequence of theorems 3.1, 3.2 and lemma 2.2.

Theorem 3.3 A Gaussian process $(\{j_t\},\omega)$ is stationary and has a unique non-anticipating minimal Gaussian Markov extension if and only if the underlying H-process $\{X_t\}$ is stationary and

$$[T_t, Q] = 0 = [P_{t]}, Q] \quad \text{for all } t \text{ in } \mathbb{R}. \tag{3.8}$$

Remark Since for a stationary H-process we have

$$T_t P_{s]} = P_{s+t]} T_t \quad \text{for all } s, t \text{ in } \mathbb{R}, \tag{3.9}$$

the condition (3.8) is very restrictive in the non-commutative case $(Q \neq 0)$. In particular (3.8) is not satisfied in the quantum case when $\omega$ is KMS for $\mathcal{U}_t = W(T_t)$ for some inverse temperature $\beta$ $(\neq 0)$. Then Q is explicitly given as a function of $T_t$ [28], and $[P_{t]}, Q] \neq 0$ (unless $P_{t]} = 1$ for all t); so the process is not Markov because of the lack of conditional expectations compatible with $\omega$. This is the reason why the quantum Ford-Kac-Mazur model [5] at non-zero temperature fails to provide an example of a Markov process, whereas the corresponding classical process (with Q=0) is Markov. The classical situation is recovered in the quantum case in the limit of infinite temperature ($\beta \to 0$).

The failure of the Markov property at finite temperatures is reflected in the fact that the covariance function is not a semigroup [5.29]. Indeed we have the following generalization of Doob's theorem [17]:



Theorem 3.4 A Gaussian process $(\{j_t\}, \omega)$ is Gaussian Markov if and only if its covariance function satisfies

$$K(s,t)K(t,u) = K(s,u) \qquad \text{for all } s \leq t \leq u \quad \text{in } \mathbb{R}, \tag{3.10a}$$

and

$$K^Q(s,t) = -Q_M K(s,t) \qquad \text{for all } s \leq t \quad \text{in } \mathbb{R}. \tag{3.10b}$$

Proof (cf [3]) If the process is Gaussian Markov, the underlying H-process is Markov by Theorem 3.2, then (3.10a) holds by Lemma 2.1; moreover

$$K^Q(s,t) = -X_s^* Q X_t = -X_s^* P_{s]} Q X_t = -X_s^* Q P_{s]} X_t$$
$$= -X_s^* Q X_s X_s^* X_t = -Q_M K(s,t) \qquad \text{for all } s \leq t,$$

where we have used (2.3), (3.4) and (3.5); so that (3.10b) holds. Conversely, if (3.10a) holds, $K(s,t)$ is the covariance function of a Markov H-process by Lemma 2.1; then (3.10b) tells us that

$$\langle QX_s m, [1 - X_t X_t^*] X_u m' \rangle = \langle Q_M m, K(s,u) m' \rangle - \langle Q_M m, K(s,t) K(t,u) m' \rangle = 0$$

for all m, m' in M and $s \leq t \leq u$ in $\mathbb{R}$: hence Q maps $H_{t]}$ into the orthogonal complement of $D_t^+$, which is $H_{t]}$; and since Q is skew-adjoint it commutes with $P_{t]}$. Then by Theorem 3.2 the process is Gaussian Markov.

Corollary A Gaussian process is stationary and Gaussian Markov if and only if (3.8) holds and the covariance function $K(s,t)$ is given by a semigroup $\{S_t : t \geq 0\}$ of contractions on M, as in (2.4).

Let M be a Hilbert space, $Q_M$ a skew-adjoint contraction on M, $\{S_t : t \geq 0\}$ a strongly continuous contraction semigroup on M, $\omega_M$ the state on W(M) defined by $\omega_M(W(m)) = \exp(-\frac{1}{2}\|m\|^2)$. Then, by Theorem 1.2,

$$Z_t W(m) = W(S_t m) \exp\{\tfrac{1}{2}\|S_t m\|^2 - \tfrac{1}{2}\|m\|^2\}, m \in M, t \geq 0, \qquad (3.11)$$

defines a semigroup $\{Z_t : t \geq 0\}$ of completely positive identity preserving normal maps of W(M) into itself if and only if (1.12) holds with T replaced by $S_t$ for all $t \geq 0$, and $\omega_M \circ Z_t = \omega_M$ for all $t \geq 0$.
So far, we have only proved the statement for the case $[S_t, Q_M] = 0$, and in the general case we only know that there is a semigroup of completely positive identity preserving maps of the C*-algebra generated by $\{W(m) : m \in M\}$ into itself if (1.12) holds. The following Theorem will allow us to complete the proof of Theorem 1.2.

Theorem 3.5 (Reconstruction theorem) Let $\{Z_t : t \geq 0\}$ be as above. Then there exists a stationary Gaussian Markov process $(\{j_t\}, \omega)$ over W(M) such that

$$j_0 Z_t = E_{0]} u_t j_0 \qquad (3.12)$$

for all $t \geq 0$, and the process is unique up to equivalence in the class of Gaussian Markov processes.

Proof Since $Z_t$ is completely positive and $\omega_M \circ Z_t = \omega_M$, there exists a contraction $F_t$ on $\mathcal{H}_M$ such that

$$F_t W(m)\Omega_M = Z_t W(m)\Omega_M \qquad \text{for all m in M,}$$

and $\{F_t\}$ is a semigroup of contractions [12]. Then, by [30] Chapter I Section 8; we have

$$\sum \bar{c}_i c_j \langle x_i, F_{t_j - t_i} x_j \rangle \geq 0$$

for all finite sequences $\{c_i\}$ in $\mathbb{C}$ and $\{x_i\}$ in $\mathcal{H}_M$, where



$F_t = F_{-t}^*$ for $t \leq 0$. If we let

$$x_i = \frac{d}{d\lambda} W(\lambda m_i)\Omega \Big|_{\lambda=0}, \quad m_i \in M,$$

we find explicitly

$$\sum \bar{c}_i c_j \{\langle m_i, K(t_i, t_j) m_j\rangle + i\langle m_i, K^Q(t_i, t_j) m_j\rangle\} \geq 0$$

where

$$K(s,t) = \begin{cases} S_{t-s}, & t \geq s, \qquad (3.13a) \\ S_{s-t}^*, & t \leq s, \qquad (3.13b) \end{cases}$$

$$K^Q(s,t) = \begin{cases} -Q_M S_{t-s}, & t \geq s, \qquad (3.13c) \\ -S_{s-t}^* Q_M, & t \leq s. \qquad (3.13d) \end{cases}$$

Then the pair $K(\cdot,\cdot), K^Q(\cdot,\cdot)$ determines a (unique) Gaussian process by Lemma 3.1, and the process is stationary and Gaussian Markov by the corollary to Theorem 3.4.
A straightforward verification proves that a Gaussian Markov process satisfies (3.12 with $Z_t$ given by (3.11), if and only if its covariance function is given by (3.13a-d). This concludes the proof.

Remarks Notice that $S_t$ need not commute with $Q_M$, but $T_t$ and $P_{t]}$ commute with Q. Then, by Theorems 1.1 and 1.2, $j_0$, $u_t$ and $E_{0]}$ are normal maps, and also $Z_t$ extends to a normal map of W(M), by (3.12). The same construction obviously works for a discrete semigroup $\{S_n = S^n : n \in \mathbb{Z}_+\}$, S being a contraction on M satisfying (1.12); this allows us to complete the proof of Theorem 1.2 to the general case. If $S_t$ commutes with $Q_M$, then (3.10b) holds for all s, t in $\mathbb{R}$, and Q commutes with $P_t = X_t X_t^*$ for all t, by the same reasoning as in the proof of Theorem 3.4; then there is a conditional expectation $E_0 = W(P_0)$ of W(H) onto $W(H_0)$, compatible with $\omega$, and $j_0 Z_t = E_0 u_t j_0$; conversely, if Q commutes with all $T_t$ and $P_t$, then $S_t$ commutes with $Q_M$. Similar constructions have been used in [31,32,21] for the dilation of dynamical semigroups on the CCR algebra, for related results, on the CAR(or Clifford) algebra, see [33,3].

<u>Definition</u> A W*-stochastic process $(\{j_t\}, \omega)$ over $\mathcal{B}$ evolving in $\mathcal{A}$ is said to be <u>regular</u> if

$$\bigwedge \{\mathcal{A}_{t]} : t \in \mathbb{R}\} = \mathbb{C}1 .$$

Note that the property of regularity only depends on the localization $\{\mathcal{A}_{t]}\}$ induced on $\mathcal{A}$ by $\{j_t\}$. Regular W*-stochastic processes with $\omega$ faithful and with a family $\{E_{t]}\}$ of conditional expectations compatible with $\omega$ correspond to generalized K-flows in the sense of Emch [34,31].

<u>Theorem 3.6</u> Let $(j_t, \omega)$ be a stationary Gaussian Markov process over W(M) evolving in W(H), and let $\{Z_t\}$ be the associated semigroup on W(M). Then if $\omega$ is faithful the following are equivalent:

(i) the process is regular;

(ii) $\varphi \circ Z_t$ tends weakly to $\omega_M^0$ as $t \to \infty$ for all states $\varphi$ on the C*-algebra $W^0(M)$ generated by $\{W(m) : m \in M\}$ such that $m \mapsto \varphi(W(m))$ is continuous, where $\omega_M^0$ is the restriction of $\omega_M$ to $W^0(M)$.

<u>Proof</u> By Lemma 2.3, it suffices to show that $(\{j_t\}, \omega)$ is regular if and only if the underlying H-process $\{X_t\}$ is regular, and that (ii) holds if and only if $\lim_{t\to\infty} \| S_t m \| = 0$ for all m. Let $\{X_t\}$ be not regular: then there is a nonzero h in $\bigwedge\{H_{t]} : t \in \mathbb{R}\}$, and W(h) in $\bigwedge\{\mathcal{A}_{t]} : t \in \mathbb{R}\}$ is not a multiple of the identity; so $(\{j_t\}, \omega)$ is not regular. Let $\{X_t\}$ be regular: then $\lim_{t\to\infty} \| P_{-t]} h \| = 0$ for all h in H. Let $\hat{E}_{t]}$ be the orthogonal projection in H such that

$$\hat{E}_{t]}(a\Omega) = E_{t]}(a)\Omega \quad \text{for all a in } \mathcal{A} .$$

We have, in particular,

$$\hat{E}_{t]} W(h)\Omega = W(P_{t]}h)\Omega \exp[-\tfrac{1}{2} \|(1-P_{t]})h\|^2] \xrightarrow[t\to-\infty]{} \langle \Omega, W(h)\Omega \rangle \Omega$$

in norm. The linear span of vectors of the form $W(h)\Omega : h \in H$ is norm dense in $\mathcal{H}$, hence $\lim_{t\to-\infty} \hat{E}_{t]}\psi = \langle \Omega, \psi \rangle \Omega$ for all $\psi$ in $\mathcal{H}$. If $a \in \bigwedge\{\mathcal{A}_{t]} : t \in \mathbb{R}\}$ then $\hat{E}_{t]} a\Omega = a\Omega$ for all t, hence $a\Omega = \langle \Omega, a\Omega \rangle \Omega$; and since $\Omega$ is assumed to be separating, a is a multiple of the identity, and $(\{j_t\}, \omega)$ is regular. The second equivalence has been shown by Vanheuverzwijn in [35]. We sketch the proof for the sake of completeness: if $\| S_t m \| \to 0$ as $t \to \infty$, then $\varphi(Z_t W(m)) \to \omega(W(m))$ if $\varphi(W(m))$ is a continuous function of m; conversely $\varphi(W(m)) = e^{-\|m\|^2}$ is a state on W(M) such that $m \mapsto \varphi(W(m))$ is continuous, and $\varphi(Z_t W(m)) \to \omega(W(m))$ holds if and only if $\| S_t m \| \to 0$.

<u>Remark 1</u> It is clear from the above proof that (i) implies (ii) also when $\omega$ is not faithful.

<u>Remark 2</u> Taking into account the faithfullness of $\omega$, it follows that if the process is regular then $\varphi \circ Z_t$ tends weakly to $\omega_M$ for all normal states $\varphi$ on the W*-algebra W(M), but the converse is not true in general.

We introduce now the unbounded field operators R(h) (see [6,7]). Since $\lambda \mapsto W(\lambda h)$ is a strongly continuous group of unitaries, there exists a self-adjoint operator R(h) in $\mathcal{H}$ such that

$$W(\lambda h) = \exp[i\lambda R(h)] , \quad \lambda \in \mathbb{R}, \ h \in H.$$

It is known that the $\{R(h) : h \in H\}$ are essentially self-adjoint on a common dense domain $\mathcal{D} \subset \mathcal{H}$, the map $h \mapsto R(h)$ is linear, R(h) = 0 if and only if h=0, and

$$R(h)R(k) - R(k)R(h) = 2i\sigma(h,k)1 = 2i\langle Qh, k \rangle. \tag{3.23}$$

The following result is a consequence of Lemma 2.4 and theorem 3.1, 3.2, 3.6. We use the notation introduced in Lemma 2.4.

<u>Theorem 3.7</u> A stationary Gaussian Markov process is regular if and only if

$$W(H) = V\{W(\xi_t^G m) : t \in R,\ m \in D(g)\} \tag{3.24}$$

and the $R(X_t)$ satisfy the following Langevin equation

$$R(X_t m) - R(X_s m) = R(\int_s^t X_u G m\, du) + R((\xi_t^G - \xi_s^G) m) \tag{3.25}$$

for all $s \leq t$ in $\mathbb{R}$, m in $\mathcal{D}(G)$. Moreover, we have

$$[R(X_t m), R(X_t m')] = [R(m), R(m')] = 2i \langle Q_M m, m' \rangle \tag{3.26}$$

for all m, m' in M, t in $\mathbb{R}$,

$$[R(X_s m), R((\xi_u^G - \xi_t^G) m')] = 0 \quad \text{for all } s \leq t \leq u \text{ in } \mathbb{R},\ m, m' \text{ in } M. \tag{3.27}$$

Proof (Sketch) (3.24), (3.25) are equivalent to (2.7), (2.8) respectively. (3.26) follows from $[T_t, Q] = 0$ and (3.27) from $[P_{t]}, Q] = 0$ .

The Boson Wiener process has been studied by Hudson [36,37].



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