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Mean-Field Dynamical Semigroups on C*-Algebras.

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Abstract. We study a notion of the mean-field limit of a sequence of dynamical semigroups on the n-fold tensor products of a C*-algebra $\mathcal{A}$ with itself. In analogy with the theory of semigroups on Banach spaces we give abstract conditions for the existence of these limits. These conditions are verified in the case of semigroups whose generators are determined by the successive resymmetrizations of a fixed operator, as well as generators which can be approximated by generators of this type. This includes the time evolutions of the mean-field versions of quantum lattice systems. In these cases the limiting dynamical semigroup is given by a continuous flow on the state space of $\mathcal{A}$. For a class of such flows we show stability by constructing a Liapunov function. We also give examples where the limiting evolution is given by a diffusion, rather than a flow on the state space of $\mathcal{A}$.

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1. Introduction.

In this paper we consider the mean-field limit of quantum dynamical semigroups on C*-algebras. The setup is as follows. Starting with an arbitrary unital C*-algebra $\mathcal{A}$, one constructs a sequence of algebras $\mathcal{A}^n$ by taking the n-fold tensor product of $\mathcal{A}$ with itself, completed in the minimal C*-cross-norm. Let $j_{nm} : \mathcal{A}^m \to \mathcal{A}^n$ denote the mean of all embeddings of the form $\mathcal{A}^m \ni X_m \mapsto \pi(X_m \otimes \mathbb{1}_{n-m})$, the mean being over all automorphisms $\pi \in \mathrm{Aut}(\mathcal{A}^n)$ induced by permutations of the factors of the tensor product $\mathcal{A}^n$. The spaces $j_{nn}\mathcal{A}^n$ together with the maps j_{nm} form an inductive system of vector spaces. The completion of the inductive system in an appropriate topology is homomorphic with $\mathcal{C}(K(\mathcal{A}))$, the space of weak*-continuous functions on the state space $K(\mathcal{A})$ of $\mathcal{A}$. Now assume that a quantum dynamical semigroup $(T_{t,n})_{t\geq 0}$ is specified on each $\mathcal{A}^n$ (i.e. each $T_{\cdot,n}$ is a strongly continuous semigroup of completely positive unit preserving contractions on $\mathcal{A}^n$). Roughly speaking, we will say that the sequence of semigroups $(T_{\cdot,n})_{n\in\mathbb{N}}$ has a mean-field limit, or that it is a mean-field semigroup, if it has a well defined limit as a strongly continuous contraction semigroup on $\mathcal{C}(K(\mathcal{A}))$. The main results of this paper are as follows. We give a general theory of mean-field dynamical semigroups and their limits, along the lines of the standard theory of contraction semigroups on Banach spaces. We establish some very general conditions under which a sequence of quantum dynamical semigroups will have a mean-field limit, and such that the limit is implemented by a weak*-continuous flow on $K(\mathcal{A})$. We verify these conditions for the mean-field versions of quantum lattice systems. For a class of mean-field limits we examine dynamical stability. Finally, we exhibit classes of mean-field limits which are implemented by diffusions on $K(\mathcal{A})$, rather than flows.

Quantum dynamical semigroups have been studied extensively, and a complete characterization of their generators exists in the norm-continuous case **[Lin,CE]**. One motivation for obtaining a general theory of mean-field limits of quantum dynamical semigroups comes from the physical literature, where they have been used, implicitly or explicitly, to analyse the dynamics of dissipative quantum systems in the thermodynamic limit. For examples, see **[HL,BM,BSP1,BSP2,D1,D2,UR]**, and the review in **[Sp]**. In these examples $\mathcal{A}^n$ is the observable algebra of a quantum system comprising n sites, to each of which is attached the algebra $\mathcal{A}$. Furthermore, the generators G_n of the semigroups $T_{\cdot,n}$ are polynomial in the sense that each G_n is obtained by symmetrization over n sites of some fixed generator acting on the algebra of a finite number of sites. In the treatment of specific models, the emphasis has been on demonstrating the existence of the limits $\lim_{n\to\infty}\langle\sigma^n, T_{t,n}j_{n1}X\rangle$ for all states σ on $\mathcal{A}$ and observables X in $\mathcal{A}$, and on finding a closed set of differential equations to describe the evolution of such quantities for all X in some finite subset of $\mathcal{A}$. (Here, σ^n denotes the n-fold tensor product state $\sigma\otimes\ldots\sigma$ on $\mathcal{A}^n$, and $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form between a C*-algebra and its dual). Furthermore, one shows that this limit factorizes in the sense that $\lim_{n\to\infty}\langle\sigma^n, T_{n,t}j_{n2}X\otimes Y\rangle = \lim_{n\to\infty}\langle\sigma^n, T_{n,t}j_{n1}X\rangle \lim_{n\to\infty}\langle\sigma^n, T_{n,t}j_{n1}Y\rangle$.

Finally, it is found that the evolution is implemented by a flow F_t on the state space of $\mathcal{A}$ i.e. $\lim_{n\to\infty}\langle\sigma^n, T_{t,n}j_{n1}X\rangle = \langle F_t\sigma, X\rangle$. General theorems to this effect **[AM,Dn]** have been established only for the class of models where the generators are polynomial as described above, and are bounded (except for a one-site term which may be unbounded, but not dissipative). This class of models has also been used to understand the limiting evolutions in representations of the quasi-local algebra when $\mathcal{A}$ is a finite dimensional matrix algebra containing a representation of the Lie algebra of a compact Lie group **[Bon1,Bon2,Unn]**.

Our perspective on mean-field dynamical limits is at the outset somewhat different to that described in the previous paragraph. Our aim is to provide a general theory of mean-field limits, rather than simply to establish various properties for classes of models. Following **[RW1]** we shall see that the inductive limit described above provides a natural setting in which to describe the asymptotic commutativity of observables and the existence of the mean-field dynamical limits seen in the examples.

The inductive limit will be described fully at the start of section 2. For the moment we can summarize it as follows. Amongst all sequences $\{X = (X_n)_{n\in\mathbb{N}} \mid X_n \in \mathcal{A}^n\}$ we naturally single out those in the inductive system, that is those for which for some $n_0 \in \mathbb{N}$ and $X_{n_0} \in \mathcal{A}^{n_0}$, $X_n = j_{nn_0}X_{n_0}$ for all $n \le n_0$. Such sequences will be called strictly symmetric. The seminorm $\|X\| = \lim_{n\to\infty}\|X_n\|$ is well defined on such sequences. Sequences in the closure of the set of strictly symmetric sequences with respect to this seminorm will be called approximately symmetric: they can be approximated uniformly for large n by strictly symmetric sequences. Under the seminorm, n-wise multiplication of approximately symmetric sequences is commutative:

$$\lim_{n\to\infty}\|X_nY_n - Y_nX_n\| = 0$$

for any approximately symmetric X and Y. In fact, the algebra of approximately symmetric sequences is isometric and homomorphic with $\mathcal{C}(K(\mathcal{A}))$, the commutative algebra of weak*-continuous functions on the state space of $\mathcal{A}$. This homomorphism assigns to each approximately symmetric sequence X the function $\sigma \mapsto \langle\sigma^n, X_n\rangle$. We note, however, that the existence of this limit for all σ does not imply that X is approximately symmetric. The image of the strictly symmetric sequences under the homomorphism is the dense subalgebra of polynomials on the state space of $\mathcal{A}$.

We shall say that a family of $(T_{\cdot,n})_{n\in\mathbb{N}}$ of strongly continuous unit preserving completely positive contractions has a mean-field limit if (modulo a technical condition) for all approximately symmetric X, the sequence $n \mapsto T_{t,n}X_n$ is approximately symmetric. In a more physical terminology, the approximately symmetric sequences can be viewed as intensive observables **[HL]**. We can then rephrase the above by saying that mean-field dynamical semigroups preserve the set of intensive

observables. Through the homomorphism of approximately symmetric sequences with functions in $\mathcal{C}(K(\mathcal{A}))$, the sequence of transformations $T_{t,\cdot}$ becomes a contraction semigroup on $\mathcal{C}(K(\mathcal{A}))$. We shall call this the mean-field limit of the sequence $T_{t,\cdot}$. We shall obtain a theory of mean-field dynamical semigroups which parallels the standard theory of contraction semigroups on Banach spaces. This is done in section 2. We emphasize that since the existence of $\lim_{n\to\infty}\langle\sigma^n, X_n\rangle$ does not imply approximate symmetry for the sequence X, the approximate preservation of symmetry is a stronger property than the convergence of expectation values on product states. This allows us to control limits of the form $\lim_n\langle\omega_n,\ T_{t,n}(X_n)\rangle$, when ω_n is a general "classical state" in the sense of **[HL]**, i.e. when $\lim_n\langle\omega_n, X_n\rangle$ exists for all approximately symmetric sequences. The restrictions of a translation invariant state on a lattice system to an increasing family of regions have this property **[Ru1,KR]**, but they cannot in general be expressed as an integral over product states.

If the semigroups $T_{\cdot,n}$ are asymptotically homomorphisms in the sense that

$$\lim_{n\to\infty}\|T_{t,n}(X_nY_n) - T_{t,n}(X_n)T_{t,n}(Y_n)\| = 0 \quad ,$$

then the limiting dynamics will be a homomorphism on $\mathcal{C}(K(\mathcal{A}))$, and will hence be implemented by a flow on the state space of $\mathcal{A}$. We demonstrate in section 3 that this is what happens for the bounded polynomial generators of the physical literature. If we move beyond this class of examples this will not necessarily be the case. We note however, that if a sequence of *-automorphism groups has a mean-field limit, then it is always implemented by a flow.

Of course, the theory we give is not intended merely to be an abstract specification. In section 4 we treat a class of approximately polynomial generators. These can be viewed in some sense as the closure of bounded polynomial generators within the class of bounded generators. For such generators one hopes to construct their evolutions as a limit of the evolutions of bounded polynomial generators, the limiting evolution being implemented by the limit of the flows determined by these generators. We show that this happens when the generator sequence is a derivation in the sense that

$$\lim_{n\to\infty}\|(G_nX_n)Y_n - X_n(G_nY_n) - G_n(X_nY_n)\| = 0$$

for all strictly symmetric X and Y. If each G_n is itself a derivation, then this is always true provided that the sequence $n \mapsto G_nX_n$ is approximately symmetric for all strictly symmetric X. The proof of some results on convergence of flows and their generators are deferred to section 5.

In section 4 we also give an application of the results for approximately polynomial generators to quantum lattice systems. We work on d-dimensional lattice $\mathbb{Z}^d$, to each site of which is attached a copy of a unital C*-algebra $\mathcal{A}$. Let a translation invariant lattice potential Φ be given. i.e. for each $\alpha \subset \mathbb{Z}^d$ of finite cardinality $|\alpha|$, a potential $\Phi(\alpha) \in \mathcal{A}^{|\alpha|}$ is specified. It has been shown **[Rob]** that the dynamics generated by the family of Hamiltonians $H_\Lambda = \sum_{\alpha\subset\Lambda} \Phi(\alpha)$ for a sequence of regions $\Lambda \subset \mathbb{Z}^d$ growing infinity in a suitable sense has a thermodynamic limit as a strongly continuous automorphism group on the quasi-local algebra generated by the one-site algebras provided that

$$\sum_{\alpha\subset \mathbb{Z}^d:\alpha\ni 0} e^{|\alpha|} \|\Phi(\alpha)\| < \infty \quad .$$

A mean-field version of any such model is obtained by symmetrizing the Hamiltonians over all sites of the regions in the sequence. We are able to prove that the dynamics has a mean-field limit provided that

$$\sum_{\alpha\subset \mathbb{Z}^d:\alpha\ni 0} |\alpha| \, \|\Phi(\alpha)\| < \infty \quad ,$$

a considerably weaker condition. In models like this it is interesting to examine the connection between the limiting Hamiltonian dynamics and the equilibrium statistical mechanics. We leave this to a later work **[DW]**. The general theory of the present study does not require the generators to be Hamiltonian.

As well as the existence theory given above, we are interested in the qualitative properties of the mean-field limits. Of course, matters such as stability, recurrence, and ergodicity will depend on detailed properties of the generator. However, in section 6 we are able to obtain a Liapunov functional for a class of limiting flows.

We emphasize that mean-field limits need not in general be implemented by flows on $K(\mathcal{A})$, but may in general be given by a Markov process on $K(\mathcal{A})$. In section 7 we construct a class of examples from sequences of representations of some group which have a mean-field limit, by integrating with respect to a convolution semigroup. In some cases the limiting dynamics is then given by a diffusion on the state space, in others by a jump process.

We note that in all the foregoing, we are not restricted to homogeneous mean-field models. If $\mathcal{A}$ is a one-site algebra in quantum lattice system, the sites of which are labelled by points in some compact space X, then under suitable conditions on the limiting distribution of the labels in X, one can carry through the theory using the algebra $\mathcal{C}(X,\mathcal{A})$ in place of $\mathcal{A}$. This idea is used in **[RW2]** to treat the thermodynamics of such inhomogeneous mean-field models. These frequently appear in physical applications. In **[DRW]** we treat their dynamics.

2. The Mean-Field Limit of a Sequence of Semigroups.

We shall start this section with a more detailed account of the theory of approximately symmetric sequences following **[RW1]**. This is summarized in Theorem 2.1. This prepares the way for the theory of mean field dynamical semigroups, which is contained in Theorem 2.3. In what follows, for any unital C*-algebra $\mathcal{A}$, $\mathcal{A}^*$ denotes the dual of $\mathcal{A}$, $\langle\cdot,\cdot\rangle : \mathcal{A}^* \times \mathcal{A} \to \mathbb{C}$ denotes the canonical bilinear form between $\mathcal{A}^*$ and $\mathcal{A}$, $K(\mathcal{A}) = \{\rho \in \mathcal{A}^* \mid \rho \geq 0,\ \langle\rho, \mathbb{1}\rangle = 1\}$ is the state space of $\mathcal{A}$, and $\mathcal{C}(K(\mathcal{A}))$ denotes the space of weak*-continuous functions on $K(\mathcal{A})$, with the supremum norm.

From now on, let $\mathcal{A}$ be a fixed unital C*-algebra. For any $n \in \mathbb{N}$ we define $\mathcal{A}^n$ to be the tensor product of $\mathcal{A}$ with itself n times, completed in the minimal C*-cross-norm **[Tak]**. This choice of completion has important consequences for the continuity of certain linear functionals on the tensor products. For any collection $\{\omega_1, \ldots \omega_n\} \subset K(\mathcal{A})$, the linear functional $\omega_1 \otimes \ldots \omega_n$ on the algebraic tensor product $\mathcal{A}^{\odot n}$ has an extension to $\mathcal{A}^n$ which is a state. If all ω_i are equal to some ω we will write the corresponding state on $\mathcal{A}^n$ as ω^n. It follows from Corollary 4.25 of **[Tak]** that for any $n \in \mathbb{N}$ and $X \in \mathcal{A}^n$, the map $(K(\mathcal{A}))^n \ni (\omega_1, \ldots \omega_n) \mapsto \langle\omega_1 \otimes \ldots \omega_n, X\rangle$ is weak*-continuous.

For $n \geq m$ we define the operator $j_{nm} : \mathcal{A}^m \to \mathcal{A}^n$ by

$$j_{nm}X_m = \frac{1}{n!}\sum_\pi \pi(X_m \otimes \mathbb{1}_{n-m}) \quad ,$$

where the sum is over all automorphisms π induced by permutations of the factors of the tensor product in $\mathcal{A}^n$. In other words, $j_{nm}X_m$ is the average over embeddings of X_m into m of the n factors of $\mathcal{A}^n$. The averaging operators j_{nm} are consistent in that they satisfy $j_{nm} \circ j_{mr} = j_{nr}$ for $n \geq m \geq r$. Restricted to the symmetric part of $j_{mm}\mathcal{A}^m$ of $\mathcal{A}^m$, each j_{nm} is injective. Therefore we can consider abstractly the spaces $j_{nn}\mathcal{A}^n$ together with the maps j_{nm} as an inductive system of vector spaces. Since each j_{nm} is a contraction for the given norms on $\mathcal{A}^m$ and $\mathcal{A}^n$ the inductive limit carries a natural seminorm: for an arbitrary sequence $(X_n)_{n\in\mathbb{N}}$ with $X_n \in \mathcal{A}^n$ we write

$$\|X\| = \limsup_{n\to\infty} \|X_n\| \quad . \tag{2.1}$$

Within the set of all sequences $(X_n)_{n\in\mathbb{N}}$ we single out those in the inductive limit space: those sequences $X. : n \mapsto j_{nn}\mathcal{A}^n$ for which for some $m_0 \in \mathbb{N}$, $X_n = j_{nm_0}X_{m_0}$ for all $n \geq m_0$. Such sequences will be called **strictly symmetric**, and the number m_0 will be called the **degree** of the sequence X as defined above. The set of all such sequences will be denoted by $\mathcal{Y}$. For $X_n \in \mathcal{A}^n$ we define $j_{\infty n}X_n \in \mathcal{C}(K(\mathcal{A}))$ by $(j_{\infty n}X_n)(\rho) = \langle\rho^n, X_n\rangle$ for all $\rho \in K(\mathcal{A})$. For strictly symmetric sequences this is independent of n for n sufficiently large. Trivially, $X_\infty \equiv \lim_{n\to\infty} j_{\infty n} X_n$ exists as an element of $\mathcal{C}(K(\mathcal{A}))$.

A sequence $X. : n \mapsto \mathcal{A}^n$ is called **approximately symmetric** if if for all $\varepsilon > 0$ there is an n_ε such that $\|X_n - j_{nm}X_m\| \le \varepsilon$ for all $n \ge m \ge n_\varepsilon$. For the latter statement we also write

$$\lim_{n\ge m\to\infty} \|X_n - j_{nm}X_m\| = 0 \quad . \tag{2.2}$$

The set of approximately symmetric sequences will be denoted by $\tilde{\mathcal{Y}}$.

We now state without proof the results from **[RW1]** which establish the relationship between the inductive limit space and $\mathcal{C}(K(\mathcal{A}))$.

Theorem 2.1. [RW1]

(1) For all $X \in \tilde{\mathcal{Y}}$, $\|X\| = \lim_{n\to\infty} \|X_n\|$ exists, and $\tilde{\mathcal{Y}}$ is the completion of $\mathcal{Y}$ in the seminorm (2.1). Furthermore, $\tilde{\mathcal{Y}}$ is closed within the set of all sequences $n \mapsto X^n \in \mathcal{A}^n$ in this seminorm.

(2) $\tilde{\mathcal{Y}}$ is an algebra with the operations of n-wise addition $(X., Y.) \mapsto X. + Y.$ and n-wise multiplication $(X., Y.) \mapsto X.Y.$. $\tilde{\mathcal{Y}}$ is commutative under the seminorm (2.1) in the sense that

$$\|XY - YX\| = \lim_{n\to\infty} \|X_nY_n - Y_nX_n\| = 0 \quad .$$

(3) For all $X \in \tilde{\mathcal{Y}}$, $X_\infty(\rho) = \lim_{n\to\infty}(j_{\infty n}X_n)(\rho)$ exists uniformly for $\rho \in K(\mathcal{A})$.

*(4) The map $\tilde{\mathcal{Y}} \to \mathcal{C}(K(\mathcal{A})) : X \mapsto X_\infty$ is an isometric *-homomorphism from $\tilde{\mathcal{Y}}$ onto $\mathcal{C}(K(\mathcal{A}))$.*

Some remarks and illustrations are now in order. First, note that the statement that the completion of $\mathcal{Y}$ is $\tilde{\mathcal{Y}}$ in (1) above is just a restatement of the definition of approximate symmetry. Second, for an arbitrary sequence $(X_n)_{n\in\mathbb{N}}$ the existence of $\lim_{n\to\infty} j_{\infty n}X_n$ is not a sufficient condition for X to be approximately symmetric. Third, although the algebra $\tilde{\mathcal{Y}}$ is non-commutative, we see from item (2) above that all commutator sequences in $\tilde{\mathcal{Y}}$ are null in the seminorm (2.1). Thus they are contained in the kernel of the homomorphism which maps $\tilde{\mathcal{Y}}$ onto the abelian algebra $\mathcal{C}(K(\mathcal{A}))$. The proof of (2) given in **[RW1]** is based on a decomposition of the product of two strictly symmetric sequences $X_n = j_{nx}X_x$ and $Y_n = j_{ny}Y_y$ as

$$X_nY_n = \sum_r c_n(x, y; r) j_{n(x+y-r)}((X_x \otimes \mathbb{1}_{y-r})(\mathbb{1}_{x-r} \otimes Y_y)) \quad , \tag{2.3}$$

where $c_n(x, y; r)$ is the proportion of permutations γ of $\{1, \ldots n\}$ such that the intersection of $\{1, \ldots x\}$ and $\{\gamma(1), \ldots \gamma(y)\}$ has r elements. The result (2) follows from the observation that except for "overlap" $r = 0$ all c_n go to zero. Similar decompositions for operators acting on elements of $\mathcal{A}^n$ will be used in sections 3 and 4. Fourth, item (4) can be rephrased by saying that $\mathcal{C}(K(\mathcal{A}))$ is the Hausdorff completion of $\mathcal{Y}$. As a corollary we obtain Størmers theorem **[Stø]** about permutation symmetric states on the C*-inductive limit algebra $\mathcal{A}^\infty = \overline{\bigcup_n \mathcal{A}^n}$, where the inductive limit is taken with the embeddings $\tilde{j}_{nm} : \mathcal{A}^m \to \mathcal{A}^n : X_m \mapsto X_m \otimes \mathbb{1}_{n-m}$, *without* symmetrizing over permutations. For permutation invariant states Φ on $\mathcal{A}^\infty$ and $X \in \tilde{\mathcal{Y}}$ the limit $\lim_n \Phi(X_n)$ exists, and defines a state on $\tilde{\mathcal{Y}}$, vanishing on the kernel of $X \mapsto X_\infty$. Therefore, it is given by a state on $\mathcal{C}(K(\mathcal{A}))$, i.e. a measure μ on $K(\mathcal{A})$, and one easily checks that $\Phi = \int \mu(d\varphi)\varphi^\infty$, where φ^∞ denotes the infinite product state of φ on $\mathcal{A}^\infty$.

For any $\mathcal{S} \subset \tilde{\mathcal{Y}}$ we shall define $\mathcal{S}_\infty \subset \mathcal{C}(K(\mathcal{A}))$ to be the set $\{X_\infty \mid X \in \mathcal{S}\}$. A subset $\mathcal{D} \subset \tilde{\mathcal{Y}}$, will be called **dense**, if all elements of $\tilde{\mathcal{Y}}$ can be approximated in seminorm by elements of $\mathcal{D}$. This is equivalent to saying that $\mathcal{D}_\infty$ is dense in $\mathcal{C}(K(\mathcal{A}))$. We define $\mathcal{P} = \mathcal{Y}_\infty$. Clearly $\mathcal{P}$ is an algebra in $\mathcal{C}(K(\mathcal{A}))$ and since $\mathcal{Y}$ is dense in $\tilde{\mathcal{Y}}$, $\mathcal{P}$ is dense in $\mathcal{C}(K(\mathcal{A}))$.

The objects of principal interest in this paper are sequences of quantum dynamical semigroups on the $\mathcal{A}^n$. These are semigroups of the following kind: for each $n \in \mathbb{N}$, $t \in \mathbb{R}^+$ we have a completely positive, identity preserving contraction $T_{t,n} : \mathcal{A}^n \to \mathcal{A}^n$, such that for fixed n, $(T_{t,n} = e^{tG_n})_{t\ge 0}$ is a strongly continuous one-parameter semigroup on $\mathcal{A}^n$ with generator G_n. Our objective is to define a suitable notion of mean-field limit $n \to \infty$ of a sequence of semigroups, and to establish some properties of the "limit" when it exists. We have postulated that the family $(T_{t,n})_{n\in\mathbb{N}}$ should have good mean-field properties if it takes approximately symmetric sequences into approximately symmetric sequences. We shall also want to consider this property for the resolvents of the G_n, defined for each $s \in \mathbb{C}$ with $\Re(s) > 0$ by

$$R_n(s) = (s - G_n)^{-1} = \int_0^\infty dt\, e^{-st} T_{t,n} \quad .$$

For this reason we will state the following general result. If for each $n \in \mathbb{N}$, $T_n : \mathcal{A}^n \to \mathcal{A}^n$ is a bounded linear operator, and $X_n \in \mathcal{A}^n$, we write $T.X.$ for the sequence $(T.X.)_n = T_nX_n$.

Lemma 2.2. *Let $\mathcal{A}$ be a unital C*-algebra, and let $T_n : \mathcal{A}^n \to \mathcal{A}^n$ be a uniformly bounded sequence of operators, which intertwine the permutation automorphisms. Then the following conditions are equivalent:*

(1) For all $X \in \tilde{\mathcal{Y}}$: $\quad T.X. \in \tilde{\mathcal{Y}}$.

(2) *For all* $X \in \tilde{\mathcal{Y}}$:
$$\lim_m \lim_n \|(j_{nm}T_m - T_n j_{nm})X_m\| = 0 \quad .$$

(3) *For all* $X \in \tilde{\mathcal{Y}}$:
$$\lim_{n \geq m \to \infty} \|(j_{nm}T_m - T_n j_{nm})X_m\| = 0 \quad .$$

Furthermore, if these conditions are satisfied $T_\infty : X_\infty \mapsto (T.X.)_\infty$ *is well defined, and the sequence* $(T_n)_{n\in\mathbb{N}}$ *will be called* **approximate symmetry preserving** .

Proof: Note first that because $\|T_n\|$ is uniformly bounded, and because $\mathcal{Y}$ is dense in $\tilde{\mathcal{Y}}$, all three statements are equivalent to their counterparts with $X \in \mathcal{Y}$. Then the norm appearing in conditions (2) and (3) is equal to $\|T_n X_n - j_{nm}T_m X_m\|$. Writing out condition (1) in terms of equation (2.2) defining approximate symmetry for $T.X.$ gives (3). The implication (3)⇒(2) is trivial. Condition (2) says that for sufficiently large m, the sequence $T.X.$ is approximated in the seminorm (2.1) by the strictly symmetric sequence $n \mapsto j_{nm}(T_m X_m)$. Hence $T.X.$ is approximately symmetric, by the closedness of $\tilde{\mathcal{Y}}$. This proves the equivalence of all three conditions.

To see that T_∞ is well defined, let $X \in \tilde{\mathcal{Y}}$ with $X_\infty = 0$. Then
$$\|(T.X.)_\infty\| = \lim_{n\to\infty} \|T_n X_n\| \leq \sup_{n\in\mathbb{N}} \|T_n\| \lim_{n\to\infty} \|X_n\| = 0 \quad .$$
■

For sequences of unbounded linear operators the existence of a limiting operator is not so clear. For each $n \in \mathbb{N}$ let P_n be an (unbounded) linear operator on $\mathcal{A}^n$ with domain $\mathrm{Dom}(P_n)$. We denote by $\mathcal{D}om(P)$ the sequence space
$$\mathcal{D}om(P) = \{X \in \tilde{\mathcal{Y}} \mid X_n \in \mathrm{Dom}\, P_n \text{ for all } n \text{ and } P.X. \in \tilde{\mathcal{Y}}\} \quad .$$
It is not clear that $P_\infty : X_\infty \mapsto (P.X.)_\infty$ with domain $(\mathcal{D}om(P))_\infty$ is well defined: there may exist an X in $\mathcal{D}om(P)$ with $X_\infty = 0$ such that $(P.X.)_\infty \neq 0$. However, during the proof of the main theorem of this section which follows, it is shown that G_∞ is well defined whenever $T_{t,\cdot}$ is approximate symmetry preserving and $\mathcal{D}om(G)$ dense. This theorem can be viewed as the mean-field dynamical analogue of the standard theory of contraction semigroups on Banach spaces.

Theorem 2.3. *For each* $n \in \mathbb{N}$ *let* $(T_{t,n} = e^{tG_n})_{t\geq 0}$ *be a strongly continuous one-parameter semigroup of contractions on* $\mathcal{A}^n$ *with generator* G_n, *and let* $s \in \mathbb{C}$, $\Re(s) > 0$. *Then the following conditions are equivalent:*

(1) *For each* t, $T_{t,\cdot}$ *is approximate symmetry preserving, and* T *is strongly continuous in the sense that for all* $X. \in \tilde{\mathcal{Y}}$,
$$\lim_{\substack{t\to 0\\ n\to\infty}} \|T_{t,n}X_n - X_n\| = 0 \quad .$$

(2) *For each* t, $T_{t,\cdot}$ *is approximate symmetry preserving, and the set of sequences* X *with* $X_n \in \mathrm{Dom}(G_n)$ *and* $\|G_n X_n\|$ *uniformly bounded is dense in* $\tilde{\mathcal{Y}}$.

(3) $R.(s)$ *is approximate symmetry preserving, and* $\{R.(s)X. \mid X \in \tilde{\mathcal{Y}}\}$ *is dense in* $\tilde{\mathcal{Y}}$.

(4) *There is a dense linear subspace* $\mathcal{D} \subset \tilde{\mathcal{Y}}$ *such that* $\{(G. - s)X. \mid X \in \mathcal{D}\}$ *is also dense in* $\tilde{\mathcal{Y}}$.

(5) G_∞ *is well-defined, closed, and generates a semigroup of contractions on* $\mathcal{C}(K(\mathcal{A}))$.

Moreover, if these conditions are satisfied, $\mathcal{D}om(G) = \{R.(s)X. \mid X \in \tilde{\mathcal{Y}}\}$, $T_{t,\infty} = e^{tG_\infty}$, *and* $T_{t,\cdot}$ *will be said to have a* **mean-field limit**, *namely,* $T_{t,\infty}$.

Proof: (2) ⇒ (1) : Let $X \in \tilde{\mathcal{Y}}$. Then for every $\varepsilon > 0$ we can find $X^\varepsilon \in \tilde{\mathcal{Y}}$, $C < \infty$, and $n_\varepsilon \in \mathbb{N}$ such that $\|G_n X_n^\varepsilon\| \leq C$, and $\|X_n^\varepsilon - X_n\| \leq \varepsilon$ for $n > n_\varepsilon$. Consequently,
$$\begin{aligned}\|T_{t,n}X_n - X_n\| &\leq 2\|X_n^\varepsilon - X_n\| + \|T_{t,n}X_n^\varepsilon - X_n^\varepsilon\| \\ &\leq 2\varepsilon + |t| \cdot \|G_n X_n^\varepsilon\| \leq 2\varepsilon + C|t| \quad .\end{aligned}$$
This can be made small by simultaneously choosing n sufficiently large and t sufficiently small, as required.

(1) ⇒ (3, all s) : Let $X \in \tilde{\mathcal{Y}}$, and $s \in \mathbb{C}$, $\Re(s) > 0$. Then since each $T_{t,\cdot}$ is approximate symmetry preserving the sequence $Y_n^{\tau,N} := \tau \sum_{k=0}^{N-1} e^{-sk\tau} T_{k\tau,n} X_n$ is approximately symmetric. On the other hand,
$$\begin{aligned}&\|Y_n^{\tau,N} - R_n(s)X_n\| \\ &\leq \sum_{k=0}^{N-1} \| \int_0^\tau dt\, e^{-sk\tau} T_{k\tau,n}(X_n - e^{-st}T_{t,n}X_n)\| \quad + \| \int_{N\tau}^\infty dt e^{-st} T_{t,n} X_n\| \\ &\leq N\tau \sup_{0\leq t\leq\tau} \|(X_n - e^{-st}T_{t,n}X_n)\| + (\Re(s)N\tau)^{-1}\|X_n\| \\ &\leq N\tau \sup_{0\leq t\leq\tau} \|(X_n - T_{t,n}X_n)\| + N\tau^2\Re(s)\|X_n\| + (\Re(s)N\tau)^{-1}\|X_n\| \quad .\end{aligned}$$

Since $\|X_n\|$ is uniformly bounded, we may make the last term arbitrarily small by choosing N_T large enough. For fixed N_T we only have to choose τ small to make the second arbitrarily small, and the same holds for the first term by assumption (1). Hence the sequence $R.(s)X.$ is uniformly approximated by approximately symmetric sequences and consequently is itself approximately symmetric.

It is also clear from the strong continuity that $\|X_n - s'R_n(s')X_n\| \to 0$ as $s' \to \infty$, uniformly for large n. By the resolvent equation $R.(s') = R.(s)(\mathbb{1}-(s-s')R.(s'))$ the sequences $s'R.(s')X.$ are all in the range of $R.(s)$, and since $X \in \tilde{\mathcal{Y}}$ was arbitrary, the range of $R.(s)$ is dense.

(3, some s) $\Rightarrow$ (2) : For any s we have that $X \in \mathcal{Dom}(G)$ iff for all n $X_n \in \mathrm{Dom}(G_n)$, and both X and $(s-G.)X.$ are approximately symmetric. This is equivalent to the existence of $Y_n \in \mathcal{A}^n$, such that $X_n = R_n(s)Y_n$, with the additional properties that $Y. = (s-G.)X. \in \tilde{\mathcal{Y}}$ and $X \in \tilde{\mathcal{Y}}$. Therefore, if $R.(s)$ is approximate symmetry preserving, the equation $\mathcal{Dom}(G) = R.(s)\tilde{\mathcal{Y}}$ holds, and since the latter set is assumed to be dense, so is $\{X \in \tilde{\mathcal{Y}} \mid G.X.$ uniformly bounded$\} \supset \mathcal{Dom}(G)$.

It remains to be shown that $T_{t,\cdot}$ is approximate symmetry preserving for all t. First we show that if (3) holds for one value of s it holds for all values. This follows by noting that for s' such that $|s'-s| < \Re(s)$, the series expansion of $R_n(s') = R_n(s)(1+(s-s')R_n(s))^{-1}$ converges uniformly in n, and that that each approximant is approximate symmetry preserving. Furthermore, we see that for any such s' the ranges of $R.(s)$ and $R.(s')$ are contained within each other, and so are equal. By iteration, the argument extends to the whole positive half-plane.

For $X. \in \mathcal{Dom}(G^2)$ and $\mu \in \mathbb{N}$ we have by **[Kat,Sect.IX-§1.2]** that

$$\|T_{t,n}X_n - ((t/\mu)R_n(\mu/t))^\mu X_n\| \le \frac{t^2}{2\mu}\|G_n^2 X_n\| \quad . \tag{2.4}$$

Thus $T_{t,n}X_n$ is approximated uniformly in n by approximately symmetric sequences, and so is itself approximately symmetric. If $\mathcal{Dom}(G^2)$ is dense in $\tilde{\mathcal{Y}}$, then since the $T_{t,n}$ are contractions, $T_{t,\cdot}$ will be approximate symmetry preserving. But $\{R.(s)X. \mid X \in \mathcal{Dom}(G)\} \subset \mathcal{Dom}(G^2)$ is dense in $\tilde{\mathcal{Y}}$ so that, thus $\mathcal{Dom}(G^2)$ is itself dense.

(3) $\Rightarrow$ (5) : $\|R_n(s)\|$ is uniformly bounded by $\Re(s)^{-1}$, so by Lemma 2.2 we can define $R_\infty(s)X_\infty = (R.(s)X.)_\infty$ on $\mathcal{C}(K(\mathcal{A}))$. Clearly the operators $R_\infty(s)$ satisfy the resolvent equation, are norm bounded by $1/\Re(s)$, and by assumption have dense range. Thus **[Dav]** they are resolvents of a closed, densely defined operator $\tilde{G}_\infty$ which generates a semigroup of contractions $\tilde{T}_{t,\infty}$ on $\mathcal{C}(K(\mathcal{A}))$. Let $X \in \mathcal{Dom}(G)$ with $X_\infty = 0$. Then $(G.X.)_\infty = ((G.-s)X.)_\infty = (s-\tilde{G}_\infty)R_\infty(s)((G.-s)X.)_\infty = (s-\tilde{G}_\infty)(R.(s)(G.-s)X.)_\infty) = (\tilde{G}_\infty - s)X_\infty = 0$. Thus G_∞ is well-defined. Now for any $X. \in \mathcal{Dom}(G)$, $X. = R.(s)(sX. - G.X.) \in R.(s)\tilde{\mathcal{Y}}$, while for

any $Y. \in \tilde{\mathcal{Y}}$, $R.(s)Y. \in \mathcal{Dom}(G)$. Hence $\mathcal{Dom}(G) = R.(s)\tilde{\mathcal{Y}}$, and so $\mathrm{Dom}(G_\infty) = R_\infty(s)\mathcal{C}(K(\mathcal{A})) = \mathrm{Dom}(\tilde{G}_\infty)$. Furthermore, $R_\infty(s)(s-G_\infty)$ is the identity on the common domain of G_∞ and $\tilde{G}_\infty$, so these two operators are equal.

(5) $\Rightarrow$ (4) : First note that a subset $\mathcal{D}$ of $\tilde{\mathcal{Y}}$ is dense if and only if $\mathcal{D}_\infty$ is dense in $\mathcal{C}(K(\mathcal{A}))$. By hypothesis, $\mathrm{Dom}(G_\infty)$ and $(s-G_\infty)\,\mathrm{Dom}(G_\infty)$ are dense in $\mathcal{C}(K(\mathcal{A}))$. Since the latter set is just $((s-G.)\mathcal{Dom}(G))_\infty$, (4) holds with $\mathcal{D} = \mathcal{Dom}(G)$.

(4) $\Rightarrow$ (3) : By hypothesis, for every $Y \in \tilde{\mathcal{Y}}$ and $\varepsilon > 0$ there is an $X^\varepsilon \in \mathcal{D}$ such that $\limsup_{n\to\infty}\|Y_n - (s-G_n)X_n^\varepsilon\| < \varepsilon$. Consequently, $\limsup_{n\to\infty}\|R_n(s)Y_n - X_n^\varepsilon\| \le \varepsilon/\Re(s)$. Hence the sequence $R.(s)Y.$ is uniformly approximated by approximately symmetric sequences and is consequently itself approximately symmetric. To show that the range of $R.(s)$ is dense we note since by hypothesis $\mathcal{D}$ is dense in $\tilde{\mathcal{Y}}$, then for every $X \in \tilde{\mathcal{Y}}$ and $\varepsilon > 0$ there is an $X^\varepsilon \in \mathcal{D}$ with $\|X - X^\varepsilon\| < \varepsilon$ so that

$$\limsup_{n\to\infty}\|X_n - R_n(s)(s-G_n)X_n^\varepsilon\| < \varepsilon \quad ,$$

which tells us that any sequence $X.$ in $\tilde{\mathcal{Y}}$ can be uniformly approximated by sequences in $R.(s)\tilde{\mathcal{Y}}$.

When conditions (1) to (5) are satisfied, we can calculate $T_{t,\infty}$ from equation (2.4). For $X \in \mathcal{Dom}(G^2)$

$$\|T_{t,\infty}X_\infty - ((t/\mu)R_\infty(\mu/t))^\mu X_\infty\| \le \frac{t^2}{2\mu}\sup_{n\in\mathbb{N}}\|G_n^2 X_n\| \quad .$$

Taking the limit as $\mu \to \infty$, $T_{t,\infty}X_\infty = e^{t\tilde{G}_\infty}X_\infty = e^{tG_\infty}X_\infty$. Since $\mathcal{Dom}(G^2)$ is dense in $\tilde{\mathcal{Y}}$, this relation extends to the whole of $\mathcal{C}(K(\mathcal{A}))$.

■

In sections 3 and 4 we shall be concerned with demonstrating that mean-field limits of certain models exist, and have a limit which is implemented by a flow on $\mathcal{C}(K(\mathcal{A}))$. We recall the following general result from the theory of semigroups which gives an equivalence between continuous flows on a compact space X and strongly continuous semigroups of *-homomorphisms on $\mathcal{C}(X)$, and their generators.

Proposition 2.4.[Sch]. *Let X be a compact space. Then there is an correspondence between members of the following classes of objects:*

*(1) Strongly continuous contraction semigroups $(T_t)_{t\ge0}$ on $\mathcal{C}(X)$ which (a) preserve the identity function; and (b) are *-homomorphisms.*

(2) *Linear operators Z on $\mathcal{C}(X)$ which (a) generate contraction semigroups on $\mathcal{C}(X)$; and (b) are *-derivations.*

(3) *Continuous flows $(F_t)_{t\geq 0}$ on X.*

The correspondence is that for all $f \in \mathcal{C}(X)$, $\quad e^{tZ}f = T_t f = f \circ F_t$.

We can combine this proposition with our general Theorem 2.3 to obtain more detailed information about mean-field limits of automorphism groups.

Proposition 2.5. *For each $n \in \mathbb{N}$ let $T_{t,n} = e^{tG_n}$ be a group of *-automorphisms on $\mathcal{A}^n$, and let $T_{t,\cdot}$ have a mean-field limit $T_{t,\infty}$. Then*

(1) *The semigroup $(T_{t,\infty})_{t\geq 0}$ is implemented by some weak*-continuous flow $(F_t)_{t\geq 0}$ on $\mathcal{C}(K(\mathcal{A}))$.*

(2) *For each $t \geq 0$ the implementing flow F_t is onto.*

Proof: For (1) we verify that condition (1) of Proposition 2.4 holds. First, $T_{t,\infty}$ is a homomorphism:

$$\begin{aligned} T_{t,\infty}(X_\infty Y_\infty) &= \lim_{n\to\infty} j_{\infty n} T_{t,n}(X_n Y_n) = \lim_{n\to\infty} j_{\infty n} T_{t,n} X_n T_{t,n} Y_n \\ &= \lim_{n\to\infty} j_{\infty n} T_{t,n} X_n \lim_{n\to\infty} j_{\infty n} T_{t,n} Y_n = T_{t,\infty} X_\infty T_{t,\infty} Y_\infty \quad . \end{aligned}$$

Second, since $T_{t,n}\mathbb{1}_n = \mathbb{1}_n$ for all n, the identically one function $1_{K(\mathcal{A})} = j_{\infty n}\mathbb{1}_n$ is preserved by $T_{t,\infty}$, and since each $T_{t,n}$ is *-preserving, so is $T_{t,\infty}$.

For (2), note first that $T_{t,\infty}$ is an isometry: $\|T_{t,\infty}X_\infty\| = \lim_{n\to\infty}\|T_{t,n}X_n\| = \lim_{n\to\infty}\|X_n\| = \|X_\infty\|$. Suppose that for some $t > 0$, F_t is not onto. Since F_t is continuous we can find a function $f \in \mathcal{C}(K(\mathcal{A}))$ with $\|f\| \neq 0$, such that $f(x) = 0$ for all $x \in F_t(K(\mathcal{A}))$. But since $T_{t,\infty}$ is an isometry, $\|f\| = \|T_{t,\infty}f\| = \sup_{\rho\in K(\mathcal{A})}\|f(F_t\rho)\| = 0$, a contradiction.

∎

We remark that injectivity of the flow F_t is not decided here: the flow may not be reversible. To put it another way, even though the sequence of *-automorphisms $T_{t,\cdot}$ has a mean-field limit for $t \geq 0$, we can not conclude that it also has one for $t < 0$.

3. Bounded Polynomial Generators.

In this section we deal with the simplest class of examples of mean-field dynamical semigroups, namely those with bounded polynomial generators. In what follows, for any C*-algebra $\mathcal{A}$, $\mathcal{B}(\mathcal{A})$ will denote the set of bounded linear operators on $\mathcal{A}$.

We will start by defining a symmetrization operator on the $\mathcal{B}(\mathcal{A}^n)_{n\in\mathbb{N}}$. For each n we define $\mathrm{Sym}_n : \bigcup_{m\leq n}\mathcal{B}(\mathcal{A}^m) \to \mathcal{B}(\mathcal{A}^n)$ by

$$\mathrm{Sym}_n G_g = \frac{1}{n!}\sum_\pi \pi(G_g \otimes \mathrm{id}_{n-g})\pi^{-1}$$

for all $m \leq n$ and $G_m \in \mathcal{B}(\mathcal{A}^m)$, where the sum is over all automorphisms π induced by permutations of the factors of the tensor product in $\mathcal{A}^n$.

Definition 3.1. *A sequence of operators $G_\cdot = (G_n)_{n\geq g}$, with $G_n \in \mathcal{B}(\mathcal{A}^n)$ will be called a* **bounded polynomial generator** *of degree g if*

$$G_n = \frac{n}{g}\,\mathrm{Sym}_n G_g \quad ,$$

where G_g is the generator of a norm-continuous semigroup of completely positive unital maps on $\mathcal{A}^g$.

In this definition G_n is calculated as the resymmetrization of G_g, with the scaling factor n/g. This factor is chosen in accordance with the equilibrium statistical mechanics of mean-field systems **[RW1]**. These are defined by a sequence of Hamiltonians $H_n \in \mathcal{A}^n$ such that the sequence of densities $h_n = (1/n)H_n$ is approximately symmetric. We obtain a bounded polynomial generator of degree g if h is strictly symmetric of the same degree, and if we set $G_n(A) = i[H_n, A]$. Note that a generator of degree g is also a generator of any larger degree, since for $n \geq m \geq g$ we have $(n/m)\,\mathrm{Sym}_n(G_m) = (n/m)(m/g)\,\mathrm{Sym}_n(\mathrm{Sym}_m(G_g)) = (n/g)\,\mathrm{Sym}_n(G_g)$. The terminology "bounded" stems from the fact that each G_n is a bounded operator: clearly $\|G_n\| \leq (n/g)\|G_g\|$. However, we shall see that for bounded polynomial generators the limiting operator G_∞ exists and is equal to a derivation on the abelian algebra $\mathcal{C}(K(\mathcal{A}))$, and as such is unbounded. We remark that as a consequence of the definition all G_n generate completely positive semigroups. For any $n > g$, and any permutation automorphism π on $\mathcal{A}^n$, $\pi(e^{tG_g} \otimes \mathrm{id}_{g-n})\pi^{-1}$ is completely positive. By means of the Trotter product formula e^{tG_n} can be written as the limit of products of such operators is hence completely positive.

We will fix a dense subset of $\tilde{\mathcal{Y}}$ with which it will be convenient to work. Define $\hat{\mathcal{Y}} \subset \tilde{\mathcal{Y}}$ by

$$\hat{\mathcal{Y}} = \{X \in \tilde{\mathcal{Y}} : X_n = j_{nm}(X^n_m), n \geq m, \lim_{n\to\infty} X^n_m = \hat{X}_m \in \mathcal{A}^m\} \quad .$$

We shall call m the degree of $X \in \hat{\mathcal{Y}}$ and $\hat{X}_m$ its limiting element, as so defined. Let $n \in \mathbb{N}$, let γ, γ' be arbitrary permutations of the set $\{1,2,\dots n\}$, and denote the corresponding permutation automorphisms on $\mathcal{A}^n$ by π_γ and $\pi_{\gamma'}$. Let $U, V \in \mathcal{A}^n$ be of the form $U = \pi_\gamma(U_u \otimes \mathbb{1}_{n-u})$, $V = \pi_{\gamma'}(V_v \otimes \mathbb{1}_{n-v})$ for arbitrary $u, v \leq n$ and $U_u \in \mathcal{A}^u$ and $V_v \in \mathcal{A}^v$. As was briefly mentioned in section 2, we define the **overlap** of U and V to be the number of elements in the intersection of $\{\gamma(1), \dots \gamma(u)\}$ and $\{\gamma'(1), \dots \gamma'(v)\}$. In other words, the overlap is an upper bound for the number of factors of the tensor product on which neither U nor V has an identity factor. The following proposition tells us that in the limit $n \to \infty$ the action of a bounded polynomial generator on an element of $\hat{\mathcal{Y}}$ is determined by terms where the overlap is 1.

Proposition 3.2.

(1) Let $G.$ be a bounded polynomial generator of degree g. Then $\hat{\mathcal{Y}} \subset \mathcal{D}om(G)$ and $G\hat{\mathcal{Y}} \subset \hat{\mathcal{Y}}$.

(2) Additionally, let $X \in \hat{\mathcal{Y}}$ be of degree x. Then

$$(G.X.)_\infty = x j_{\infty(x+g-1)}(G_g \otimes \mathrm{id}_{x-1})(\mathbb{1} \otimes \hat{X}_x) \quad ,$$

where $\hat{X}_x$ is the limiting element of X. Thus $G.X.$ is of degree $(g + x - 1)$ and $\|(G.X.)_\infty\| \leq x\|G_g\|\|\hat{X}_x\|$.

Proof: Let $n \geq g + x - 1$. Then

$$\begin{aligned} G_n X_n &= \frac{n}{g} j_{nn}((G_g \otimes \mathrm{id}_{n-g}) j_{nx} X^n_x) \\ &= \frac{n}{g} \sum_{r=1}^{\min(g,x)} c_n(g,x;r) j_{n\ (g+x-r)}(G_g \otimes \mathrm{id}_{x-r})(\mathbb{1}_{g-r} \otimes X_x) \quad , \end{aligned} \tag{3.1}$$

where

$$c_n(g,x;r) = \frac{g!x!(n-g)!(n-x)!}{n!r!(g-r)!(x-r)!(n+r-g-x)!}$$

is the proportion of permutations in $j_{nx}X_x$ for which the overlap with $G_g \otimes \mathrm{id}_{n-g}$ is r. The terms with 0-overlap are zero because $G_g \mathbb{1}_g = 0$. Now for fixed x and g, $c_n(g,x;r) \sim n^{-r} g!x!/(r!(g-r)!(x-r)!)$ so that

$$\lim_{n\to\infty} (n/g) c_n(g,x;r) = \begin{cases} x & \text{if } r = 1 \\ 0 & \text{if } r \geq 2. \end{cases}$$

Hence

$$G_n X_n - x j_{n\ (g+x-1)}(G_g \otimes \mathrm{id}_{x-1})(\mathbb{1}_{g-1} \otimes X^n_x) = j_{n\ (g+x-1)}(Y^n_{m+l-1})$$

with $\lim_{n\to\infty} \|Y^n_{m+l-1}\| = 0$. Hence $X \in \mathcal{D}om(G)$ and $G.X.$ is in $\hat{\mathcal{Y}}$. This proves (1).

For (2), the form of $(G.X.)_\infty$ follows by taking the limit $n \to \infty$ in the above equation, and the norm bound follows by taking the limit of norms.

■

Proposition 3.3. *Let G be a bounded polynomial generator. Then $T_{t,\cdot}$ is approximate symmetry preserving.*

Proof: Let g be the degree of $G.$ and let $X \in \hat{\mathcal{Y}}$ be of degree x. Iterating the integral equation for $T_{t,n}$, we write

$$T_{t,n} X_n = X^{(m)}_{t,n} + R^{(m)}_{t,n} \quad ,$$

where

$$X^{(m)}_{t,n} = \sum_{p=0}^{m-1} \frac{t^p}{p!} (G_n)^p X_n \quad ,$$

and

$$R^{(m)}_{t,n} = \int_0^t ds_m \dots \int_0^{s_2} ds_1 T_{s_1,n} (G_n)^m X_n \quad .$$

By Prop. 3.2 $(G.^m)X. \in \hat{\mathcal{Y}}$ for all $m \in \mathbb{N}$. $T_{t,n}$ is a contraction. Thus

$$\begin{aligned} \lim_{n\to\infty} \|T_{t,n} X_n - X^{(m)}_{t,n}\| &\leq \frac{t^m}{m!} \lim_{n\to\infty} \|(G_n)^m X_n\| \\ &\leq \frac{t^m}{m!} \|G_g\|^m \|\hat{X}_x\| \prod_{p=0}^{m-1} (x + p(g-1)) \quad . \end{aligned}$$

Now for $a, b, m \in \mathbb{N}$

$$\frac{1}{m!}\prod_{p=0}^{m-1}(a+pb) \leq \frac{1}{m!}\prod_{p=0}^{m-1}(a+(m-1)b-p)$$
$$= \binom{a+(m-1)b}{m} \leq 2^{a+(m-1)b} \quad .$$

Thus

$$\lim_{n\to\infty}\|T_{t,n}X_n - X_{t,n}^{(m)}\| \leq 2^{x+g-1}(2^{g-1}t\|G_g\|)^m\|\hat{X}_x\| \quad . \tag{3.2}$$

For $t < \tau = (2^{g-1}\|G_g\|)^{-1}$, we can take the limit $m \to \infty$ and conclude that $T_{t,n}X_n$ can be approximated uniformly for large n by approximately symmetric sequences and so $(T_{t,n}X_n) \in \tilde{\mathcal{Y}}$. Note that τ is independent of X.

We extend to the whole of $\tilde{\mathcal{Y}}$ by continuity, and finally for all $t \in \mathbb{R}^+$ by joining together the solutions on successive intervals of length less that τ.

■

In fact, not only is $T_{t,\cdot}$ approximate symmetry preserving, but it also has a mean-field limit which is implemented by a flow on $K(\mathcal{A})$.

Proposition 3.4. *Let $G_\cdot$ be a bounded polynomial generator of degree g. Then:*

(1) G_∞ is well-defined and is the generator of a contraction semigroup $T_{t,\infty}$ on $\mathcal{C}(K(\mathcal{A}))$ which is the mean-field limit of $T_{t,\cdot}$.

(2) The dense subset in $\mathcal{C}(K(\mathcal{A}))$ of polynomial functions $\mathcal{P} = \mathcal{Y}_\infty$ is a core for G_∞.

*(3) G_∞ is a *-derivation on $\mathcal{P}$.*

*(4) $T_{t,\infty}$ is a strongly continuous semigroup of *-homomorphisms on $\mathcal{C}(K(\mathcal{A}))$, and is thus implemented by weak*-continuous flow F_t on $K(\mathcal{A})$. F_t satisfies the differential equation*

$$(d/dt)\langle F_t\rho, A\rangle|_{t=0} = (j_{\infty g}G_g j_{g1}A)(\rho) \quad .$$

Proof:

(1) Since $\mathcal{Y} \subset \mathcal{D}om(G)$ is dense in $\tilde{\mathcal{Y}}$, this follows from Prop. 3.3 above, and the implication (2) ⇒ (5) of Theorem 2.3.

(2) By taking the limits n and then $m \to \infty$ in equation (3.2) we conclude from the power series approximation that all polynomials $p \in \mathcal{P}$ are analytic for G_∞ in the sense that $T_{t,\infty}p$ can be expressed as the convergent power series $\sum_{r=0}^{\infty}(r!)^{-1}(tG_\infty)^r p$. Each term in this sum is itself a polynomial, so the partial sums of the series are polynomials approximating $T_{t,\infty}p$. Replacing p with the polynomial $G_\infty p$ in the series, we we find that

$$G_\infty T_{t,\infty} = \lim_{m\to\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}G_\infty^{r+1}p = \lim_{m\to\infty} G_\infty \sum_{r=0}^{\infty}\frac{t^r}{r!}G_\infty^r p \quad .$$

So the action of G_∞ on the partial sums which approximate $T_{t,\infty}p$ yields a sequence which approximates $G_\infty T_{t,\infty}p$.

The set $\Omega = \bigcup_{t\geq 0} T_{t,\infty}\mathcal{P}$ is a dense $T_{t,\infty}$-invariant subset of $\mathrm{Dom}(G_\infty)$, and is hence a core for G_∞. The above argument shows that for every $q \in \Omega$ there is a sequence of polynomials p^α for which $\lim_\alpha p^\alpha = q$ and $\lim_\alpha G_\infty p^\alpha = G_\infty q$. Hence $\mathcal{P}$ is itself a core for G_∞.

(3) By Proposition 3.2(2), $G_\infty j_{\infty x}X_x = x j_{\infty(g+x-1)}(G_g \otimes \mathrm{id}_{x-1})(\mathbb{1}_{g-1} \otimes X_x)$. Thus

$$G_\infty(j_{\infty x}X_x j_{\infty y}Y_y) = G_\infty j_{\infty\, x+y}(X_x \otimes Y_y)$$
$$= (G_\infty j_{\infty\, x}X_x)j_{\infty\, y}Y_y + j_{\infty\, x}X_x(G_\infty j_{\infty\, y}Y_y) \quad .$$

The last equality follows from the fact that the action of G_∞ on $\mathcal{P}$ is given only by terms with overlap equal to 1. Thus there are no "mixed" terms which involve G acting on both X and Y. Hence G_∞ is a derivation on $\mathcal{P}$. That it is a *-derivation on $\mathcal{P}$ follows from the fact that the G_n are *-preserving.

(4) Since G_∞ acts as a *-derivation on $\mathcal{P}$ (which is a core), it is a *-derivation on its whole domain. Thus by Proposition 2.4 $T_{t,\infty}$ is a *-homomorphism, and is implemented by some weak*-continuous flow $(F_t)_{t\geq 0}$ on $\mathcal{C}(K(\mathcal{A}))$. To obtain the generator of the flow we note that

$$\frac{d}{dt}\langle F_t\rho, A\rangle\Big|_{t=0} = \frac{d}{dt}(T_{t,\infty}j_{\infty 1}A)(\rho)\Big|_{t=0} = (G_\infty j_{\infty 1}A)(\rho) = (j_{\infty g}G_g j_{g1}A)(\rho).$$

■

Proposition 3.3 has a generalization to the following case. Let Z be the (unbounded) generator of S_t, a strongly continuous semigroup of completely positive unital maps on $\mathcal{A}$. For all $n \in \mathbb{R}$ define the semigroup $S_{t,n}$ to be the completely positive extension of the n-fold tensor product $S_t \otimes \ldots S_t$ to $\mathcal{A}^n$ as given in Prop. 4.23

of [Tak]. Since $S_{t,n}$ is completely positive and unital, it is a contraction. Clearly for X in the algebraic tensor product $\mathcal{A}^{\odot n} \subset \mathcal{A}^n$ we have that $t \mapsto S_{t,n}X$ is norm continuous. Approximating any member of $\mathcal{A}^n$ by a sequence in $\mathcal{A}^{\odot n}$ we see that $t \mapsto S_{t,n}$ is strongly continuous. We denote its generator by Z_n. Finally, observe that each $S_{t,n}$ is degree preserving i.e. $S_{t,n} j_{nm} = j_{nm} S_{t,m}$.

If $G.$ is a bounded polynomial generator, then for each $n \in \mathbb{N}$ we can use the standard theory of bounded perturbations (see e.g [Dav]) to form the contraction semigroup $T_{t,n}$ whose generator is $Z_n + G_n$ with domain $\mathrm{Dom}(Z_n)$. From the Trotter product formula we see that $T_{t,n}$ is completely positive. In the following proposition we essentially repeat the estimates in the proof of Proposition 3.3.

Proposition 3.5. *The sequence of semigroups $T_{t,n}$ as defined above has a mean-field limit.*

Proof: We shall show that condition (2) of Theorem 2.3 is satisfied. First we show for each $t \geq 0$ that $T_{\cdot,\infty}$ is approximate symmetry preserving. For $X \in \mathcal{Y}$ we use the standard perturbation expansion on $T_{t,n}$ and write

$$T_{t,n}X_n = X_{t,n}^{(m)} + R_{t,n}^{(m)} \quad ,$$

where for $m \geq 1$,

$$X_{t,n}^{(m)} = S_{t,n}X_n + \sum_{p=1}^{m-1} \int_0^t dt_p \dots \int_0^{t_2} dt_1 S_{t-t_p,n} G_n S_{t_p-t_{p-1},n} \dots G_n S_{t_1,n} X_n \quad ,$$

and

$$R_{t,n}^{(m)} = \int_0^t dt_m \dots \int_0^{t_2} dt_1 T_{t-t_p,n} G_n S_{t_p-t_{p-1},n} \dots G_n S_{t_1,n} X_n \quad .$$

Since the $S_{t,n}$ preserve degree, and since the $T_{t,n}$ and $S_{t,n}$ are contractions, we can repeat the estimates of Prop. 3.3 to conclude that for $t < \tau$, and hence for all $t \in \mathbb{R}$, the sequence $(T_{t,n})_{n\in\mathbb{N}}$ is approximate symmetry preserving.

To complete the proof we show that $Z + G : X. \mapsto Z.X. + G.X.$ has dense domain in $\tilde{\mathcal{Y}}$. Since S_t is unit preserving, $\mathbf{1} \in \mathrm{Dom}(Z_1)$ so we can define $\mathcal{D} = \{X \in \mathcal{Y} \mid X_n \in (\mathrm{Dom}(Z_1))^{\odot n}\}$ Since $S_{t,n}$ is a product we see that for X of degree x in $\mathcal{D}$, $Z_n X_n = j_{nm} Z_m X_m$ for $n \geq m \geq x$, and hence that $\mathcal{D} \subset \mathcal{D}om(Z)$. Since $\mathcal{D}$ contains only strictly symmetric sequences it is also in $\mathcal{D}om(G)$. Lastly $\mathcal{D}$ is dense in $\mathcal{Y}$: since $\mathrm{Dom}(Z_n)$ is dense in $\mathcal{A}^n$ for all $n \in \mathbb{N}$, then for any $x \in \mathbb{N}$ the subset of $\mathcal{D}$ comprising strictly symmetric sequences of degree x is dense in the set of degree x sequences in $\mathcal{Y}$.

■

4. Approximately Polynomial Generators.

In this section we obtain an extension of the results of the previous section to a class of *approximately* polynomial generators. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of generators of norm-continuous semigroups of completely positive contractions on $(\mathcal{A}^n)_{n\in\mathbb{N}}$. As was stated in section 2, even if $G.$ is approximate symmetry preserving on a dense set in $\tilde{\mathcal{Y}}$, it is not clear that G_∞ is even well defined, let alone a generator. For the class of generators studied in this section we shall tackle this problem in two steps. First, we associate with every sequence $G.$ of generators and each $m \in \mathbb{N}$ a polynomial generator sequence $(G_n^m)_{n\geq m}$ by $G_n^m = (n/m)\,\mathrm{Sym}_n\, G_m$. From the results of section 3 we see that each of these polynomial generators $G.^m$ gives rise to a mean-field limit which is implemented by a flow on $\mathcal{C}(K(\mathcal{A}))$. We denote this flow by $F_{m,t}$. We define the sequence $G.$ to be approximately polynomial, if the limiting generators G_∞^m converge to a generator Z in a sense to be specified, from which we deduce the convergence of the flows $F_{m,t}$ to a limiting flow F_t. The natural question to ask in this case is whether G_∞ is well-defined and equal to Z. So the second step is to show that $T_{t,\infty} = \exp(tZ)$. But for this, more properties of the sequence G_n are needed. By Proposition 2.4 G_∞ would have to be a derivation in order to generate a dynamics given by a flow. At this point we can again take up the analogy with the standard semigroup theory: we show in Theorem 4.9 that if $G.$ acts asymptotically as a derivation on strictly symmetric sequences, then the mean-field limit of $T_{t,n}$ exists and is generated by Z. Of course, if each G_n is a derivation this is (modulo domain questions) always the case. It is an interesting question to characterize all flows which can be obtained in the mean-field limit of quantum dynamical semigroups. For this task the derivation condition in the second step can be ignored: we show in Proposition 4.10 that each approximately polynomial sequence of generators can be modified without changing the limiting flow to a sequence for which the mean-field limit exists. In Theorem 4.11 we describe explicitly a class of generator sequences which generate mean-field semigroups by virtue of Theorem 4.9, and give Example 4.12 as an illustration of its utility in quantum lattice systems.

In the first part of the section we state some general results on the convergence of flows, their generators and their Jacobians; the proofs will be deferred until section 5. We first fix the relevant notion of a derivative on $\mathcal{C}(K(\mathcal{A}))$ in our present context.

Definition 4.1. *We will say that $f \in \mathcal{C}(K(\mathcal{A}))$ is* **differentiable** *if*

(1) for all $\rho \in K(\mathcal{A})$ there exists an element $\mathrm{d}f(\rho)$ of $\mathcal{A}$ such that for all $\sigma \in K(\mathcal{A})$ the derivative

$$\langle \sigma - \rho, \mathrm{d}f(\rho) \rangle = \lim_{t \searrow 0} \frac{1}{t} \big(f((1-t)\rho + t\sigma) - f(\rho) \big)$$

exists as a weak-continuous affine functional of σ.*

(2) *The maps $\rho \mapsto \langle \sigma - \rho, \mathrm{d}f(\rho)\rangle$ are weak*-continuous, uniformly for $\sigma \in K(\mathcal{A})$.*

Our definition of differentiability is more stringent than usual: the derivative $df(\rho)$ is required to be in $\mathcal{A}$, not just in $\mathcal{A}^{**}$. Note that $\mathrm{d}f(\rho)$ is defined only up to a multiple of the identity. To fix it as a unique element of $\mathcal{A}$ we must adopt a convention. We will do this by specifying that $\langle \rho, \mathrm{d}f(\rho)\rangle = 0$, so that we can always write the derivative in the form $\langle \sigma, \mathrm{d}f(\rho)\rangle$. With this convention, item (2) above can be restated as saying that the map $\rho \mapsto \mathrm{d}f(\rho)$ is weak*-to-norm continuous.

As an example, let $X_x \in \mathcal{A}^x$ and let us calculate the derivative of $j_{\infty x}X_x$ with our convention.

$$\begin{aligned}\langle \sigma - \rho, \mathrm{d}j_{\infty x}X_x\rangle &= x\langle(\sigma - \rho)\otimes \rho^{x-1}, X_x\rangle \\ &= x\big\langle \sigma \otimes \rho^{x-1}, X_x - \langle \rho^x, X_x\rangle \mathbf{1}_x\big\rangle \quad .\end{aligned}$$

As we noted as the beginning of section 3, this is a weak*-continuous function of σ and ρ. From the above expression we obtain the bound

$$\|(\mathrm{d}j_{\infty x}X_x)(\rho)\| \le 2x\|X_x\| \quad .$$

We proceed with a definition of a class of generators of flows on $\mathcal{C}(K(\mathcal{A}))$. The motivation for this is that the generators of flows obtained from bounded polynomial generators fall into this class.

Definition 4.2. *Let $L : K(\mathcal{A}) \to \mathcal{A}^*$ be the generator of a weak*-continuous flow Q_t on $K(\mathcal{A})$, i.e for all $A \in \mathcal{A}$, $\langle L(\rho), A\rangle \equiv (d/dt)\langle Q_t\rho, A\rangle|_{t=0}$. L is said to be* **regular** *if it satisfies the following conditions:*

(1) *$\rho \mapsto \langle L(\rho), A\rangle$ is differentiable in the sense that*

$$\langle \sigma - \rho, \mathrm{d}L(\rho)A\rangle \equiv \frac{d}{d\lambda}\langle L((1-\lambda)\rho + \lambda\sigma), A\rangle|_{\lambda=0}$$

exists for all ρ, σ, A, for some $\mathrm{d}L(\rho) \in \mathcal{B}(\mathcal{A})$.

(2) *$\rho \mapsto \mathrm{d}L(\rho)A$ is weak*-to-norm continuous.*

(3) *$\|\mathrm{d}L(\rho)A\| \le \delta\|A\|$ with $\delta > 0$ independent of A and ρ.*

As in Definition 4.1 a convention must be adopted to fix $\mathrm{d}L(\rho)$ as an element of $\mathcal{B}(\mathcal{A})$. But 4.2(1) can be rewritten in the form

$$\langle \sigma - \rho, \mathrm{d}L(\rho)A\rangle = \big\langle \sigma - \rho, \mathrm{d}\langle L(\rho), A\rangle\big\rangle \quad .$$

Thus our the convention $\langle \rho, \mathrm{d}f(\rho)\rangle = 0$ already fixes the convention $\langle \rho, \mathrm{d}L(\rho)A\rangle = 0$ for all $A \in \mathcal{A}$. The terminology "regular" is rather bland, but is at least brief. Condition (2) states that $\rho \mapsto L(\rho)$ is continuously differentiable if the final topology is the strong operator topology, so that the corresponding flows are twice differentiable in this sense.

In the next proposition it is determined how the derivatives of functions transform under composition with the flows of regular generators. The proof is deferred until section 5.

Proposition 4.3. *Let L be a regular generator with associated flow Q_t. Then*

(1) *the differential equation*

$$\dot{J}_t(\rho) = J_t(\rho)\mathrm{d}L(Q_t\rho) \qquad \textit{with} \quad J_0(\rho) = \mathrm{id} \tag{4.1}$$

for the Jacobian $J_t(\rho) : \mathcal{A} \to \mathcal{A}$ has a unique solution (the differential equation being understood in the strong operator topology on $\mathcal{B}(\mathcal{A})$). Moreover

$$\mathrm{d}(f \circ Q_t)(\rho) = J_t(\rho)\mathrm{d}f(Q_t\rho) \tag{4.2}$$

and

$$\|J_t(\rho)\| \le e^{\delta t} \tag{4.3}$$

In particular, the set $\mathcal{C}^1(K(\mathcal{A}))$ of differentiable functions is invariant under the flow.

(2) *Let Z be the generator of the strongly continuous semigroup on $\mathcal{C}(K(\mathcal{A}))$ which is (according to Proposition 2.4) implemented by Q_t. Then any $f \in \mathcal{C}^1(K(\mathcal{A}))$ is also in $\mathrm{Dom}(Z)$ with $(Zf)(\rho) = \langle L(\rho), \mathrm{d}f(\rho)\rangle$, and $\mathcal{C}^1(K(\mathcal{A}))$ is a core for Z.*

Lemma 4.4. *The flows which implement the *-homomorphism semigroups with bounded polynomial generators have regular generators.*

Proof: Let $G.$ be a bounded polynomial generator of degree g. $\langle L(\rho), A\rangle$ is given in Proposition 3.4(4) to be equal to $(j_{\infty g}G_g j_{g1}A)(\rho) = \rho^g(G_g j_{g1}A)$. The derivative $\mathrm{d}L(\rho)$ is calculated as

$$\begin{aligned}\langle \sigma - \rho, \mathrm{d}L(\rho)A\rangle &= g\langle(\sigma - \rho)\otimes \rho^{g-1}, G_g j_{g1}A\rangle \\ &= g\big\langle \sigma \otimes \rho^{g-1}, G_g j_{g1}A - \langle \rho^g, G_g j_{g1}A\rangle \mathbf{1}_g\big\rangle \quad .\end{aligned}$$

Since for all n and $X_n \in \mathcal{A}^n$ the function from $(K(\mathcal{A}))^{\times n}$ to $\mathbb{R}^+$ given by $(\sigma_1, \dots \sigma_n) \mapsto (\sigma_1 \otimes \dots \sigma_n)(X_n)$ is weak*-continuous, we have that $dL(\rho)A \in \mathcal{A}$ for every ρ. Likewise, $\rho \mapsto dL(\rho)A$ is weak*-to-norm continuous for every $A \in \mathcal{A}$. This establishes properties (1) and (2) of Definition 4.2. The norm estimate $\|dL(\rho)A\| \leq 2g\|G_g\|\|A\|$ yields property (3).

■

We now come consider convergence of sequences of regular generators of flows. In Definition 4.5 which follows we will specify a notion of convergence, and in Proposition 4.6 (whose proof is deferred to section 5) we will find its consequences. We will then explain the relevance to the mean-field dynamics.

Definition 4.5. *Let L_n be a sequence of regular generators. The sequence is said to be* **regularly Cauchy** *if*

(1) $\|dL_n(\rho)\| \leq \delta$, uniformly in n and ρ; and

(2) For each $A \in \mathcal{A}$ and all $\varepsilon > 0$ there is an $n(\varepsilon)$ such that for all ρ and $n,m \geq n(\varepsilon)$,

$$\|dL_n(\rho)A - dL_m(\rho)A\| \leq \varepsilon \quad .$$

(3) For each $A \in \mathcal{A}$ and all $\varepsilon > 0$ there is an $n(\varepsilon)$ such that for all ρ and $n,m \geq n(\varepsilon)$,

$$\|\langle L_n(\rho), A\rangle - \langle L_m(\rho), A\rangle\| \leq \varepsilon \quad .$$

(4) There exists an $\omega \in K(\mathcal{A})$ and $\hat{\delta} > 0$ such that for all $n \in \mathbb{N}$, $\|L_n(\omega)\| \leq \hat{\delta}$.

Proposition 4.6. *Let a sequence of regular generators L_m, with associated flows $Q_{m,t}$, be regularly Cauchy. Then*

(1) $L(\rho) \equiv \text{weak*} - \lim_{m\to\infty} L_m(\rho)$ exists for all $\rho \in K(\mathcal{A})$ and is regular.

(2) A **limiting flow** *Q_t given by $Q_t\rho \equiv \text{weak*} - \lim_{m\to\infty} Q_{m,t}\rho$ exists uniformly for $\rho \in K(\mathcal{A})$ and t in compact intervals of $\mathbb{R}^+$. Furthermore, the limit defines a weak*-continuous flow on $K(\mathcal{A})$ which satisfies the differential equation*

$$\frac{d}{dt}\langle Q_t\rho, A\rangle = \langle L(Q_t\rho), \mathcal{A}\rangle \tag{4.4}$$

for every $A \in \mathcal{A}$.

(3) The Jacobians $J_{m,t}(\rho)$ for the flows $Q_{m,t}$ converge in the strong operator topology to the Jacobian $J_t(\rho)$ of the flow Q_t, the convergence being uniform for $\rho \in K(\mathcal{A})$ and t in compact intervals.

(4) Let Z^n (respectively Z) be the generator of the strongly continuous homomorphism semigroup on $\mathcal{C}(K(\mathcal{A}))$ which is implemented by $Q_{n,t}$ (resp. Q_t). Then $\lim_{n\to\infty} \|Z^n f - Zf\| = 0$ for all $f \in \mathcal{C}^1(K(\mathcal{A}))$.

We now return to the mean-field limits. At the start of this section we defined the operators $G^m_n = (n/m)\,\mathrm{Sym}_n\, G_m$ for all $n \geq m$ For each m the sequence $(G^m_n)_{n\geq m}$ is a bounded polynomial generator, and so by Theorem 3.4 the corresponding family contraction semigroups $(T^m_{t,n})_{n\in\mathbb{N}}$ is approximate symmetry preserving with a limit $T^m_{t,\infty} = \exp(tG^m_\infty)$. We denote by $F_{m,t}$ the implementing flow: $T^m_{t,\infty} f = f \circ F_{m,t}$, and L_m its generator. Lemma 4.4 establishes each L_m is a regular generator. If the L_m are regularly Cauchy we will say that G is **approximately polynomial**. If this the the case, we will call the $F_{m,t}$ the **approximating flows**, which according to Proposition 4.6 have a well-defined **limiting flow** F_t which implements some strongly continuous semigroup of *-homomorphisms $(e^{tZ})_{t\geq 0}$ on $\mathcal{C}(K(\mathcal{A}))$. Z will be called the **limiting generator** on $\mathcal{C}(K(\mathcal{A}))$ of G. If we can show that $(G.X.)_\infty$ is equal to ZX_∞ for all X with X_∞ in a core for Z, then one can use the general machinery of Theorem 2.3 to infer that G_∞ well-defined, equal to Z, and generating. We shall see that when $\mathcal{A}$ is finite dimensional this is relatively easy.

We make some definitions. We define $\mathcal{L} \subset \mathcal{Y}$ to be the set of strictly symmetric sequences of degree 1, i.e. $\mathcal{L} = \{X. \in \mathcal{Y} \mid X_n = j_{n1}A, \text{ some } A \in \mathcal{A}\}$. Let $\mathcal{P}^{\odot}$ be the algebra in $\mathcal{C}(K(\mathcal{A}))$ which is generated by $\mathcal{L}_\infty$. Let $\mathcal{Y}^{\odot}$ be the algebra in $\tilde{\mathcal{Y}}$ generated from $\mathcal{L}$ by finitely many n-wise additions $(X., Y.) \mapsto X. + Y.$ and multiplications $(X., Y.) \mapsto X.Y.$ Since $(X.Y.)_\infty = X_\infty Y_\infty$ for $X, Y \in \tilde{\mathcal{Y}}$ it is clear that $\mathcal{Y}^{\odot}{}_\infty = \mathcal{P}^{\odot}$. Define the C^1-norm on $\mathcal{C}^1(K(\mathcal{A}))$ $\|\cdot\|_1$ by by

$$\|f\|_1 = \max\{\|f\|, \|df\|\} \quad ,$$

where $\|df\| = \sup_{\rho\in K(\mathcal{A})} \|df(\rho)\|$.

Proposition 4.7. *Let $\mathcal{A}$ be finite dimensional, let $(Q_t)_{t\geq 0}$ be a continuous flow on $K(\mathcal{A})$ with regular generator L, and let Z be the generator of the contraction semigroup of *-homomorphisms implemented by Q_t. Then $\mathcal{P}^{\odot}$ is a core for Z.*

Proof: First note that since $\mathcal{A}$ is finite dimensional, our Definition 4.1 of the derivative is identical with the standard notion of the derivative on a manifold. We have seen in Proposition 4.3(2) that $C^1(K(\mathcal{A}))$ is a core for Z. Since L is bounded, it is enough to show that $\mathcal{P}^{\odot}$ is dense in $C^1(K(\mathcal{A}))$ in C^1-norm. Now note that any f in $\mathcal{L}_\infty$ is of the form $f(\rho) = \langle \rho, A\rangle$ for some $A \in \mathcal{A}$. Thus $\mathcal{L}_\infty$ separates the points of $K(\mathcal{A})$. Furthermore $(\mathrm{d}j_{\infty 1}A)(\rho) = A - \langle\rho, A\rangle\mathbf{1}$ for all $\rho \in K(\mathcal{A})$, so that for all $\sigma \neq \rho$ (i.e. all non-zero vectors tangent to $K(\mathcal{A})$ at the point ρ) we can find an $f \in \mathcal{L}_\infty$ such that $\langle \sigma - \rho, \mathrm{d}f(\rho)\rangle \neq 0$. So, by Nachbin's Theorem stated in Theorem 1.2.1 of **[Lla]**, $\mathcal{P}^{\odot}$ is $\|\cdot\|_1$ dense in $C^1(K(\mathcal{A}))$, as required.

■

The extension of Nachbin's Theorem to infinite dimensions is non-trivial, and is known not to be valid in general **[Lla]**. We have been unable to determine whether or not it applies in the present case. However, we are able to furnish an independent proof that the conclusions of Proposition 4.7 remain true when $\mathcal{A}$ is infinite dimensional, whenever Q_t is the limiting flow of an approximately polynomial generator.

Proposition 4.8. *Let G. be approximately polynomial with limiting flow F_t. Let F_t have generator L, and let Z be the limiting generator for G. Then $\mathcal{P}^{\odot}$ is a core for Z.*

Proof: Suppose we can show for all bounded polynomial generators $\tilde{G}$ with implementing flow $\tilde{Q}_t$ that for all $p \in \mathcal{P}$, all $t \geq 0$ and all $\varepsilon \geq 0$ we can find a $p^{\varepsilon} \in \mathcal{P}$ such that

$$\|e^{t\tilde{G}_\infty}p - p^{\varepsilon}\|_1 = \|p \circ \tilde{Q}_t - p^{\varepsilon}\|_1 < \varepsilon \tag{4.5a}$$

$$\|\mathrm{d}e^{t\tilde{G}_\infty}p - \mathrm{d}p^{\varepsilon}\|_1 = \|\mathrm{d}(p \circ \tilde{Q}_t) - \mathrm{d}p^{\varepsilon}\|_1 < \varepsilon \quad . \tag{4.5b}$$

Now let $(F_{m,t})_{m\in\mathbb{N}}$ be the approximating flows for G. Then by Proposition 4.6(2 & 3) we have that for all $p \in \mathcal{P}$, all $t \geq 0$ and all $\varepsilon' > 0$, we can find an $m \in \mathbb{N}$ such that $\|p \circ F_t - p \circ F_{m,t}\| < \varepsilon'$ and $\|\mathrm{d}(p \circ F_t) - \mathrm{d}(p \circ F_{m,t})\| = \sup_{\rho\in K(\mathcal{A})} \|J_t(\rho)\mathrm{d}p(F_t\rho) - J_{t,m}(\rho)\mathrm{d}p(F_{m,t}\rho)\| < \varepsilon'$. Thus for all $\varepsilon, \varepsilon' > 0$ we can choose first $m \in \mathbb{N}$, then $p^{\varepsilon} \in \mathcal{P}$, such that

$$\|p \circ F_t - p^{\varepsilon}\| < \|p \circ F_t - p \circ F_{m,t}\| + \|p \circ F_{m,t} - p^{\varepsilon}\| < \varepsilon + \varepsilon'$$

and

$$\|\mathrm{d}(p \circ F_t) - \mathrm{d}p^{\varepsilon}\| < \|\mathrm{d}(p \circ F_t) - \mathrm{d}(p \circ F_{m,t})\| + \|\mathrm{d}(p \circ F_{m,t}) - \mathrm{d}p^{\varepsilon}\| < \varepsilon + \varepsilon' \quad .$$

Thus $\mathcal{P}$ is $\|\cdot\|_1$-dense in $\Omega = \bigcup_{t\geq 0}\{p \circ F_t \mid p \in \mathcal{P}\}$. Now since $\|\mathrm{d}j_{\infty x}X_x\| \leq 2x\|X_x\|$ and $\mathcal{A}^{\odot n}$ is (by definition) dense in $\mathcal{A}^n$, we see that $\mathcal{P}^{\odot}$ is also $\|\cdot\|_1$-dense in Ω. Being

a dense invariant subset of $\mathrm{Dom}(Z)$, Ω is a core for Z. Finally, since the limiting flow generator L is bounded, any sequence in $\mathcal{P}^{\odot}$ which converges to an element of Ω in the $\|\cdot\|_1$-norm also converges in the norm $\mathrm{Dom}(Z) \ni f \mapsto \|f\| + \|Zf\|$. Hence $\mathcal{P}^{\odot}$ is a core for Z.

It remains to be shown that the approximations in equation (4.5) are possible. Let $\tilde{G}$. be a bounded polynomial generator of degree g, and let $\tilde{T}_{t,\infty} = e^{t\tilde{G}_\infty}$. The existence of polynomials approximating the $\tilde{T}_{t,\infty}$ evolutes of polynomials has already been shown during the proof of Proposition 3.3. Letting $j_{\infty x}X_x = p \in \mathcal{P}$ and taking the limit $n \to \infty$ in equation (3.1) we can write

$$\tilde{T}_{t,\infty}p = \sum_{r=0}^{s-1} \frac{(t\tilde{G}_\infty)^r}{r!}p + \int_0^t dt_s \dots \int_0^{t_2} dt_1 \tilde{T}_{t_1,\infty}(\tilde{G}_\infty)^s p \quad .$$

Since, by Prop. 4.3 $\tilde{T}_{t,\infty}\mathcal{P} \subset C^1(K(\mathcal{A}))$, and $(\tilde{G}_\infty)^r p \in \mathcal{P} \subset C^1(K(\mathcal{A}))$, we can use the bound of equation (4.3) to write

$$\|\mathrm{d}\tilde{T}_{t,\infty}p - \mathrm{d}\sum_{r=0}^{s-1} \frac{(t\tilde{G}_\infty)^r}{r!}p\| \leq \frac{t^s e^{\delta t}}{s!}\|\mathrm{d}(\tilde{G}_\infty)^s p\| \quad . \tag{4.6}$$

Since the derivatives of polynomials are themselves polynomials, the proposition is proved if the limit of the RHS of equation (4.6) is zero.

Using the bound $\|\mathrm{d}j_{\infty y}Y_y\| \leq 2y\|Y_y\|$ and the bound in Prop 3.2(2) again, we see that the RHS of (4.6) is bounded by $((2t)^s e^{\delta t}/s!)\|\tilde{G}_g\|^s\|X_x\|\prod_{r=0}^{s}(x + rg)$. In a similar way to the proof of Prop. 3.3 one sees that this bound is less than $e^{\delta t}\|X_x\|\|\tilde{G}_g\|^s x 2^x 2^{s(g-1)} t^s$. This converges to zero as $s \to \infty$, provided that $t < \tau = (2^{m-1}\|\tilde{G}_g\|)^{-1}$. Note that τ is independent of x.

What we have just proved is equivalent to saying that for all $t < \tau$, for all $p = j_{\infty x}X_x \in \mathcal{P}$, there exists a approximately symmetric sequence $n \mapsto Y_{t,n}(X_x)$, depending on X_x, such that

$$\tilde{T}_{t,\infty}p = \lim_{n\to\infty} j_{\infty n}Y_{t,n}(X_x) \quad \text{and} \quad \mathrm{d}\tilde{T}_{t,\infty}p = \lim_{n\to\infty} \mathrm{d}j_{\infty n}Y_{t,n}(X_x) \quad .$$

We use this observation to obtain the stated result for all $t \in \mathbb{R}^+$

For any $t' \in \mathbb{R}^+$ choose $N \in \mathbb{N}$ and $t \in [0, \tau)$ such that $t' = Nt$. Let $p = j_{\infty x}X_x$ as before. Then by virtue of the continuity of $f \mapsto \tilde{T}_{t,\infty}f$

$$\lim_{n_N\to\infty} \cdots \lim_{n_1\to\infty} j_{\infty n_1}Y_{t,n_1}(Y_{t,n_2}(\dots Y_{t,n_N}(X_x)\dots))$$
$$= \lim_{n_N\to\infty} \cdots \lim_{n_2\to\infty} \tilde{T}_{t,\infty}j_{\infty n_2}(Y_{t,n_2}(\dots Y_{t,n_N}(X_x)\dots)) \quad ,$$

which by iteration is found to be

$$= (\tilde{T}_{t,\infty})^N p = \tilde{T}_{t',\infty} p \quad .$$

Similarly

$$\begin{aligned}
&\lim_{n_N\to\infty} \cdots \lim_{n_1\to\infty} \mathrm{d} j_{\infty n_1} Y_{t,n_1}(Y_{t,n_2}(\ldots Y_{t,n_N}(X_x)\ldots)) \\
&= \mathrm{d} \lim_{n_N\to\infty} \cdots \lim_{n_2\to\infty} \tilde{T}_{t,\infty} j_{\infty n_2}(Y_{t,n_2}(\ldots Y_{t,n_N}(X_x)\ldots)) \\
&= \mathrm{d}\tilde{T}_{t',\infty} p \quad .
\end{aligned}$$

where at each stage we have used the property that since $\|\mathrm{d}\tilde{T}_{t,\infty} f\| \le e^{\delta t}\|\mathrm{d}f\|$ for $f \in \mathcal{C}^1(K(\mathcal{A}))$,

$$\lim_{n\to\infty} \|\mathrm{d}\tilde{T}_{s,\infty} j_{\infty n} Y_n(Z_z) - \mathrm{d}\tilde{T}_{t+s,\infty} j_{\infty z} Z_z\| \le e^{\delta s} \lim_{n\to\infty} \|\mathrm{d} j_{\infty n} Y_n(Z_z) - \mathrm{d}\tilde{T}_{\infty,t} j_{\infty z} Z_z\| = 0$$

for all $z \in \mathbb{N}$ and $Z_z \in \mathcal{A}^z$.

■

We are now ready to prove the main theorem of this section, which establishes conditions under which approximately polynomial generators have mean-field limits which are implemented by flows.

Theorem 4.9. *Let $G.$ be approximately polynomial with limiting flow F_t. Let Z be the generator of the semigroup of *-homomorphisms implemented by F_t. Then amongst the following conditions we have the implications (1) ⇒ (2) ⇒ (3) ⇒ (4):*

*(1) $\mathcal{L} \subset \mathcal{D}om(G)$ and each G_n is a *-derivation.*

(2) $\mathcal{Y}^{\odot} \subset \mathcal{D}om(G)$ and $(G.X.Y.)_\infty = (G.X.)_\infty Y_\infty + X_\infty (G.Y.)_\infty$ for all $X, Y \in \mathcal{Y}^{\odot}$.

(3) $\mathcal{Y}^{\odot} \subset \mathcal{D}om(G)$ and $(G.X.)_\infty = Z X_\infty$ for all $X \in \mathcal{Y}^{\odot}$.

(4) G_∞ is well defined and equal to Z; $T_{t.}$ has a mean-field limit $T_{t,\infty} = e^{tG_\infty}$.

Proof: (1) ⇒ (2): For $X, Y \in \mathcal{L}$ we have

$$G_n(X_n Y_n) = (G_n X_n) Y_n + X_n (G_n Y_n) \tag{4.7}$$

and so $(X.Y.) \in \mathcal{D}om(G)$. Similarly, finite n-wise sums and products of sequences from $\mathcal{L}$ are in $\mathcal{D}om(G)$. Hence $\mathcal{Y}^{\odot} \subset \mathcal{D}om(G)$ and equation (4.7) holds for all $X, Y \in \mathcal{Y}^{\odot}$. Operating with $j_{\infty n}$ and taking the limit $n \to \infty$ we conclude statement (2).

(2) ⇒ (3): $\mathcal{L} \subset \mathcal{Y}^{\odot} \subset \mathcal{D}om(G)$. Let $\mathcal{L} \ni X$ with $X_n = j_{n1} A$ some $A \in \mathcal{A}$. Since $c_n(m, 1; 1) = (m/n)$, we conclude from equation (3.1) that $G_n^m j_{n1} A = j_{nm} G_m j_{m1} A$. Hence

$$\begin{aligned}
Z X_\infty &= \lim_{m\to\infty} G_\infty^m X_\infty = \lim_{m\to\infty} \lim_{n\to\infty} j_{\infty n} G_n^m j_{n1} A \\
&= \lim_{m\to\infty} j_{\infty m} G_m j_{m1} A = (G.X.)_\infty \quad .
\end{aligned}$$

For any $x \in \mathbb{N}$ pick an arbitrary collection $\{X^{(y)} : y = 1, \ldots x\}$ in $\mathcal{L}$. Then by assumption

$$\begin{aligned}
\left(G.(X.^{(1)} \ldots X.^{(x)})\right)_\infty &= \sum_{y=1}^{x} X_\infty^{(1)} \ldots X_\infty^{(y-1)} (G.X.^{(y)})_\infty X_\infty^{(y+1)} \ldots X^{(x)}{}_\infty \\
&= \sum_{y=1}^{x} X_\infty^{(1)} \ldots X_\infty^{(y-1)} (Z X_\infty^{(y)})_\infty X_\infty^{(y+1)} \ldots X^{(x)}{}_\infty \\
&= Z(X_\infty^{(1)} \ldots X_\infty^{(x)}) \quad .
\end{aligned}$$

We remind the reader that this equality does not require G_n to be a derivation for each n.

(3) ⇒ (4): Let $s \in \mathbb{C}$. For $X \in \mathcal{Y}^{\odot}$ we have $((s - G.)X.)_\infty = (s - Z)X_\infty$. So $\left((s - G.)\mathcal{Y}^{\odot}\right)_\infty = (s - Z)\mathcal{Y}^{\odot}{}_\infty = (s - Z)\mathcal{P}^{\odot}$. If $\Re(s) > 0$ then s is in the resolvent set. Furthermore, since by Proposition 4.8 $\mathcal{P}^{\odot}$ is a core for Z, $(s - Z)\mathcal{P}^{\odot}$ is dense in $\mathcal{C}(K(\mathcal{A}))$. Thus $(s - G.)\mathcal{Y}^{\odot}$ is dense in $\tilde{\mathcal{Y}}$: condition (4) in Theorem 2.3 is satisfied. From (5) of the same theorem we conclude that G_∞ is well defined, that $T_{t,.}$ is approximate symmetry preserving, and that $T_{t,\infty} = e^{tG_\infty}$. Since G_∞ and Z agree on a common core, they are equal.

■

For demonstrating that a given sequence $T_{t,n}$ has a mean-field limit via this theorem, we have to show that $G.$ is approximately polynomial, and also verify one of the conditions (1),(2), and (3). The first step implies a constraint on the set of flows on $K(\mathcal{A})$, which can be obtained in the mean-field limit of a sequence $T_{t,n}$ by this procedure: the generator of the flow has to be in the completion of the set of polynomial generators with respect to "regular" limits in the sense of Definition 4.5. We shall show now that condition (2) above adds no new constraint to this. In contrast, condition (1) does impose an additional constraint: for a finite dimensional matrix algebra $\mathcal{A}$ it implies that $G_n(A) = i[H_n, A]$ for a sequence of Hamiltonians H_n. It is easy to see from this that the flows $F_{m,t}$, and consequently F_t respect unitary equivalence **[DW]**, i.e. for all $t \in \mathbb{R}$ and $\rho \in K(\mathcal{A})$ there is a unitary $U_t^\rho \in \mathcal{A}$ such that $(F_t\rho)(A) = \rho(U_t^\rho A U_t^{\rho *})$. This is clearly not the case for general dynamical semigroups.

Proposition 4.10. *Let $(F_t)_{t\geq 0}$ be the limiting flow of an approximately polynomial generator $G.$. Then there is another approximately polynomial sequence of generators $\tilde{G}.$ such that $\tilde{T}_{t,n}$ has the mean-field limit $\tilde{T}_{t,\infty}f = f \circ F_t$.*

Proof: We shall set $\tilde{G}_n = (n/n')\,\mathrm{Sym}_n\, G_{n'}$, where $n \in \mathbb{N} \mapsto n' \in \mathbb{N}$ is a non-decreasing function with $n' \leq n$. This form ensures that apart from possible omissions and repetitions the sequence of generators $\tilde{G}^m_\infty$ is precisely the same as G^m_∞. Therefore, $\tilde{G}.$ is approximately polynomial with the same limiting flow. It remains to be shown that condition (2) of Theorem 4.9 holds if we make n' increase sufficiently slowly. We shall do this by showing that the terms with overlap $r = 1$ dominate the expression in equation (3.1) for $\tilde{G}_n X_n$. With $g = n'$ in that equation we find for the sum of all terms with $r \geq 2$ the norm bound

$$\frac{n}{n'}\|G_{n'}\|\|X_x\| \sum_{r=2}^{\min(g,x)} c_n(g,x;r) \quad .$$

Since $\sum_r c_n(n',x;r) = 1$ this sum is equal to $1 - c_n(n_i,x;0) - c_n(n_i,x;1)$. Using the explicit expression for $c_n(x,y;r)$ we see that

$$c_n(x,y;1) = \frac{xy}{n} c_{n-1}(x-1,y-1;0) \quad .$$

In Lemma IV.1 of **[RW1]** it is shown that $c_n(x,y;0) \geq 1 - xy/n$ Thus the sum is bounded by

$$\sum_{r\geq 2} c_n(g,x;r) \leq 1 - \left(1 - \tfrac{xy}{n}\right) - \tfrac{xy}{n}\left(1 - \tfrac{(x-1)(y-1)}{n-1}\right) = \tfrac{x(x-1)y(y-1)}{n(n-1)} \quad .$$

Hence the terms of overlap ≥ 2 are collectively bounded by

$$x(x-1)\frac{n'-1}{n}\|G_{n'}\|\|X_x\| \quad .$$

Clearly, we can define n' such that expression goes to zero for all X.

We can now decompose the product of two strictly symmetric sequences as in equation (2.3). We can apply the above estimate to each of the strictly symmetric sequences $j_{n(x+y-r)}((X_x \otimes \mathbb{1}_{y-r})(\mathbb{1}_{x-r} \otimes Y_y))$ of degree less than $x+y$ appearing in the decomposition (2.3). Retaining only the $r = 0$ term of (2.3) means an error of at most $(1 - c_n(x,y;r))\|X_x\|\,\|Y_y\| \leq xy/n\|X_x\|\,\|Y_y\|$. Hence we may neglect for large n all terms in $G_n(X_n Y_n)$ for which either X and Y have a non-zero overlap, or G_n has total overlap ≥ 2 with X and Y. Neglecting the same terms on the right hand side, we obtain the derivation relation up to errors, which are small in norm.

■

We now demonstrate a class of approximately polynomial generators for which the existence of mean-field limits can be proved by use of Theorem 4.9. Let I be some countable index set. Suppose that to each $i \in I$ is associated an integer n_i, the algebra $\mathcal{A}^{n_i}$, and a sequence $(\Gamma^n_i)^{n\in\mathbb{N}}$ of bounded operators such that

(a) $\Gamma^n_i = 0$ for all $n < n_i$,

(b) $\Gamma_i = \lim_{n\to\infty} \Gamma^n_i$ exists in the strong operator topology.

(c) the bounds $\gamma_i \equiv \sup_{n\in\mathbb{N}} \|\Gamma^n_i\|$ are summable so that $\sum_{i\in I} n_i\gamma_i \equiv \gamma < \infty$

(d) For each $n \in \mathbb{N}$

$$G_n \equiv \sum_{i\in I} \frac{n}{n_i}\,\mathrm{Sym}_n(\Gamma^n_i) \tag{4.8}$$

is the generator of a norm-continuous semigroup of completely positive unital contractions on $\mathcal{A}^n$.

Thus for each n we construct G_n as a sum (over those $i \in I$ such that $n_i \leq n$) of symmetrized generators on $\mathcal{A}^{n_i}$. Of course, a sufficient condition for each G_n to generate is that each Γ^n_i generates on $\mathcal{A}^{n_i}$. Perhaps the simplest type of generators in this class are obtained when $I = \{1\}$, so that each of the $G_n = (n/g)\,\mathrm{Sym}_n\,\Gamma^n_g$ for some $g \in \mathbb{N}$ and some convergent sequence $n \mapsto \Gamma^n_g$ of generators on $\mathcal{A}^g$. Such generators can be thought of as lying in the generator analogue of the $\hat{\mathcal{Y}}$ class of sequences.

Theorem 4.11. *Let the family of sequences $(n \mapsto \Gamma^n_i)_{i\in I}$ satisfy conditions (a), (b), (c) and (d) above, and let $G.$ be defined as in equation (4.8). Then the generator sequence $G.$ is approximately polynomial, $G.$ acts a derivation on $\mathcal{Y}$, and hence $(e^{tG_n})_{n\in\mathbb{N}}$ has a mean-field limit which is implemented by a flow on $\mathcal{C}(K(\mathcal{A}))$.*

Proof: First we show that $\|L_n(\rho)\|$ and $\|dL_n(\rho)\|$ are uniformly bounded in $n \in \mathbb{N}$ and $\rho \in K(\mathcal{A})$. For any $A \in \mathcal{A}$, $\rho \in K(\mathcal{A})$

$$|\langle L_n(\rho), A\rangle| = |\langle \rho^n, G_n j_{n1} A\rangle| \leq \sum_{i\in I} |\langle \rho^{n_i}, \Gamma^n_i j_{n_i 1} A\rangle| \leq \sum_{i\in I} \|\Gamma^n_i\|\|A\| \leq \gamma\|A\| \quad .$$

Similarly, with $\sigma \in \mathcal{A}^*$,

$$|\langle \sigma, dL_n(\rho)A\rangle| \leq \sum_{i\in I} |\langle \sigma \otimes \rho^{n_i-1}, j_{n_i n_i}\Gamma^n_i j_{n_i 1} A\rangle| \leq \|\sigma\| \sum_{i\in I} n_i\|\Gamma^n_i\|\|A\| \leq \|\sigma\|\gamma\|A\| \quad .$$

This demonstrates properties (1) and (4) of Definition 4.5. To show that $\langle L_n(\rho), A\rangle$ is Cauchy, calculate

$$\begin{aligned}|\langle L_n(\rho) - L_m(\rho), A\rangle| &= |\sum_{i\in I}\langle \rho^{n_i}, \Gamma_i^n j_{n_i 1}A - \Gamma_i^m j_{n_i 1}A\rangle| \\ &\le \sum_{i\in I}\|(\Gamma_i^n - \Gamma_i^m) j_{n_i 1}A\| \quad ,\end{aligned}$$

where in the last step we have used the bound $\|\mathrm{d}j_{\infty y}Y_y\| \le 2y\|Y_y\|$ for any $Y_y \in \mathcal{A}^y$. Each term in the sum is bounded by the $2\|A\|$ multiplied by the appropriate γ_i, and the sum of these bounds is $2\|A\|\sum_{i\in I}\gamma_i \le \gamma$. By condition (b) each sequence $n \mapsto \Gamma_i^n j_{n_i 1}A$ is convergent. We conclude by dominated convergence that $\langle L_n(\rho), A\rangle$ is Cauchy, uniformly for $\rho \in K(\mathcal{A})$. Similarly, for $\sigma \in \mathcal{A}^*$

$$\begin{aligned}|\langle \sigma, (\mathrm{d}L_n(\rho) - \mathrm{d}L_m(\rho))A\rangle| &= |\sum_{i\in I}\langle \sigma, \mathrm{d}j_{\infty n_i}\Gamma_i^n j_{n_i 1}A - \mathrm{d}j_{\infty n_i}\Gamma_i^m j_{n_i 1}A\rangle| \\ &\le \|\sigma\|\sum_{i\in I} 2n_i\|(\Gamma_i^n - \Gamma_i^m) j_{n_i 1}A\| \quad .\end{aligned}$$

Each term in is bounded by the appropriate $4n_i\gamma_i\|A\|$, and the sum of these bounds is less than $4\gamma\|A\|$. By dominated convergence we conclude that $n \to \mathrm{d}L_n(\rho)$ is Cauchy, uniformly for $\rho \in K(\mathcal{A})$ and $A \in \mathcal{A}$. Hence conditions (2) and (3) of Definition 4.5 are satisfied: $G.$ is approximately polynomial.

We now show that $\mathcal{Y}^{\odot}$ is in $\mathcal{Dom}(G)$ and that $G.$ satisfies the derivation property $(G.X.Y.)_\infty = (G.X.)_\infty Y_\infty + X_\infty(G.Y.)_\infty$ for $X, Y \in \mathcal{Y}^{\odot}$. Since $\mathcal{Y}^{\odot} \subset \hat{\mathcal{Y}}$ it suffices as in the proof of Proposition 4.10 to show for each $X \in \hat{\mathcal{Y}}$ that $G.X. \in \tilde{\mathcal{Y}}$ and is given by terms of overlap 1. Then the result will follow by the implication (2) $\Rightarrow$ (4) of Theorem 4.9.

We let $X \in \hat{\mathcal{Y}}$ be of degree x with limiting element $\hat{X}_x$ i.e. $X_n = j_{nx}X_x^n$ with $\lim_{n\to\infty} X_x^n = \hat{X}_x$. Using the decomposition over overlaps from Proposition 3.2 we write

$$G_n X_n = R_n^1 + R_n^2 + R_n^3 \quad ,$$

where

$$R_n^1 = \sum_{i:n_i>n-x}\frac{n}{n_i}\mathrm{Sym}_n(\Gamma_i^n) j_{n_i x}X_x^n \quad ,$$

$$R_n^2 = \sum_{i\in I:n_i\le n-x}\frac{n}{n_i}\sum_{r=2}^{\min(g,x)} c_n(n_i,x;r) j_{n(n_i+x-r)}(\Gamma_i^n\otimes \mathrm{id}_{x-r})(\mathbb{1}_{n_i-r}\otimes X_x^n) \quad ,$$

and

$$R_n^3 = \sum_{i\in I:n_i\le n-x}\frac{n}{n_i}c_n(n_i,x;1) j_{n(n_i+x-1)}(\Gamma_i^n\otimes \mathrm{id}_{x-1})(\mathbb{1}_{n_i-1}\otimes X_x^n) \quad .$$

R_n^1 comprises terms generated from those $i \in I$ such that the minimum overlap of the Γ_i^n with X_x is greater than 1. For the remaining $i \in I$, R_n^2 comprises terms with overlap greater than 1, and R_n^3 comprises terms with overlap 1.

First we show that $\lim_{n\to\infty}\|R_n^1\| = 0$.

$$\begin{aligned}\limsup_{n\to\infty}\|R_n^1\| &\le \limsup_{n\to\infty}\sum_{i:n_i>n-x}\gamma_i\frac{n}{n-x}\|X_x^n\| \\ &= \limsup_{n\to\infty}\sum_{i:n_i>n-x}\gamma_i\|\hat{X}_x\| \qquad = 0 \quad .\end{aligned}$$

Since $\sum_{i\in I}\gamma_i$ exists, the last equality holds since the tail of the sum $\sum_{i:n_i>n-x}\gamma_i$ must vanish as $n \to \infty$.

Second, we show that $\lim_{n\to\infty}\|R_n^2\| = 0$. In

$$\|R_n^2\| \le \sum_{i\in I:n_i\le n-x}\frac{n}{n_i}\gamma_i\|X_x^n\|\sum_{r\ge 2}c_n(n_i,x;r) \quad ,$$

we can apply the estimate of the sum over r from the proof of Proposition 4.10 to obtain

$$\|R_n^2\| \le \sum_{i\in I;n_i\le n-x}\frac{(n_i-1)x(x-1)}{(n-1)}\gamma_i\|X_x^n\| \le \frac{x(x-1)}{(n-1)}\gamma\|X_x^n\| \quad ,$$

and so $\lim_{n\to\infty}\|R_n^2\| = 0$.

Finally we examine R_n^3. Note that

$$\frac{n}{n_i}c_n(n_i,x;1) = x\prod_{\alpha=0}^{x-2}\frac{n-n_i-\alpha}{n-1-\alpha} \le \lim_{n\to\infty}\frac{n}{n_i}c_n(n_i,x;1) = x \quad .$$

Since $n \mapsto \Gamma_i^n$ is convergent for each $i \in I$, the sequence $(Y_{i,n})$ defined by

$$Y_{i,n} = \frac{n}{n_i}c_n(n_i,x;1) j_{n(n_i+x-1)}(\Gamma_i^n\otimes \mathrm{id}_{x-1})(\mathbb{1}_{n_i-1}\otimes X_x^n)$$

if $n \ge n_i + x$ and $Y_{i,n} = 0$ otherwise, is approximately symmetric with norm bound $x\gamma_i\|X\|$. Since the γ_i are summable we can use an argument identical with that used to treat R_n^1 to show that by the sequence $n \mapsto R_n^3$ can be approximated uniformly in n by finite sums approximately symmetric sequences. Specifically, for every $\varepsilon > 0$ there exists an $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$\|R_n^3 - \sum_{i\in I:n_i\le m}Y_{i,n}\| \le \varepsilon \quad .$$

Thus $G.X. \in \tilde{\mathcal{Y}}$ and

$$(G.X.)_\infty(\rho) = (R.^3)_\infty(\rho) = \sum_{i \in I} \langle \rho^{n_i+x-1}, (\Gamma_i \otimes \mathrm{id}_{x-1})(\mathbb{1}_{n_i-1} \otimes \hat{X}_x) \rangle \quad .$$

■

We conclude with an example of a class of mean-field quantum lattice systems which satisfy conditions (a) to (d) above.

Example 4.12. Consider the lattice $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. For any $\alpha \subset \mathbb{Z}^d$ let $|\alpha|$ be the cardinality of α, and define $\mathcal{S} = \{\alpha \subset \mathbb{Z}^d \mid |\alpha| < \infty\}$ to be the set of subsets of $\mathbb{Z}^d$ of finite cardinality. To each $x \in \mathbb{Z}^d$ is associated a copy of some unital C^*-algebra $\mathcal{A}$. For any $\alpha \subset \mathcal{S}$ we will write $\mathcal{A}^\alpha$ for the minimal C^*-tensor product of the algebra associated with the elements of α. For any $x \in \mathbb{Z}^d$ and $\alpha \subset \mathbb{Z}^d$ we will write $\alpha + x$ as the translation of the set α by the lattice vector x. For each $x \in \mathbb{Z}^d$ we can define the translation operator τ_x consistently as a map on $\cup_{\alpha \in \mathcal{S}} \mathcal{A}^\alpha$, so that, for example, $\tau_x \mathcal{A}^\alpha = \mathcal{A}^{x+\alpha}$.

Our models are defined as follows. First, we have the interactions. We assume that a family of bounded operators $\{\Gamma_\alpha \mid \alpha \in \mathcal{S}\}$ is specified, each Γ_α being the generator of a norm-continuous semigroup of completely positive unital contractions on $\mathcal{A}^\alpha$. We assume that $\Gamma.$ is translation invariant in the sense that for any $x \in \mathbb{Z}^d$ and $\alpha \subset \mathbb{Z}^d$, $\tau_x \circ \Gamma_\alpha \circ \tau_{-x} = \Gamma_{\alpha+x}$. Furthermore we impose the summability condition

$$\sum_{\alpha \ni 0} |\alpha| \, \|\Gamma_\alpha\| < \infty \quad . \tag{4.9}$$

Second, we have a sequence of regions $(\Lambda_n)_{n \in \mathbb{N}}$ converging to infinity in the sense of van Hove (see e.g. **[Ru2]**). For any $\Lambda, \alpha \subset \mathbb{Z}^d$, let

$$N_\Lambda(\alpha) = \left|\{x \in \mathbb{Z}^d \mid \alpha + x \subset \Lambda\}\right|$$

be the number of translates of α which lie inside Λ. Then $N_\Lambda(\alpha) \leq \Lambda$ and van Hove convergence has the consequence that for each fixed $\alpha \in \mathcal{S}$

$$\lim_{n \to \infty} \frac{N_{\Lambda_n}(\alpha)}{|\Lambda_n|} = 1 \quad . \tag{4.10}$$

Given the family $\{\Gamma_\alpha \mid \alpha \in \mathcal{S}\}$ then for each $\Lambda \subset \mathcal{S}$ we define the mean-field generator

$$G_\Lambda = \mathrm{Sym}_\Lambda \sum_{\alpha \subset \Lambda} \Gamma_\alpha \quad , \tag{4.11}$$

where Sym_Λ denotes the symmetrization operator from $\mathcal{B}(\mathcal{A}^\alpha)$ to $\mathcal{B}(\mathcal{A}^\Lambda)$ when $|\alpha| \leq |\Lambda|$. If, for example, $\Gamma_\alpha(\cdot) = i[\Phi_\alpha, \cdot]$ for some potential $\Phi_\alpha \in \mathcal{A}^\alpha$, then G_Λ is just the generator one gets by commutation with $\sum_{\alpha \subset \Lambda} \mathrm{Sym}_\Lambda \Phi_\alpha$, the sum of symmetrized potentials over subsets of Λ. By a rearrangement we can write G_Λ in the following way

$$\begin{aligned} G_\Lambda &= \sum_{\alpha \ni 0} \frac{1}{|\alpha|} \sum_{\substack{x \in \mathbb{Z}^d: \\ \alpha + x \subset \Lambda}} \mathrm{Sym}_\Lambda \Gamma_{\alpha+x} \\ &= \sum_{\alpha \ni 0} \frac{N_\Lambda(\alpha)}{|\Lambda|} \frac{|\Lambda|}{|\alpha|} \mathrm{Sym}_\Lambda \Gamma_\alpha \end{aligned}$$

by translation invariance of Γ.

With this framework in place, the connection to Theorem 4.11 is straightforward. The only difference is that our sequences will take values in $(\mathcal{A}^{\Lambda_n})_{n \in \mathbb{N}}$ rather than $(\mathcal{A}^n)_{n \in \mathbb{N}}$. This presents no problem since the former is a subsequence of the latter. We identify I with the set of finite subsets of $\mathbb{Z}^d$ which contain the origin. $G_{\Lambda_n} = \sum_{\alpha \in I} (|\Lambda|/|\alpha|) \mathrm{Sym}_\Lambda \Gamma_\alpha^n$ where for each $\alpha \in \mathcal{S}$, $n \mapsto \Gamma_\alpha^n$ is the generator sequence specified by $\Gamma_\alpha^n = (N_{\Lambda_n}(\alpha)/|\Lambda_n|)\Gamma_\alpha$. We can verify that conditions (a) to (d) above are satisfied. Condition (a) is trivial, since $N_\Lambda(\alpha) = 0$ when $|\alpha| > |\Lambda|$. Equation (4.10) guarantees for each $\alpha \in I$ and $A \in \mathcal{A}$ that $\Gamma_\alpha^n j_{n;1} A$ converges as $n \to \infty$: condition (b) is satisfied. Since $N_\Lambda(\alpha)/\Lambda \leq 1$, equation (4.9) means that the condition (c) is satisfied. Finally, each Γ_α is assumed to generate, so condition (d) is satisfied. Thus we conclude from Theorem 4.11 that the sequence of semigroups $T_{t,n} = \exp(tG_{\Lambda_n})$ has a mean-field limit.

One can compare the stringency of equation (4.9) with conditions which have been used to establish the existence of a limiting dynamics for generators which are not of a mean-field type i.e. if the symmetrization operator in equation (4.11) is omitted. Let $\Gamma_\alpha(\cdot) = i[\Phi_\alpha, \cdot]$ for some family of self adjoint elements $\{\Phi_\alpha = \Phi_\alpha^* \mid \alpha \in \mathcal{S}\}$ and let $G_\Lambda = \sum_{\alpha \subset \Lambda} \Gamma_\alpha$. It has been shown **[Rob]** that whenever

$$\|\Phi\|_{(1)} \equiv \sum_{\alpha \in I} e^{|\alpha|} \|\Phi_\alpha\|$$

is finite, the family (e^{tG_Λ}) has a strong limit (as $\Lambda \to \infty$ in the sense of van Hove) as a strongly continuous group of *-automorphisms on the quasi-local algebra generated by $\mathcal{S}$. But this is a strictly stronger requirement than is necessary in our model: in the Hamiltonian case equation (4.9) is implied by

$$\sum_{\alpha \in I} |\alpha| \, \|\Phi_\alpha\| < \infty \quad ,$$

a far weaker condition.

Let ω be a translation invariant state on the quasi-local algebra, i.e. on the inductive limit of the algebras $\mathcal{A}^{\Lambda_n}$ as $\Lambda_n \nearrow \mathbb{Z}^d$. Then by **[KR,Ru1]** ω is a classical state in the sense of [HL]. In the terminology of this paper this means that for the restrictions ω_n of ω to the algebras $\mathcal{A}^{\Lambda_n}$, the limit $\langle \omega_n, X_n \rangle$ exists for any sequence $X \in \mathcal{Y}^{\odot}$. Since $\mathcal{Y}^{\odot}$ is dense in $\tilde{\mathcal{Y}}$, the limit therefore exists for all $X \in \tilde{\mathcal{Y}}$, and is of the form $\lim_n \langle \omega_n, X_n \rangle = \int \mu_\omega(d\varphi) X_\infty(\varphi)$, for a unique probability measure μ_ω on $K(\mathcal{A})$. Hence for all $t \geq 0$ the limit $\langle \omega_n, T_{t,n}(X_n) \rangle$ exists as well, and is equal to $\int \mu_\omega(d\varphi) X_\infty(F_t\varphi)$, where $(F_t)_{t\geq 0}$ is the flow which implements $T_{t,\infty}$.

5. Limits of Flows and Jacobians for Convergent Generators.

In this section the proofs of Propositions 4.3 and 4.6 are supplied. In order to prove the convergence of flows and Jacobians for a sequence regularly Cauchy generators L_m, we will find it convenient to characterize the weak*-topology on $K(\mathcal{A})$ and the strong operator topology on $\mathcal{B}(\mathcal{A})$ with a special family of seminorms. We construct this family to have the particular property that for each seminorm the functions $\rho \mapsto L_m(\rho)$ satisfy a Lipschitz condition, uniformly for $\rho \in K(\mathcal{A})$ and $m \in \mathbb{N}$.

Proposition 5.1.

(1) Let (L_n) be regularly Cauchy and $\Lambda_0 \subset \mathcal{A}$ compact. Then there is a compact set $\Lambda \supset \Lambda_0$ and a constant $\delta' > 0$ such that

$$A \in \Lambda,\ \rho \in K(\mathcal{A}),\ n \in \mathbb{N} \implies \mathrm{d}L_n(\rho)A \in \delta'\Lambda \quad . \tag{5.1}$$

(2) Let Υ denote the collection of compact subsets of $\mathcal{A}$ for which equation (5.1) holds. Then the family $\{D_\Lambda \mid \Lambda \in \Upsilon\}$ of seminorms on $\mathcal{A}^*$ given by

$$D_\Lambda(\rho) = \sup_{A\in\Lambda} |\langle \rho, A \rangle|$$

generates the weak-topology on $K(\mathcal{A})$, while the family $\{\Delta_\Lambda \mid \Lambda \in \Upsilon\}$ of seminorms on $\mathcal{B}(\mathcal{A})$ given by*

$$\Delta_\Lambda(X) = \sup_{A\in\Lambda} \|X(A)\|$$

generates the strong-operator topology on $\mathcal{B}(\mathcal{A})$.

Proof: (1) We first show that for any precompact Λ_0 the set

$$\tilde{\Lambda}_1 = \bigcup_{n\in\mathbb{N}} \bigcup_{A\in\Lambda_0} \bigcup_{\rho\in K(\mathcal{A})} \mathrm{d}L_n(\rho)A$$

is precompact. Since Λ_0 is precompact, then for all $\varepsilon_1 > 0$ we may find a finite set $\dot{\Lambda}_0$ that approximates it to within ε_1, i.e. for all $A \in \Lambda_0$ there exists a $\dot{A} \in \dot{\Lambda}_0$ such that $\|A - \dot{A}\| \leq \varepsilon_1$. Pick ε_2, and $n(\varepsilon_2)$ as the maximum over the finite set $\dot{\Lambda}_0$ of the $n(\varepsilon_2)$ specified in Definition 4.5(2). Since $\rho \mapsto dL_n(\rho)A$ is weak*-continuous, the set $\hat{\Lambda}_1 \equiv \bigcup_{\dot{A}\in\dot{\Lambda}_0} \bigcup_{n\leq n(\varepsilon_2)} \bigcup_{\rho\in K(\mathcal{A})} \mathrm{d}L_n(\rho)\dot{A}$ is compact. Let $\dot{\Lambda}_1$ be a finite set approximating $\hat{\Lambda}_1$ within norm-distance ε_3. Then for any n, ρ, A we may pick first $\dot{A} \in \dot{\Lambda}_0$, then $\dot{B} \in \dot{\Lambda}_1$ such that

$$\begin{aligned}\|\mathrm{d}L_n(\rho)A - \dot{B}\| \leq& \|\mathrm{d}L_n(\rho)(A - \dot{A})\| \\ &+ \|\mathrm{d}L_n(\rho)\dot{A} - \mathrm{d}L_{n(\varepsilon_2)}(\rho)\dot{A}\| + \|\mathrm{d}L_{n(\varepsilon_2)}(\rho)\dot{A} - \dot{B}\| \\ \leq& \delta\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad .\end{aligned}$$

Thus $\tilde{\Lambda}_1$ is precompact.

We can iterate this construction to yield a sequence of precompact sets $(\Lambda_n)_{n\in\mathbb{N}}$. Define $\Lambda \subset \mathcal{A}$ by

$$\Lambda = \{A = \sum_{i=0}^{\infty} (\delta')^{-i} A_i \mid A_i \in \Lambda_i\} \qquad \text{for some } \delta' > \delta \ .$$

Such sums are norm-convergent, since

$$\begin{aligned}\sup\{\|A\| \mid A \in \Lambda_{i+1}\} &= \sup\{\|\mathrm{d}L_n(\rho)A\| \mid A \in \Lambda_i, \rho \in K(\mathcal{A}), n \in \mathbb{N}\} \\ &\leq \delta \sup\{\|A\| \mid A \in \Lambda_i\}\end{aligned}$$

so that $\|A_i\| \leq \delta^i \sup\{\|A\| \mid A \in \Lambda_0\}$. Thus Λ is a bounded set in $\mathcal{A}$ which for all $\varepsilon > 0$ can be approximated to within ε by a finite sum of terms from the precompact sets $(\Lambda_n)_{n\in\mathbb{N}}$: Λ is precompact. Furthermore, by construction, if $A = \sum_{i=0}^{\infty} (\delta')^{-i} A_i$ then $\mathrm{d}L_n(\rho)A_i = B_i$ for some $B_i \in \Lambda_{i+1}$. Consequently, $\mathrm{d}L_n(\rho)A_i = \sum_{i=0}^{\infty} (\delta')^{-i} B_i \in \delta'\Lambda$. Finally, we may replace Λ by its closure, which is compact, since this does not spoil the property stated in the Proposition.

(2) Weak*-convergence in $K(\mathcal{A})$ implies uniform convergence on each compact subset of $\mathcal{A}$, and so uniform convergence on each $\Lambda \in \Upsilon$. Conversely, each $A \in \mathcal{A}$ is an element of some $\Lambda_A \in \Upsilon$, (just use the set which is generated using $\Lambda_0 = \{A\}$ in the first part of the theorem), so convergence in the seminorm $\rho \mapsto D_{\Lambda_A}(\rho)$ implies convergence in the seminorm $\rho \mapsto |\langle \rho, A \rangle|$.

Similarly, strong operator in $\mathcal{B}(\mathcal{A})$ implies uniform convergence on each compact subset of $\mathcal{A}$, and so uniform convergence on each $\Lambda \in \Upsilon$. Conversely, each $A \in \mathcal{A}$ is an element of some $\Lambda_A \in \Upsilon$, (just use the set which is generated using $\Lambda_0 = \{A\}$ in the first part of the theorem), so convergence in the seminorm $\rho \mapsto \Delta_{\Lambda_A}(\rho)$ implies convergence in the seminorm $\mathcal{B}(\mathcal{A}) \ni X \mapsto \|X(A)\|$.

■

Proof of Proposition 4.3:

Let $\delta > 0$ be as in Def. 4.2(3). For any $\rho \in K(\mathcal{A})$ consider the set of norm continuous $J_\cdot(\rho) : \mathbb{R}^+ \to \mathcal{B}(\mathcal{A})$ with $J_0(\rho) = \mathrm{id}$, equipped with the norm $\|J_\cdot(\rho)\| = \sup_{t\geq 0} e^{-\delta' t}\|J_t(\rho)\|$ for some $\delta' > 0$. Construct the iteration

$$(\mathcal{I}J(\rho))_t = \mathrm{id} + \int_0^t ds J_s(\rho)\mathrm{d}L(Q_s\rho) \quad .$$

From Definition 4.2(2) we see that this integral must be understood in the sense of the strong operator topology on $\mathcal{B}(\mathcal{A})$. However, since by assumption $\|\mathrm{d}L(\rho)\| \leq \delta$ for all $\sigma \in K(\mathcal{A})$, the map $t \mapsto (\mathcal{I}J(\rho))_t$ will be norm continuous whenever the map $t \mapsto J_t(\rho)$ is. The iteration operator satisfies

$$\|\mathcal{I}J(\rho) - \mathcal{I}\tilde{J}(\rho)\| \leq \frac{\delta}{\delta'}\|J(\rho) - \tilde{J}(\rho)\| \quad .$$

Thus $\mathcal{I}$ is contractive if $\delta' > \delta$: the differential equation has a unique solution. The exponential bound (4.3) follows from the fact that $\|\dot{J}_t(\rho)\| \leq \delta\|J_t(\rho)\|$.

It remains to be shown that equation (4.2) is satisfied. We fix an element $\sigma \in K(\mathcal{A})$ and for all $\rho \in K(\mathcal{A})$ and $h \in [0,1]$ define the perturbation $\rho^h = (1-h)\rho + h\sigma$. Then for all $A \in \mathcal{A}$

$$\begin{aligned}
&\langle Q_t\rho^h - Q_t\rho - h(\sigma - \rho)\circ J_t(\rho), A\rangle \\
&\quad = \int_0^t ds\Big\langle L(Q_s\rho^h) - L(Q_s\rho) - h(\sigma - \rho)\circ J_s(\rho)\circ J_s'(Q_s\rho), A\Big\rangle \\
&\quad = \Xi_t^{(1)}(h, A) + \Xi_t^{(2)}(h, A) \quad ,
\end{aligned}$$

where we have defined

$$\Xi_t^{(1)}(h, A) \quad = \int_0^t ds\Big\langle Q_s\rho^h - Q_s\rho - h(\sigma - \rho)\circ J_s(\rho), \mathrm{d}L(Q_s\rho)A\Big\rangle$$

and

$$\Xi_t^{(2)}(h, A) \quad = \int_0^t ds\langle L(Q_s\rho^h) - L(Q_s\rho), A\rangle - \langle Q_s\rho^h - Q_s\rho, \mathrm{d}L(Q_s\rho)A\rangle \quad .$$

Suppose we can show that for all $\Lambda \in \Upsilon$, $\lim_{h\to 0} h^{-1}\Xi_t^{(2)}(h, A) = 0$, uniformly for t in compact intervals and $A \in \Lambda$. Then for all $\tau > 0$ and for all $\varepsilon > 0$ there exists an $h(\varepsilon)$ such that for all $h < h(\varepsilon)$ and $t \leq \tau$, $h^{-1}D_\Lambda(\Xi_s^{(2)}(h, \cdot)) < \varepsilon$. Applying Theorem 5.1 with the L_n there all set equal to the present L, it is seen that for the δ of Def. 4.2(3) and all $h < h(\varepsilon)$ and $t \leq \tau$,

$$D_\Lambda(h^{-1}(Q_t\rho^h - Q_t\rho) - (\sigma-\rho)\circ J_t(\rho)) \leq \delta\int_0^t ds D_\Lambda(h^{-1}(Q_s\rho^h - Q_s\rho) - (\sigma-\rho)\circ J_s(\rho)) + \varepsilon.$$

By Gronwall's Lemma (see e.g. [HS])

$$D_\Lambda(h^{-1}(Q_t\rho^h - Q_t\rho) - (\sigma - \rho)\circ J_t(\rho)) \leq \varepsilon\exp(\delta t)$$

for all $h < h(\varepsilon)$. Taking the superior limit as $h \to 0$ and using the fact that ε is arbitrary we see that $(\sigma-\rho)\circ J_t(\rho)$ will indeed be the weak*-derivative of the function $\rho \to Q_t\rho$. The statement of the theorem will follow by taking the differential of the composition of any function $f \in \mathcal{C}^1(K(\mathcal{A}))$ with Q_t. For then

$$\begin{aligned}
\langle \sigma - \rho, \mathrm{d}(f\circ Q_t)(\rho)\rangle &= \lim_{h\to 0} h^{-1}(f(Q_t\rho^h) - f(Q_t\rho)) \\
&= \lim_{h\to 0} h^{-1}\langle Q_t\rho^h - Q_t\rho, \int_0^1 du f(uQ_t\rho^h + (1-u)Q_t\rho)\rangle \\
&= \langle \sigma - \rho, J_t(\rho)\mathrm{d}f(Q_t\rho)\rangle \quad .
\end{aligned}$$

It remains to be shown that $\Xi_t^{(2)}(\rho, A)$ has the desired properties. Integrating up the difference of $L(Q_s\rho^h)$ and $L(Q_s\rho)$ we write

$$\Xi_t^{(2)}(\rho, A) = \int_0^t ds\Big\langle Q_s\rho^h - Q_s\rho, \int_0^1 du\{\mathrm{d}L(uQ_s\rho^h + (1-u)Q_s\rho) - \mathrm{d}L(Q_s\rho)\}A\Big\rangle \quad .$$

Now for $B \in \mathcal{A}$

$$\begin{aligned}
\langle Q_t\rho^h - Q_t\rho, B\rangle - \langle \rho^h - \rho, B\rangle &= \int_0^t \langle L(Q_s\rho^h) - L(Q_s\rho), B\rangle \\
&= \int_0^t \langle Q_s\rho^h - Q_s\rho, \int_0^1 du\mathrm{d}L(uQ_s\rho^h + (1-u)Q_s\rho)B\rangle \quad .
\end{aligned}$$

Taking the supremum over $B \in \mathcal{A}$ and using Gronwall's Lemma again

$$\|Q_t\rho^h - Q_t\rho\| \leq e^{\delta t}\|\rho^h - \rho\| \leq he^{\delta t}\|\rho - \sigma\| \quad . \tag{5.2}$$

Thus $h^{-1}\Xi_t^{(2)}(h, A) \leq \|\rho - \sigma\|\int_0^t ds\int_0^1 du C_s(h, u; A)$ where we set $C_s(h, u; A) = e^{\delta s}\|\mathrm{d}L(uQ_s\rho^h + (1-u)Q_s\rho)A - \mathrm{d}L(Q_s\rho)A\|$. By equation (5.2) above, $Q_s\rho^h$ converges weak* to $Q_s\rho$ for all s as $h \to 0$, and so by continuity of $\rho \mapsto \mathrm{d}L(\rho)A$, $C_s(h, u; A)$ converges to 0 as $h \to 0$, for each for s and u. Since $C_s(h, u; A) \leq e^{\delta s}\delta\|A\|$, we conclude by dominated convergence that $\lim_{h\to 0} h^{-1}\Xi_t^{(2)}(h, A) = 0$. Now the maps $A \mapsto h^{-1}\Xi_s^{(2)}(h, A)$ are linear and norm-bounded for $s \in [0, t]$ by $2\delta te^{\delta t}$. Thus these maps are uniformly continuous for $h \geq 0$ and t in compact intervals. Hence $h^{-1}\Xi_t^{(2)}(h, A)$ converges to 0, uniformly for t in compact intervals and A in the compact set $\Lambda \in \Upsilon$, as required.

To prove (2) we calculate for $f \in \mathcal{C}^1(K(\mathcal{A}))$ and $\rho \in K(\mathcal{A})$,

$$
\begin{aligned}
(Zf)(\rho) &= \lim_{t\to 0} t^{-1}(f(Q_t\rho) - f(\rho)) \\
&= \lim_{t\to 0} t^{-1}\langle Q_t\rho - \rho, \int_0^t du\, df(uQ_t\rho + (1-u)\rho)\rangle \\
&= \langle L(\rho), df(\rho)\rangle
\end{aligned}
$$

by differentiability of $Q_t\rho$ and continuity of $\rho \mapsto df(\rho)$. $\mathcal{C}^1(K(\mathcal{A}))$ is an invariant dense subset of $\mathrm{Dom}(Z)$ and hence a core for Z.

■

Proof of Proposition 4.6:

(1) By Definition 4.5(2) (respectively Def. 4.5(3)) we see for each $A \in \mathcal{A}$ that the functions $\rho \mapsto dL_n(\rho)A$ (resp. $\rho \mapsto \langle L_n(\rho), A\rangle$) converge uniformly on $K(\mathcal{A})$ to some limit which we call $f(\cdot, A)$ (resp. $g(\cdot, A)$). These limits are clearly linear functions of A. We can see that they are also continuous. For $\|g(\rho, A) - g(\rho, B)\| \leq \lim_{n\to\infty} \|dL_n(\rho)A - dL_n(\rho)B\| \leq \delta\|A - B\|$. Likewise we see that $|f(\rho, A) - f(\rho, B)| \leq \lim_{n\to\infty} \|L_n(\rho)\|\|A - B\|$. So $A \mapsto f(\rho, A)$ is a continuous linear functional $L(\rho) \in \mathcal{A}^*$ provided that $\|L_n(\rho)\|$ is uniformly bounded in n. But let ω, $\hat{\delta}$ be as in Def. 4.5(4). Then for any $\rho \in K(\mathcal{A})$,

$$
\|L_n(\rho)\| \leq \|L_n(\omega)\| + \sup_{A:\|A\|\leq 1} \int_0^1 du |\langle \rho - \omega, dL_n(u\rho + (1-u)\omega)A\rangle| \leq \hat{\delta} + 2\delta
$$

as required. Finally we show that dL exists with $g(\rho, A) = dL(\rho)A$ for all $A \in \mathcal{A}$. Let $\rho^h = (1-h)\rho + h\sigma$ as before. Then

$$
\begin{aligned}
\lim_{h\to 0} h^{-1}\langle L(\rho^h) - L(\rho), A\rangle &= \lim_{h\to 0}\lim_{n\to\infty} h^{-1}\langle L_n(\rho^h) - L_n(\rho), A\rangle \\
&= \lim_{h\to 0}\lim_{n\to\infty} \langle \sigma - \rho, \int_0^1 du\, dL_n(u\rho^h + (1-u)\rho)A\rangle \\
&= \langle \sigma - \rho, g(\rho, A)\rangle \quad .
\end{aligned}
$$

For (2) we show that $Q_{m,t}$ is Cauchy in the weak* sense. For any $A \in \mathcal{A}$

$$
\begin{aligned}
\langle Q_{n,t}\rho - Q_{m,t}\rho, A\rangle &= \int_0^t ds \langle L_n(Q_{n,s}\rho) - L_m(Q_{m,s}\rho), A\rangle \\
&= \int_0^t ds \{\Gamma^{(1)}_{nm}(s, A) + \Gamma^{(2)}_{nm}(s, A)\} \quad ,
\end{aligned}
$$

where

$$
\Gamma^{(1)}_{nm}(s, A) = \langle L_n(Q_{m,s}\rho) - L_m(Q_{m,s}\rho), A\rangle \quad ,
$$

and

$$
\Gamma^{(2)}_{nm}(s, A) = \langle L_n(Q_{n,s}\rho) - L_n(Q_{m,s}\rho), A\rangle \quad .
$$

Integrating the difference of the derivatives in $\Gamma^{(2)}_{nm}(s, A)$, we write

$$
\Gamma^{(2)}_{nm}(s, A) = \langle Q_{n,s}\rho - Q_{m,s}\rho, \int_0^1 du\, dL_n(uQ_{n,s}\rho + (1-u)Q_{m,s}\rho)A\rangle \quad .
$$

So for all $\Lambda \in \Upsilon$, $D_\Lambda(\Gamma^{(2)}_{nm}(s,\cdot)) \leq \delta D_\Lambda(Q_{n,s}\rho - Q_{m,s}\rho)$. Now by Def. 4.5(3) for all $A \in \mathcal{A}$ and $\varepsilon > 0$ we can find an $n(\varepsilon)$ such that if $n, m > n(\varepsilon)$, $|\Gamma^{(1)}_{nm}(s, A)| < \varepsilon$. Since $|\Gamma^{(1)}_{nm}(s, A)| \leq (\|L_n(Q_{m,s}\rho)\| + \|L_m(Q_{m,s}\rho)\|)\|A\| \leq (2\hat{\delta} + 4\delta)\|A\|$, the linear maps $A \mapsto \Gamma^{(1)}_{nm}(s, A)$ are uniformly bounded in n, m and for t compact intervals of $\mathbb{R}^+$. This means that the bound $|\Gamma^{(1)}_{nm}(s, A)| < \varepsilon$ can be made to hold uniformly in compact sets of $\mathcal{A}$ and $\mathbb{R}^+$. Thus for all $\Lambda \in \Upsilon$ and $\varepsilon > 0$ we can find $n(\varepsilon)$ such that for all $n, m > n(\varepsilon)$,

$$
D_\Lambda(Q_{n,t}\rho - Q_{m,t}\rho) \leq \varepsilon t + \delta \int_0^t ds D_\Lambda(Q_{n,s}\rho - Q_{m,s}\rho) \quad .
$$

Applying Gronwall's Lemma we see that $D_\Lambda(Q_{n,t}\rho - Q_{m,t}\rho) \leq \varepsilon t e^{\delta t}$, so that $Q_{n,t}\rho$ is Cauchy as stated. As a weak*-limit of states, $Q_t\rho$ is a state, and by calculating the limits of compositions $Q_{n,t}Q_{n,s}\rho$ it is seen that $(Q_t)_{t\geq 0}$ is a flow on $K(\mathcal{A})$. Taking the limit of the integral equation one sees that Q_t satisfies the limiting differential equation (4.4). It only remains to be shown that solutions of equation (4.4) are weak*-continuous in the initial condition. But $\langle Q_t\rho - Q_t\sigma\rangle = \langle \rho - \sigma, A\rangle + \int_0^t ds \langle Q_s\rho - Q_s\sigma, \int_0^1 du\, dL(uQ_s\rho + (1-u)Q_s\sigma)A\rangle$, so that for all $\Lambda \in \Upsilon$,

$$
D_\Lambda(Q_t\rho - Q_t\sigma) \leq D_\Lambda(\rho - \sigma) + \delta \int_0^t ds D_\Lambda(Q_s\rho - Q_s\sigma) \quad .
$$

Hence by Gronwall's Lemma, $D_\Lambda(Q_t\rho - Q_t\sigma) \leq e^{\delta t} D_\Lambda(\rho - \sigma)$.

The proof of (3) takes a similar form. We have proved above that Q_t has a regular generator, so that by Prop. 4.3 its Jacobian J_t exists and satisfies the differential equation $J_t'(\rho) = J_t(\rho) dL(Q_t\rho)$. By the now familiar decomposition we have

$$
\|(J_{n,t}(\rho) - J_{m,t}(\rho))A\| \leq \int_0^t ds \|(J_{n,s} - J_{m,s}) dL_n(Q_s\rho)A\| + \int_0^t ds \chi_{nm}(s, A) \quad ,
$$

where

$$
\chi_{nm}(s, A) = \|J_{m,s}(dL_n(Q_{n,s}\rho) - dL_m(Q_{m,s}\rho))A\| \quad .
$$

Consider first $\chi_{nm}(s,A)$. Note that $\|J_t(\rho)\| \leq e^{\delta t}$. Furthermore, for each $s \in \mathbb{R}^+$, the sequence $Q_{n,s}\rho$ is convergent. Since for each $A \in \mathcal{A}$, $\rho \to \mathrm{d}L_n(\rho)A$ is uniformly convergent, we see that for each s and ε there exists an $n(\varepsilon)$ such that $\chi_{nm}(s,A) < \varepsilon$ for all $n,m > n(\varepsilon)$. Since $A \mapsto \chi_{nm}(s,A)$ is norm-bounded by $e^{\delta s}\delta$ this bound can be made uniform for s and A in compact sets. Thus for all $\tau > 0$, $\varepsilon > 0$ and $\Lambda \in \Upsilon$ we can choose an $n(\varepsilon)$ such that for all $t \leq \tau$ and $n,m > n(\varepsilon)$,

$$\Delta_\Lambda(J_{n,t}(\rho) - J_{m,t}(\rho)) \leq \int_0^t ds \Delta_\Lambda(J_{n,s}(\rho) - J_{m,s}(\rho)) + \varepsilon t e^{\delta t} \quad .$$

As before, we can use Gronwall's lemma and the arbitrariness of ε to conclude that $J_{n,t}(\rho)$ converges in the strong operator topology to a limit which satisfies the differential equation $J_t'(\rho) = J_t(\rho)\mathrm{d}L(Q_t\rho)$.

To prove (4) we calculate for $f \in C^1(K(\mathcal{A}))$

$$\lim_{n\to\infty} \|Z^n f - Zf\| = \lim_{n\to\infty} \sup_{\rho \in K(\mathcal{A})} |\langle L_n(\rho) - L(\rho), \mathrm{d}f(\rho)\rangle| \quad = 0 \quad .$$

■

6. Relative Entropy and Liapunov Functions.

When $T_{t,\infty}$ is implemented by a flow, F_t, we are, in special cases, able to find a Liapunov functional for F_t: i.e. a functional on $K(\mathcal{A})$, which is non-increasing along the trajectories of the flow, and which is bounded below on the whole of $K(\mathcal{A})$. It is difficult to find a function which satisfies both of these conditions. In **[AM]**, the (negative) entropy was used for a class of models. However, the entropy is not bounded above unless $\mathcal{A}$ is finite dimensional. Furthermore, there is no reason to expect that a Liapunov functional should exist in general: one would expect the presence or absence of dynamical stability to depend on the the detailed form of the generators G_n.

In the following Proposition we let $S(\varphi,\rho)$ denote the entropy of $\rho \in K(\mathcal{A})$ relative to $\varphi \in K(\mathcal{A})$ as defined for normal states on a von Neumann algebra in **[Ara]**, and extended to states on C^*-algebras in **[PW]**, **[Kos]** and also in **[Pet]**. The crucial property we shall need here is that if $\gamma : \mathcal{A}^n \to \mathcal{A}^n$ is a completely positive unital map, then $S(\varphi,\rho) \geq S(\varphi\circ\gamma, \rho\circ\gamma)$. In the particular case where both states are given by non-singular densities D_φ and D_ρ with respect to a trace Tr,

$$S(\varphi,\rho) = \mathrm{Tr}\big(D_\rho(\log D_\rho - \log D_\varphi)\big) \quad .$$

Proposition 6.1. *Let $T_{t,\cdot}$ be an approximate symmetry preserving sequence of semigroups of completely positive maps such that the limit $T_{t,\infty}$ is implemented by a flow F_t on $K(\mathcal{A})$. Furthermore, suppose that there exists a state σ in $K(\mathcal{A})$ such that for all $n \in \mathbb{N}$, $\sigma^n \circ T_{t,n} = \sigma^n$. Then for all $\rho \in K(\mathcal{A})$ the function $t \mapsto S(\sigma, F_t\rho)$ is non-increasing and bounded below by zero.*

Proof: For all $m \in \mathbb{N}$ and $X_m \in \mathcal{A}^m$, $\lim_n\langle\rho^n, T_{t,n}j_{nm}X_m\rangle = (T_{t,\infty}j_{\infty m}X_m)(\rho) = \langle(F_t\rho)^m, X_m\rangle$. In other words, $\sigma_n \circ T_{t,n}$ converges (in the sense of Def. III.2 of **[RW1]**) as $n \to \infty$ to the element $(F_t\rho)^\infty$ of the set of symmetric states on $\mathcal{A}^\infty$. Furthermore,

$$\begin{aligned} S(\sigma,\rho) &= \frac{1}{n}S(\sigma^n,\rho^n) \geq \frac{1}{n}S(\sigma^n\circ T_{t,n}, \rho^n\circ T_{t,n}) \\ &= \frac{1}{n}S(\sigma^n, \rho^n\circ T_{t,n}) \quad . \end{aligned}$$

Taking the inferior limit as $n \to \infty$ we conclude from Proposition III.4 of **[RW1]** that $S(\sigma,\rho) \geq S(\sigma, F_t\rho)$. For the boundedness simply note that $S(\omega_1,\omega_2) \geq 0$ for all $\omega_1,\omega_2 \in K(\mathcal{A})$.

■

The requirement that for some $\sigma \in K(\mathcal{A})$ $\sigma^n = T_{t,n}\circ\sigma^n$ for all $n \in \mathbb{N}$ may at first sight seem quite restrictive. However, it can be shown to hold for some inhomogeneous mean-field dynamical systems which relax to thermal equilibrium **[BSP1,BSP2]**.

7. Diffusive and other non-deterministic mean-field limits.

As we stated before, mean-field limits are not necessarily implemented by flows, and $T_{t,\infty}$ may be a more general semigroup of positive operators on $\mathcal{C}(K(\mathcal{A}))$. Such a semigroup can always be written in terms of "transition kernels" as $(T_{t,\infty}f)(\rho) = \int p_t(\rho, d\sigma)f(\sigma)$, where for each $t \geq 0$ and $\rho \in K(\mathcal{A})$, $p_t(\rho,\cdot)$ is a probability measure on $K(\mathcal{A})$. The semigroup property of $T_{t,\infty}$ then becomes the Chapman-Kolmogorov equation for the kernels p, and from these we can construct for each starting point $\rho \equiv \rho_0$ a Markov process $(\rho_t)_{t\geq 0}$ starting at ρ. If we denote the expectations with respect to this process by $\mathbf{E}^\rho$, we can write

$$(T_{t,\infty}f)(\rho) = \mathbf{E}^\rho(f(\rho_t)) \quad . \tag{7.1}$$

Hence even in the general case the limiting dynamics is given by a (not necessarily deterministic) time evolution on the one-particle state space $K(\mathcal{A})$.

From the study of approximately polynomial generators it is not at all clear whether any non-deterministic evolutions can be obtained as the mean-field limit of an approximate symmetry preserving family of semigroups. We therefore give a class of examples, where this possibility is realized.

For each n let $h \mapsto S_{h,n} \in \mathrm{Aut}(\mathcal{A}^n)$ be a representation of a locally compact group H. Suppose that $S_{h,n}$ is approximate symmetry preserving with respect to n, and strongly continuous in the sense that for $X \in \tilde{\mathcal{Y}}$, and $\varepsilon > 0$ we can find $n_\varepsilon \in \mathbb{N}$ and a neighbourhood $\mathcal{N}_\varepsilon \subset H$ of the identity such that $\|S_{h,n}X_n - X_n\| \leq \varepsilon$ for $n \geq n_\varepsilon$ and $h \in \mathcal{N}_\varepsilon$. Let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup on H, i.e. a family of probability measures on H such that $\lim_{t\to 0}\mu_t(\mathcal{N}) = 1$ for any neighbourhood $\mathcal{N}$ of the identity, and $\mu_t * \mu_s = \mu_{t+s}$, where $*$ denotes the convolution of measures **[HR]**. Then we define for each $t \geq 0$ and $n \in \mathbb{N}$ the operator

$$T_{t,n} = \int_H \mu_t(dh)\, S_{h,n} \quad \in \mathcal{B}(\mathcal{A}^n) \quad . \tag{7.2}$$

It is straightforward to check from the convolution property of μ that for fixed n $T_{t,n}$ is indeed a semigroup of completely positive unital maps, which is strongly continuous by the continuity conditions on T and μ.

Proposition 7.1. *Under the conditions stated above $T_{t,n}$ has a mean-field limit, and $T_{t,\infty} = \int \mu(dh) S_{h,\infty}$.*

Proof: We are going to verify condition (1) of Theorem 2.3. The proof that each $T_{t,\cdot}$ is approximate symmetry preserving follows the proof of the convergence of the defining integral for each $T_{t,n}$, checking uniformity in n along the way. Fix $X \in \tilde{\mathcal{Y}}$, and pick ε, and let $n_\varepsilon, \mathcal{N}_\varepsilon$ be as in the continuity condition for T. Consider a measurable partition $\{m_i \mid i \in I\}$ of H into $\mathcal{N}_\varepsilon$-small sets, i.e. there are $h_i \in H$ such that $h_i^{-1}m_i \subset \mathcal{N}_\varepsilon$. Then for $n \geq n_\varepsilon$ $T_{t,n}X_n$ is approximated in norm up to ε by the absolutely convergent sum $\sum_{i\in I}\mu_t(m_i)S_{h_i,n}X_n$. Since the finite partial sums are approximately symmetric by assumption, we have uniformly approximated $T_{t,n}X_n$ by approximately symmetric sequences, so this sequence is itself approximately symmetric. The formula for $T_{t,\infty}X_\infty$ follows by taking this approximation to the limit.

For demonstrating the strong continuity of $T_{t,n}$, let $X \in \tilde{\mathcal{Y}}, n_\varepsilon$, and $\mathcal{N}_\varepsilon$ be as before. Then for $\delta > 0$ there is some $\tau > 0$ such that for $0 \leq t \leq \tau$ we have $\mu_t(\mathcal{N}_\varepsilon) \geq (1-\delta)$. Hence for $n \geq \varepsilon$ and $t \leq \tau$ we have

$$\|T_{t,n}X_n - X_n\| = \|\Big(\int_{h\in\mathcal{N}_\varepsilon} + \int_{h\notin\mathcal{N}_\varepsilon}\Big)\mu_t(dh)(S_{h,n}X_n - X_n)\| \leq (1-\delta)\varepsilon + 2\delta\|X_n\|.$$

■

A simple example of this structure is the case $H = \mathbb{R}$. The above construction is then the analogue of the construction carried out in **[Dav]** in the context of semigroup theory on Banach spaces. Thus $(S_{h,n})_{h\in\mathbb{R}}$ could be any sequence of automorphism groups obtained from an approximately polynomial sequence of generators, so that $S_{t,\infty}$ is given by a reversible flow F_t on $K(\mathcal{A})$. An especially interesting convolution semigroup on $\mathbb{R}$ is given by the heat kernels $\mu_t(ds) = (2\pi t)^{-1/2}\exp -s^2/2t ds$. Then $S_{t,\infty} = e^{tG_\infty^2/2}$ (see **[Dav]**). The Markov process on the state space is then simply the Brownian motion along the orbits of the flow F_t. More formally, $\rho_t = F_{\tau(t)}\rho$, where $(\tau(t))_{t\geq 0}$ is the Brownian motion on the "time axis".

Proposition 7.1 allows us to generalize this example to certain sums of squares of generators, i.e. to sequences of generators of the form

$$G_n(A) = -n^2 \sum_{\alpha=1}^{r}[X_n^\alpha, [X_n^\alpha, A]] \quad , \tag{7.3}$$

where the X^α are strictly symmetric sequences. Each term in this sum is the square of a bounded polynomial generator, and with only one such term present the sequence $T_{t,n} = e^{tG_n}$ has a mean-field limit by the above remarks. It is clear, that the sequence space $\hat{\mathcal{Y}}$ defined in section 3 is in the domain of G, and even invariant. Moreover, $(G.X.)_\infty$ for $X \in \hat{\mathcal{Y}}$ is determined by acting on X_∞ with a second order differential operator, so that one expects the limiting dynamics to be given by a diffusion on the states space.

There is an instructive special case, where these assertions can be obtained directly from Proposition 7.1. We take $\mathcal{A}$ as the algebra of $d \times d$-matrices, and choose hermitian elements $X^1, \ldots X^r \in \mathcal{A}$. Let G_n be given by equation (7.3) with $X^\alpha_n = j_{n1}X^\alpha$. We shall consider the X^α as elements of the Lie algebra of SU_d. To state this formally, we denote by $d\pi$ the representation of the Lie algebra of SU_d associated with a continuous unitary representation $\pi : SU_d \to \mathcal{B}(\mathcal{H})$, i.e. $\pi(\exp \xi) = \exp(i\, d\pi(\xi))$. Thus $X^\alpha = d\pi(\xi^\alpha)$ for r elements of the Lie algebra, i.e. r left invariant vector fields on SU_d, with π chosen as the defining representation. Then $\Delta = \sum_\alpha (\xi^\alpha)^2$ is a second order elliptic differential operator, which determines the transition kernels of a left invariant diffusion process $(U_t)_{t\geq 0}$ starting at the identity (see Theorem 2 of [**AH**]). For $\mathcal{B}$ a Borel set in SU_d, $V \in SU_d$, and $t \geq s \geq 0$ these kernels are of the form

$$\mathbf{P}\big[U_t \in \mathcal{B} \mid U_s = V\big] = \mu_{t-s}(V^{-1}\mathcal{B}) \quad .$$

One checks easily that the measures μ_t determined by this equation satisfy $\mu_{t+s}(\mathcal{B}) = \int \mu_t(U^{-1}\mathcal{B})\mu_s(dU)$, and also satisfy the continuity conditions for convolution semigroups.

For any representation π the formula $T_{t,\pi}(A) = \int \mu_t(dU)\, \pi(U)A\pi(U)^*$ defines a strongly continuous, unit preserving semigroup of completely positive operators on $\mathcal{B}(\mathcal{H})$. The generator of this semigroup is simply given by

$$G_\pi(A) = -\sum_{\alpha=1}^{r} [d\pi(\xi^\alpha), [d\pi(\xi^\alpha), A]] \quad , \tag{7.4}$$

where $d\pi$ denotes the representation of the Lie algebra of SU_d associated with π, i.e. $\pi(\exp \xi) = \exp(i\, d\pi(\xi))$. Note that if π is the regular representation of SU_d, on which the Lie algebra is represented by differential operators, we recover $\Delta = G_\pi$. On the other hand, if we consider the product representation $\pi(U) \equiv S_{U,n} = U^{\otimes n} \in \mathcal{A}^n$ we get $d\pi(\xi^\alpha) = n j_{n1}X^\alpha = nX^\alpha_n$, and $G_{\pi_n} \equiv G_n$, as given in equation (7.3). We have therefore verified the integral formula for $T_{t,n}$ in terms of $S_{U,n}$. The string continuity condition for S is obvious, and $S_{U,\cdot}$ preserves not only approximate symmetry but even strict symmetry. Hence Proposition 7.1 applies, and we have identified the generator of $T_{t,n}$. The process on the state space is simply an image of the diffusion process on SU_d described above, i.e. for any starting point ρ we set $\rho_t = \rho_{U_t}$. The fact that this is again a Markov process follows from the left invariance of the process U_t.

It is evident from the form of this process, that with probability one ρ_t will be unitarily equivalent to ρ_0 for all times. Using a Poisson process rather than a diffusion one can generate non-deterministic processes violating this property. Explicitly, we can take

$$T_{t,n} = \sum_{\nu=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^\nu}{\nu!}(S_n)^\nu \quad ,$$

where S_n is an arbitrary approximate symmetry preserving sequence of completely positive unit preserving maps, and $\lambda > 0$. The mean-field limit of $T_{t,n}$ is obtained by taking $n = \infty$ in this equation, and the generators are given by $G_n = \lambda(S_n - \mathrm{id})$. Thus $\|G_n\|$ is uniformly bounded in n, whereas for the approximately polynomial generators it grows like n, and for the diffusions described above, it grows like n^2. Growth conditions on the generators are therefore not sufficient to determine the qualitative features of the limiting evolution.

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