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Finitely Correlated States on Quantum Spin Chains

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Abstract
We study a construction, which yields a class of translation invariant states on quantum spin chains, characterized by the property that the correlations across any bond can be modelled on a finite dimensional vector space. These states, which are dense in the set of all translation invariant states, can be considered as generalized valence bond states. We develop a complete theory of the ergodic decomposition of such states, including the decomposition into periodic "Néel ordered" states. Ergodic finitely correlated states have exponential decay of correlations. All states considered can be considered as "functions" of states of a special kind, so-called "purely generated states", which are shown to be ground states for suitably chosen interactions. We show that all these states have a spectral gap. Our theory does not require symmetry of the state with respect to a local gauge group, but the isotropic ground states of some one-dimensional anitferromagnets, recently studied by Affleck, Kennedy, Lieb, and Tasaki fall in this class.

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1. Introduction

In the last few years several authors have discovered and studied a certain type of Quantum Spin Hamiltonians with finite range interactions, which have ground states that can be constructed exactly **[42,5,9,22,38]**. They are called VBS-models, because of the Valence-Bond structure of the ground states. After suitable generalization one finds that the much older Majumdar-Ghosh-model **[46,47,5]**, has the same structure, although the ground states are especially simple there.

It has to be mentioned that this class of models does not coincide (and probably not even intersects) the class of integrable models which are soluble by the Bethe-Ansatz **[14,36,55,11]** or with the use of Temperly-Lieb algebras or Yang-Baxter-type methods **[12,13,43]**, and which have been the inspiring examples in many branches of theoretical physics for some decades. This means that the theory of finitely correlated states, developed in this paper, provides a new complementary technique for studying quantum spin systems.

One description of valence bond states in lattice systems **[5]** involves a contraction scheme with respect to indices of certain representations of SU(2), which can also be generalized to some other groups **[3]**. The SU(2)-valence bond states can also be expressed rather effectively in terms of homogeneous polynomials in two variables **[9,41]**. In all these studies the presence of a gauge symmetry group for the state under consideration plays a decisive rôle. In this paper, we use an abstract definition of (generalized) valence bond states, which does not involve any symmetry group. More importantly, however, we present a different perspective for the study of these states in the case of one-dimensional chains. The basic construction we use was first given in another context by Accardi, and Accardi and Frigerio **[1,2]**. It emphasizes the rôle of a family of operators, which are reminiscent of transfer matrices. Although the notion of a transfer-matrix is usually limited to the context of classical systems a generalization to quantum spin chains has been introduced by **[8]** in order to prove uniqueness and analyticity properties of Gibbs states for finite range interactions. It should be noted that, in contrast with the case of classical spins systems, such a transfer-matrix essentially lives on an infinite dimensional space. Unlike the approach of **[8]**, the fundamental difference between the quantum and the classical situation in our approach lies in the positivity properties of the transfer matrix, rather than in the structure of the space it lives on. In a specific example the utility of transfer matrix-like objects was also realized by other authors **[38,39]**.

As the essential feature characterizing the states obtainable by our construction we single out the property that the correlations across any bond of the chain can be modelled on a finite dimensional vector space. A subclass of states with this property, called C*-finitely correlated states is then shown to be identical with the class of valence bond states according to our abstract definition (Proposition 2.7). Our aim is to give a general theory of this class of translation invariant states on spin chains. Whatever the merits of the valence bond picture on lattices of higher dimension, we found the transfer-matrix point of view the more helpful representation on one-dimensional lattices, and therefore made it the starting point of our investigation. A major advantage, both for practical computations and for general considerations, is that the computation of correlation functions in our approach reduces to obtaining the spectral properties of a finite dimensional matrix. (In particular, all these states have exponential decay of correlations). For example, in the case of the state on the spin-1 chain as studied by **[5]**, which in several ways served as a paradigm for our investigations, the valence bond picture suggested a fairly involved diagrammatic technique to obtain the correlation functions **[5]**, wheras in our approach the computation reduces to evaluating one matrix element of a diagonal 4×4-matrix. At the same time our generalization embeds the state of **[5]** into a 19 dimensional manifold of states, of which we can establish essentially the same properties.

We now give a more detailed overview of the results presented in the different sections of this paper, without, however, entering into the technicalities.

Section 2. Finitely correlated states. Throughout the paper we are concerned with translation invariant states on the chain algebra $\mathcal{A}_{\mathbb{Z}} \equiv \bigotimes_{i\in\mathbb{Z}} \mathcal{A}_i$, where $\mathcal{A}_i$ denotes a copy of a fixed C*-algebra $\mathcal{A}$ "at site i". Finitely correlated states on $\mathcal{A}_{\mathbb{Z}}$ are defined by the property that the correlations across any bond can be modelled on a finite dimensional vector space $\mathcal{B}$. We show that the state can then be reconstructed from a map

$$\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \quad ,$$

and two elements $e \in \mathcal{B}, \rho \in \mathcal{B}^*$. For most of the paper we specialize to the case of "C*-finitely correlated states", for which $\mathcal{B}$ is a finite dimensional C*-algebra, and $\mathbb{E}, e, \rho$ are (completely) positive. The class of C*-finitely correlated states is shown to be a *weakly dense convex subset of the set of translation invariant states, which is important for the possibility of using these states as trial states in variational computations. We define generalized valence bond states, and show that on spin chains they coincide with the C*-finitely correlated states.

Section 3. Ergodic decompositions. Correlation functions of a C*-finitely correlated state are expressed in terms of the powers of the operator $\hat{\mathbb{E}}(B) = \mathbb{E}(\mathbb{1}_{\mathcal{A}} \otimes B)$ on $\mathcal{B}$. If $e \in \mathcal{B}$ is the only fixed point of $\hat{\mathbb{E}}$ then the state is exponentially clustering, and hence ergodic (i.e. extremal translation invariant). We show that every C*-finitely correlated state has a unique convex decomposition into finitely many ergodic C*-finitely correlated states. Using a quantum version of the classical Perron-Frobenius theory, the breaking of translation invariance, i.e. the decomposition of the given state into periodic components, can be diagnosed from the set of eigenvalues of $\hat{\mathbb{E}}$ with modulus one. All these eigenvalues are necessarily roots of unity, i.e. quasi-

periodic behaviour is excluded.

Section 4. Dilation theory and purely generated states. We continue the reduction of general C*-finitely correlated states to simpler building blocks. In classical probability theory finitely correlated states can be seen as functions of Markov Processes (see section 7.1). In this section we identify a subclass, the "purely generated states", which generate all C*-finitely correlated states by 'taking functions'. What is meant by 'taking functions' in the non-commutative context is explained there. The purely generated states are those for which the map $\mathbb{E}$ is "pure", i.e. it cannot be written as the sum of other completely positive maps. In comparison to classical probability the set of pure completely positive maps on a quantum system has a much richer structure. This strucure is essential in sections 4,5, and 6. In particular, it allows the construction of an abundance of non-trivial pure states.

Section 5. Ground state property of purely generated states. Here a crucial step for the applications is made. It is shown that each of basic building blocks identified above, i.e. every purely generated C*-finitely correlated state, is the unique ground state of some translation invariant finite range interaction. The interaction is chosen such that the energy density is equal to the lowest eigenvalue of the interaction operator, i.e. the state minimizes the energy in the strongest possible sense. As a byproduct, we prove that every purely generated state is pure, i.e. it cannot be decomposed even into non-translation invariant components, and also obtain a formula for the (finite) limiting absolute entropy density of these states (the entropy density vanishes).

Section 6. The ground state energy gap. Continuing the study of the Hamiltonians introduced in the previous section, it is shown that all these models have a spectral gap immediately above the ground state. The methods presented here are tailored to get a simple proof of the existence of the gap. Although they also allow explicit estimates, these estimates are not optimal. We do not know whether one could hope to derive exact expressions also for the gap, as is possible in the integrable models **[12]**. A short overview of our technique, stated in valence bond language, was given in **[29]**.

Section 7. Applications. We chose only a few examples to highlight the general structure developed in the main body of the paper. Further examples will be treated elsewhere **[29,30]**.

7.1. Classical systems. In order to put our results for quantum spin chains into perspective, we briefly review earlier results **[27]** for the case that all the C*-algebras appearing in the general construction are abelian. In this case C*-finitely correlated states are precisely the functions of Markov processes. A formula for the dynamical entropy (or entropy density) for such a probability measure is given.

7.2. Integrable systems. In the classical case any Gibbs state for a finite range interaction is C*-finitely correlated, and conversely any faithful Markovian measure is a Gibbs state for a well-defined nearest neighbour Hamiltonian **[51,32]**. Unfortunately, in spite of the fact that C*-finitely correlated states are dense in the translation invariant states (as noted above), this connection fails in the quantum case, even for ground states. As an example we show that the ground states of some integrable half-integer spin chains, treated by Takhtajan **[55]**, are not C*-finitely correlated. Although this can undoubtedly also be demonstrated by other methods, we show that it suffices to note that the known exact ground state energy of these models is not algebraic in the coupling constant.

7.3. Gauge invariant states. As states and models with a given group-invariance (acting on each site), certainly are of special importance, we study this situation in more detail. Some symmetry groups can be defined in terms of the state, and its generating map $\mathbb{E}$. We apply the previous results on pure states to derive simple relations between these groups. A straightforward construction for states with given symmetry is also given.

7.4. Integer spin chains. Finally, we we apply this constuction to obtain the well-known integer spin models **[5,9,28]**. By the results of section 6 all these models have a spectral gap. It is also shown how the representation theory of SU(2) can be used to carry out explicit calculations.

Appendix: Matrix order and complete positivity. Here we prove a characterization result for finitely correlated, but not C*-finitely correlated states, and collect the definitions and results about matrix ordered vector spaces needed for this purpose.

2. Finitely correlated states

In this paper we study a class of states on quantum "spin" chains. The observable algebra for a single "spin" is some fixed C*- algebra $\mathcal{A}$ with identity $\mathbb{1}_{\mathcal{A}}$. Often this algebra will be finite dimensional, or more specifically, the algebra $\mathcal{M}_d$ of $d \times d$-matrices. For each $n \in \mathbb{Z}$ we consider an isomorphic copy $\mathcal{A}_{\{n\}}$ of $\mathcal{A}$, and define for each finite subset $\Lambda \subset \mathbb{Z}$ the algebra $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_{\{x\}}$. Here and below the symbol "$\otimes$" will always refer to the minimal C*-tensor product **[54]**. For $\mathcal{A}_{\{1,\dots n\}}$ we also write $\mathcal{A}^{\otimes n}$. For infinite subsets $\Lambda \subset \mathbb{Z}$, $\mathcal{A}_\Lambda$ is defined as the C*-inductive limit of the algebras $\mathcal{A}_{\Lambda'}$ with $\Lambda' \subset \Lambda$ finite. The identification $\mathcal{A}_{\Lambda''} \hookrightarrow \mathcal{A}_{\Lambda'}$ for $\Lambda'' \subset \Lambda$ underlying this limit is by tensoring $A \in \mathcal{A}_{\Lambda''}$ with $\bigotimes_{x\in\Lambda'\setminus\Lambda''} \mathbb{1}_{\mathcal{A}_{\{x\}}}$. The most important example of this is the **chain algebra** $\mathcal{A}_\mathbb{Z}$ itself. The group $\mathbb{Z}$ acts on $\mathcal{A}_\mathbb{Z}$ by the translation automorphisms α_r, taking $\mathcal{A}_\Lambda$ into $\mathcal{A}_{\Lambda+r}$. The set of translation invariant states on $\mathcal{A}_\mathbb{Z}$ will be denoted by $\mathcal{T}$, or $\mathcal{T}(\mathcal{A})$. By grouping segments of p sites together, we obtain an isomorphism of $\mathcal{A}_\mathbb{Z}$ with $(\mathcal{A}^{\otimes p})_\mathbb{Z}$, identifying $\mathcal{A}_{\{kp,\dots kp+p-1\}}$ with $(\mathcal{A}^{\otimes p})_{\{k\}}$.

The characteristic property of the class of translation invariant states on $\mathcal{A}_\mathbb{Z}$ studied in this paper is described in (1) of the following Proposition.

2.1 Proposition. *Let $\mathcal{A}$ be a C*-algebra with unit, and let ω be a translation invariant state on the chain algebra $\mathcal{A}_\mathbb{Z}$. Then the following are equivalent:*

(1) The set of functionals $\Phi : \mathcal{A}_\mathbb{N} \to \mathbb{C}$ of the form
$$\Phi(A_1 \otimes \cdots A_n) = \omega(X \otimes A_1 \otimes \cdots A_n) \quad ,$$
with $X \in \mathcal{A}_{\mathbb{Z}\setminus\mathbb{N}}$ generates a finite dimensional linear subspace in the dual of $\mathcal{A}_\mathbb{N}$.

(2) There are a finite dimensional vector space $\mathcal{B}$, a linear map $\mathbb{E} : A \in \mathcal{A} \mapsto \mathbb{E}_A \in \mathcal{L}(\mathcal{B},\mathcal{B})$, an element $e \in \mathcal{B}$, and a linear functional $\rho \in \mathcal{B}^*$, such that $\rho \circ \mathbb{E}_\mathbb{1} = \rho$, $\mathbb{E}_\mathbb{1}(e) = e$, and for $n \in \mathbb{Z}, m \in \mathbb{N}$ and $A_i \in \mathcal{A}_{\{i\}} \simeq \mathcal{A}$:
$$\omega(A_n \otimes \cdots A_{n+m}) = \rho(e)^{-1} \; \rho \circ \mathbb{E}_{A_n} \circ \cdots \circ \mathbb{E}_{A_{n+m}}(e) \quad , \tag{2.1}$$
where the symbol "$\circ$" means composition of maps.

If in (2) $\mathcal{B}$ is chosen as minimal in the sense that
$$\mathrm{lin}\{\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(e) \mid n \in \mathbb{N}, A_1, \dots A_n \in \mathcal{A}\} = \mathcal{B}$$
and
$$\mathrm{lin}\{\rho \circ \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} \mid n \in \mathbb{N}, A_1, \dots A_n \in \mathcal{A}\} = \mathcal{B}^* \quad ,$$
then $\mathcal{B}$, $\mathbb{E}$, ρ, and e are determined by ω up to linear isomorphism.

2.2 Definition. *If the equivalent conditions of Proposition 2.1 are satisfied, ω will be called the* **finitely correlated state generated** *by $(\mathbb{E}, \rho, e)$.*

Proof of 2.1:
(1) $\Rightarrow$ (2): We abbreviate $\mathcal{A}_\sharp \equiv \mathcal{A}_{\{n|n\geq 1\}}$ and $\mathcal{A}_\flat \equiv \mathcal{A}_{\{n|n\leq 0\}}$. On $\mathcal{A}_\sharp$ we consider

the equivalence relation $X \sim Y \iff \omega(X_\flat \otimes (X - Y)) = 0$ for all $X_\flat \in \mathcal{A}_\flat$, and an analogous relation on $\mathcal{A}_\flat$. Denote by $\mathcal{B}_\natural$ the quotients of $\mathcal{A}_\natural$ by these relations and by $[X_\natural] \in \mathcal{B}_\natural$ the equivalence class of $X_\natural \in \mathcal{A}_\natural$, where $\natural$ stands for $\sharp$ or $\flat$. Obviously, there is a well defined, non-degenerate bilinear form $\eta : \mathcal{B}_\flat \times \mathcal{B}_\sharp \to \mathbb{C}$ such that $\eta([X_\flat], [X_\sharp]) = \omega(X_\flat \otimes X_\sharp)$. Clearly, $X_\flat \sim X'_\flat$ iff $X_\flat$ and $X'_\flat$ generate the same functional on $\mathcal{A}_\sharp$, hence (1) implies that $\mathcal{B}_\flat$ is finite dimensional. Since η is non-degenerate, we can identify $\mathcal{B}_\sharp$ with the dual of $\mathcal{B}_\flat$, and we shall take $\mathcal{B} = \mathcal{B}_\sharp$, $e = [\mathbb{1}] \in \mathcal{B}$, and $\rho = [\mathbb{1}] \in \mathcal{B}_\flat \equiv (\mathcal{B}_\sharp)^*$ in (2). Let $\mathbb{E}_A([X_\sharp]) = [A \otimes X_\sharp]$. We have to show that this is well defined, i.e. that $[A \otimes X_\sharp] = 0$, whenever $[X_\sharp] = 0$. But $[X_\sharp] = 0$ implies in particular that $\omega((X_\flat \otimes A) \otimes X_\sharp) = 0$ for all $X_\flat \in \mathcal{A}_{\{n | n \leq -1\}}$, and by translation invariance of ω we also have $\omega(X_\flat \otimes (A \otimes X_\sharp)) = 0$ for all $X_\flat \in \mathcal{A}_\flat$. The verification of equation(1) is straightforward.

(2) ⇒ (1): Given $\mathcal{B}$, e, ρ, $\mathbb{E}$ satisfying (2), we define the maps $\mathcal{J}_\sharp : \mathcal{A}_\sharp \to \mathcal{B}$, $\mathcal{J}_\flat : \mathcal{A}_\flat \to \mathcal{B}^*$ by

$$\mathcal{J}_\sharp(A_1 \otimes \cdots A_n) = \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(e)$$
$$\mathcal{J}_\flat(A_{-n} \otimes \cdots A_0) = \rho(e)^{-1} \rho \circ \mathbb{E}_{A_{-n}} \circ \cdots \circ \mathbb{E}_{A_0} \quad .$$

Then for $X_\natural \in \mathcal{A}_\natural$ we have $\omega(X_\flat \otimes X_\sharp) = \mathcal{J}_\flat(X_\flat)(\mathcal{J}_\sharp(X_\sharp))$. Since the range of $\mathcal{J}_\flat$ is in the finite dimensional space $\mathcal{B}^*$, (1) holds. If $\mathcal{B}$ is chosen to be minimal in the sense described, $\mathcal{J}_\flat$ is surjective. Therefore, $\mathcal{J}_\sharp(X_\sharp) = 0$ is equivalent to $\omega(X_\flat \otimes X_\sharp) = 0$ for all $X_\flat$, i.e. $X_\sharp \sim 0$. Since $\mathcal{J}_\sharp$ is also surjective, $[X_\sharp] \mapsto \mathcal{J}(X_\sharp)$ defines a linear isomorphism from $\mathcal{B}_\sharp$ to $\mathcal{B}$.

■

The Proposition gives an explicit formula (2.1) for ω in terms of the usually much simpler objects $\mathcal{B}, \mathbb{E}, \rho$, and e. We would therefore like to turn this formula into a definition of the state ω. It is clear from the structure of this formula, and from the invariance assumptions for e and ρ that the family of functionals on $\mathcal{A}_{\{n,\dots n+m\}}$ defined by (2.1) is consistent with the injections $\mathcal{A}_{\Lambda''} \hookrightarrow \mathcal{A}_{\Lambda'}$, so (2.1) defines a linear functional on $\bigcup_{\Lambda \text{ finite}} \mathcal{A}_\Lambda$. This functional is also obviously translation invariant and normalized to $\omega(\mathbb{1}) = 1$. But without further assumptions ω will rarely be positive. For this reason we had to assume positivity from the outset, by applying the Proposition only to states. In order to turn formula 2.1 into a useful tool for constructing states we need conditions, which will ensure the positivity of ω.

Necessary and sufficient conditions are given in the next Proposition, using the concept of matrix order. A matrix order for a vector space $\mathcal{B}$ is an ordering of each of the spaces $\mathcal{M}_n \otimes \mathcal{B}$ of $n \times n$-matrices with entries in $\mathcal{B}$, such that these orderings satisfy a certain consistency condition. Since a finite dimensional C*-algebra $\mathcal{A}$ is a direct sum of matrix algebras, $\mathcal{A} \otimes \mathcal{B}$ is matrix ordered in a canonical way, for any matrix ordered $\mathcal{B}$. A completely positive map $T : \mathcal{B}_1 \to \mathcal{B}_2$ between matrix ordered spaces is a linear map such that for each n, $\mathrm{id}_{\mathcal{M}_n} \otimes T$ takes positive into positive elements. In the standard case, which we shall consider almost exclusively, $\mathcal{B}$ is a C*-algebra and $\mathcal{M}_n \otimes \mathcal{B}$ is equipped with its ordering as a C*- algebra. Completely positive maps between operator algebras are well studied **[53]**. Since many of our results make use of the detailed structure theory of completely positive maps on C*-algebras, notably the Stinespring dilation theorem **[52]**, we could not extend our theory to states generated by completely positive maps on a general matrix ordered space. Therefore we collected the basic definitions and results concerning matrix order in Appendix 1, where we also prove the non-trivial direction of Proposition 2.3.

2.3 Proposition. *Let $\mathcal{A}$ be a finite dimensional C*-algebra, and $\mathcal{B}$ a finite dimensional matrix ordered space with $e \in \mathcal{B}$ positive, and $\rho \in \mathcal{B}^*$ a positive linear functional. Let $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ be a completely positive map such that $\mathbb{E}(\mathbb{1}_\mathcal{A} \otimes e) = e$, and $\rho(\mathbb{E}(\mathbb{1}_\mathcal{A} \otimes B)) = \rho(B)$ for all $B \in \mathcal{B}$. Then with $\mathbb{E}_A(B) = \mathbb{E}(A \otimes B)$, these objects generate a finitely correlated state ω, and every finitely correlated state is of this form.*

It is easy to see that complete positivity of $\mathbb{E}$ ensures positivity of ω, by introducing the "iterates" $\mathbb{E}^{(n)} : \mathcal{A}^{\otimes n} \otimes \mathcal{B} \to \mathcal{B}$ with $\mathbb{E}^{(1)} = \mathbb{E}$, and

$$\mathbb{E}^{(n+1)} = \mathbb{E} \circ (\mathrm{id}_\mathcal{A} \otimes \mathbb{E}^{(n)}) : \mathcal{A} \otimes \mathcal{A}^{\otimes n} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \quad . \tag{2.2}$$

Then $\mathbb{E}^{(n)}$ is completely positive, since this property is conserved under composition and tensoring with identity maps. Hence by equation 2.1 $A_1 \otimes \cdots A_n \mapsto \omega(A_1 \otimes \cdots A_n) = \rho(e)^{-1} \rho(\mathbb{E}^{(n)}(A_1 \otimes \cdots A_n \otimes \mathbb{1}_\mathcal{B}))$ is positive.

2.4 Definition. *Let $\mathcal{A}$ be a (not necessarily finite dimensional) C*-algebra with unit. Then if the positivity conditions of Proposition 2.3 are satisfied, and $\mathcal{B}$ is a finite dimensional C*-algebra with its canonical matrix order, ω will be called the* **C*-finitely correlated** *state generated by $(\mathbb{E}, \rho, e)$. The set of C*-finitely correlated states on $\mathcal{A}_\mathbb{Z}$ will be denoted by $\mathcal{F}$, or $\mathcal{F}(\mathcal{A})$.*

For C*-algebras the above argument that complete positivity of $\mathbb{E}$ implies positivity of ω is independent of $\mathcal{A}$ or $\mathcal{B}$ being finite dimensional. If we drop the restrictions on $\mathcal{B}$, equation 2.1 yields *every* translation invariant state ω on $\mathcal{A}_\mathbb{Z}$. To see this it suffices to take $\mathcal{B} := \mathcal{A}_\mathbb{N}$, and $\mathbb{E}(A \otimes (A_1 \otimes \cdots A_n)) = A \otimes A_1 \otimes \cdots A_n$, and to extend this map by linearity and continuity to all of $\mathcal{A} \otimes \mathcal{B}$. The state ρ is then taken as the restriction of the given translation invariant state ω on $\mathcal{A}_\mathbb{Z}$ to the subalgebra $\mathcal{A}_\mathbb{N}$. It is evident that with these definitions the original state ω satisfies equation 2.1. Hence it is mainly the finite dimension of $\mathcal{B}$, which gives a non-trivial content to Definition 2.4.

There is also a version of our construction for W*-algebras $\mathcal{A}$: the tensor product in the definition of the n-step algebra $\mathcal{A}_{\{i+1,\dots i+n\}}$ is then taken as the W*-tensor product, and the algebra $\mathcal{A}_\mathbb{Z}$ is the C*-inductive limit of these algebras. Since the category of normal completely positive maps between W*-algebras is closed under composition and tensor products, the above argument also shows that provided $\mathbb{E}$ is normal and completely positive (and if $\mathcal{B}$ is also allowed to be an infinite dimensional W*-algebra, provided also ρ is a normal state), then formula 2.1 defines a **locally normal** state on $\mathcal{A}_{\mathbb{Z}}$.

It is useful to note that the objects generating a C*-finitely correlated state ω can be chosen in the standard form described in the following Lemma. We shall use this form whenever convenient.

2.5 Lemma. *Any C*-finitely correlated state ω is also generated by some $\mathbb{E}, \rho, e$ such that $e = \mathbb{1}$ is the identity of the algebra $\mathcal{B}$, and ρ is a faithful state on $\mathcal{B}$. Moreover, $\mathcal{B}$ may be be taken either to be minimal in the sense that no proper subalgebra contains $\mathbb{1}$ and is invariant under all $\mathbb{E}_A$, or may be taken as a pure matrix algebra $\mathcal{B} = \mathcal{M}_k$*

Proof : If $0 \leq B \leq \lambda e$ for some $B \in \mathcal{B}$, and $0 \leq A \in \mathcal{A}$, then $0 \leq \mathbb{E}(A \otimes B) \leq \|A\|\mathbb{E}(\mathbb{1} \otimes B) \leq \lambda\|A\|\mathbb{E}(\mathbb{1} \otimes e) = \lambda\|A\|e$. Hence the subalgebra $\tilde{\mathcal{B}} = e\mathcal{B}e$ generated by elements dominated by e is a common invariant subspace of all operators $\mathbb{E}_A$. Hence the restriction of $\mathbb{E}$ to $\mathcal{A} \otimes \tilde{\mathcal{B}}$ also generates ω, and we may suppose that e is invertible in the algebra $\tilde{\mathcal{B}}$ generating ω. Clearly, ω is also generated from $\tilde{\mathbb{E}}, \tilde{\rho}, \mathbb{1}_{\tilde{\mathcal{B}}}$ with

$$\tilde{\mathbb{E}}(A \otimes B) = e^{-\frac{1}{2}}\mathbb{E}(A \otimes e^{\frac{1}{2}}Be^{\frac{1}{2}})e^{-\frac{1}{2}} \quad ,$$

and $\tilde{\rho}(B) = \rho(e^{\frac{1}{2}}Be^{\frac{1}{2}})$. Hence we may take $e = \mathbb{1}$.

Suppose that ρ is not strictly positive, i.e. $s := \mathrm{supp}(\rho) < \mathbb{1}$. Consider the operator $P : B \mapsto sBs$ on $\mathcal{B}$. Then since the functionals $\rho' = \rho \circ \mathbb{E}_{A_1} \circ \cdots \mathbb{E}_{A_n} \in \mathcal{B}^*$ are all dominated by ρ, we have $\rho' \circ P = \rho'$. Hence ω is also represented by $\tilde{\mathcal{B}} = s\mathcal{B}s \subset \mathcal{B}$ with $\mathbb{1}_{\tilde{\mathcal{B}}} = P\mathbb{1}_{\mathcal{B}} = s$, $\tilde{\rho} = \rho(\mathbb{1}_{\mathcal{B}})^{-1} \cdot \rho \mid \tilde{\mathcal{B}}$, and $\tilde{\mathbb{E}}_A = P \circ \mathbb{E}_a \mid \tilde{\mathcal{B}}$.

The statement about minimality is obvious. Since $\mathcal{B} = \bigoplus_\alpha \mathcal{M}_{k_\alpha}$ is a finite direct sum of matrix algebras, we may pick a representation on $\mathbb{C}^k = \bigoplus_\alpha \mathbb{C}^{k_\alpha}$. Let P_α be the projection onto the α^{th} summand and $\mathbb{P} : \mathcal{M}_k \to \mathcal{B} : \ B \mapsto \sum_\alpha P_\alpha B P_\alpha$. Then $\tilde{\mathbb{E}} := \mathbb{P} \circ \mathbb{E} \circ (\mathrm{id}_{\mathcal{A}} \otimes \mathbb{P}) : \mathcal{A} \otimes \mathcal{M}_k \to \mathcal{M}_k$ generates the same state.

■

The following Proposition lists some basic properties of the class of C*-finitely correlated states. For 2.6.(3,4) we use the identification of $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$ and $\mathcal{A}_{\mathbb{Z}}$ mentioned in the beginning of this section.

2.6 Proposition.

(1) Symmetric product states are in $\mathcal{F}$.

(2) $\mathcal{F}$ is convex.

(3) For $p \in \mathbb{N}$, $\omega \in \mathcal{F}$ is also C-finitely correlated as a state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$.*

(4) Conversely, let ω be a p-periodic state on $\mathcal{A}_{\mathbb{Z}}$, which is C-finitely correlated as a state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$. Let $\bar{\omega} = \frac{1}{p}\sum_{r=0}^{p-1} \omega \circ \alpha_r$ be the average of p consecutive translates of ω. Then $\bar{\omega} \in \mathcal{F}$.*

*(5) $\mathcal{F}$ is *weakly dense in the set $\mathcal{T}$ of translation invariant states on $\mathcal{A}_{\mathbb{Z}}$.*

Proof : (1) Let $\omega(A_n \otimes \cdots A_{n+m}) = \prod_i \eta(A_i)$. Then ω is generated by $\mathcal{B} = \mathbb{C}$, $\rho(\lambda) = \lambda$, $e = 1$, and $\mathbb{E}(A \otimes \lambda) = \lambda\eta(A)$.

(2) Let $\omega = \sum_i \lambda_i \omega_i$ with $\lambda_i > 0$ and ω_i generated by $(\mathcal{B}_i, \rho_i, \mathbb{E}_i, e_i)$. Set $\mathcal{B} = \bigoplus_i \mathcal{B}_i$, $\rho = \bigoplus_i \lambda_i \rho_i$, $e = \bigoplus_i e_i$, and $\mathbb{E} = \bigoplus_i \mathbb{E}_i$. Since $\mathbb{E}$ maps each direct summand of $\mathcal{B}$ into itself, we also have $\mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n} = \bigoplus_i \mathbb{E}_{i,A_1} \circ \cdots \circ \mathbb{E}_{i,A_n}$. Evaluating this at e and applying ρ we conclude that ω is generated by $(\mathbb{E}, \rho, e)$.

(3) If ω as a state on $\mathcal{A}_{\mathbb{Z}}$ is generated by $(\mathbb{E}, \rho, e)$, then as a state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$ it is generated by $(\mathbb{E}^{(p)}, \rho, e)$, where $\mathbb{E}^{(p)}$ is the p^{th} iterate of $\mathbb{E}$.

(4) Suppose now that the p-periodic state ω is generated by $\rho \in \mathcal{B}^*$, $e = \mathbb{1} \in \mathcal{B}$, and $\mathbb{E} : \mathcal{A}^{\otimes p} \otimes \mathcal{B} \to \mathcal{B}$. We set $\tilde{\mathcal{B}} = \bigoplus_{r=0}^{p-1} \mathcal{A}^{\otimes r} \otimes \mathcal{B}$, with the convention $\mathcal{A}^{\otimes 0} \otimes \mathcal{B} = \mathcal{B}$. We denote the r^{th} component of $\tilde{B} \in \tilde{\mathcal{B}}$ by $\tilde{B}_r$. For $A \in \mathcal{A}$ let

$$\tilde{\mathbb{E}}(A \otimes \tilde{B})_r = \begin{cases} \mathbb{E}(A \otimes \tilde{B}_{p-1}) & \text{, if } r = 0 \\ A \otimes \tilde{B}_{r-1} & \text{, if } 1 \leq r \leq p-1. \end{cases}$$

The state $\tilde{\rho} \in \tilde{\mathcal{B}}^*$ is defined by

$$\tilde{\rho}(\tilde{B}) = \frac{1}{p}\sum_{r=0}^{p-1} \rho(\mathbb{E}(\mathbb{1}_{\mathcal{A}}^{\otimes(p-r)} \otimes \tilde{B}_r)) \quad .$$

Note that by the invariance property of $\mathbb{E}$ the summand with $r = 0$ is just $\rho(\tilde{B})$. One checks that indeed $\tilde{\rho} \circ \tilde{\mathbb{E}}_{\mathbb{1}} = \tilde{\rho}$, and $\tilde{\mathbb{E}}(\mathbb{1}) = \mathbb{1}$, so $\tilde{\rho}$ and $\tilde{\mathbb{E}}$ define a translation invariant state on $\mathcal{A}_{\mathbb{Z}}$. It is clear that the p^{th} iterate of $\tilde{\mathbb{E}}$ maps each of the summands of $\tilde{\mathcal{B}}$ into itself. In fact

$$\left(\tilde{\mathbb{E}}_{A_1} \circ \cdots \circ \tilde{\mathbb{E}}_{A_p}(\tilde{B})\right)_r = A_1 \otimes \cdots A_r \otimes \mathbb{E}(A_{r+1} \otimes \cdots A_p \otimes \tilde{B}_r) \quad ,$$

and

$$\left(\tilde{\mathbb{E}}_{A_1} \circ \cdots \circ \tilde{\mathbb{E}}_{A_{np}}(\mathbb{1}_{\tilde{\mathcal{B}}})\right)_r =$$

$$= A_1 \otimes \cdots A_r \otimes \mathbb{E}_{(A_{r+1} \otimes \cdots A_{p+r+1})} \circ \cdots \circ \mathbb{E}(A_{(n-1)p+r+1} \otimes \cdots A_{np} \otimes \mathbb{1}_{\mathcal{A}}^{\otimes r} \otimes \mathbb{1}_{\mathcal{B}}) \quad .$$

Evaluating this on the state $\tilde{B}_r \mapsto \rho(\mathbb{E}(\mathbb{1}_{\mathcal{A}}^{\otimes(p-r)} \otimes \tilde{B}_r))$ gives the ω-expectation of $\mathbb{1}_{\mathcal{A}}^{\otimes(p-r)} \otimes A_1 \otimes \cdots A_{np} \otimes \mathbb{1}_{\mathcal{A}}^{\otimes r}$, i.e. the expectation of $A_1 \otimes \cdots A_{np}$ in $\omega \circ \alpha_r$. The result follows by summing over r.

(5) Let ω be a translation invariant state. Consider the product state ω' on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$ formed from the p-site restriction of ω. Let $\omega^p = \bar{\omega}'$. Then by (1) and (4) $\omega^p \in \mathcal{F}$. The states ω and $\omega' \circ \alpha_r$ coincide on observables $A = A_{i+1} \otimes \cdots A_{i+n}$ for $n < p$, unless the interval $i+1, \cdots i+n$ contains one of the "breakpoints" $np + r$. Thus

$|\omega(A) - \omega^p(A)| \leq (1 - \frac{2(n-1)}{p})\|A\|$, and $w^* - \lim_p \omega^p = \omega$. ∎

We close this section with an alternative construction for C*-finitely correlated states. It is a generalization of the "valence bond solid" states of **[18,17,19,20,21]**. For constructing a state on the chain $\mathcal{A}_{\mathbb{Z}}$ according to this scheme, we need two auxiliary finite dimensional C*-algebras $\mathcal{B}$ and $\overline{\mathcal{B}}$. The state is determined by a completely positive map $\mathbb{F} : \mathcal{A} \to \mathcal{B} \otimes \overline{\mathcal{B}}$, and a state $\Phi : \overline{\mathcal{B}} \otimes \mathcal{B} \to \mathbb{C}$, which have to satisfy the compatibility conditions

$$(\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(\mathbb{1}_{\mathcal{A}}) \otimes \mathbb{1}_{\mathcal{B}}) = \mathbb{1}_{\mathcal{B}} \qquad \text{and} \qquad (\Phi \otimes \mathrm{id}_{\overline{\mathcal{B}}})(\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(\mathbb{1}_{\mathcal{A}})) = \mathbb{1}_{\overline{\mathcal{B}}} \quad .$$

On any n consecutive sites a state ω is then defined by

$$\omega(A_1 \otimes \cdots A_n) = \underbrace{\Phi \otimes \cdots \Phi}_{n+1} \left(\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(A_1) \otimes \cdots \mathbb{F}(A_n) \otimes \mathbb{1}_{\mathcal{B}}\right) \quad .$$

Again, the compatibility conditions ensure that the hierarchy of functionals thus defined for different n determines a translation invariant state on $\mathcal{A}_{\mathbb{Z}}$. Any state, which can be obtained in this way will be called a **valence bond** state. The construction can be visualized as follows

$$\begin{array}{ccccccccccc}
 & & \mathcal{A} & \otimes & \mathcal{A} & \otimes & \mathcal{A} & \otimes\cdots\otimes & \mathcal{A} & & \\
 & & \downarrow\mathbb{F} & & \downarrow\mathbb{F} & & \downarrow\mathbb{F} & & \downarrow\mathbb{F} & & \\
\mathbb{1}_{\overline{\mathcal{B}}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes\cdots\otimes & \mathcal{B}\otimes\overline{\mathcal{B}} & \otimes & \mathbb{1}_{\mathcal{B}} \\
 & \downarrow\Phi & & \downarrow\Phi & & \downarrow\Phi & & & & \downarrow\Phi & \\
 & \mathbb{C} & & \mathbb{C} & \otimes & \mathbb{C} & & \otimes\cdots\otimes & & \mathbb{C} & \equiv \mathbb{C}
\end{array}$$

The connection to the class of C*-finitely correlated states is made in the following Proposition.

2.7 Proposition. *Every C*-finitely correlated state is a valence bond state and conversely. Moreover, in the representation of a valence bond state we may take* $\mathcal{B} \cong \overline{\mathcal{B}} \cong \mathcal{M}_k$, *and* Φ *to be a pure state with faithful restriction to either factor.*

Proof : Given a valence bond state, we define

$$\mathbb{E}(A \otimes B) = (\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(A) \otimes B) \qquad \text{and} \qquad \rho(B) = \Phi(\mathbb{1}_{\overline{\mathcal{B}}} \otimes B) \quad .$$

Then the compatibility conditions for $\mathbb{F}$ and Φ become those for $\mathbb{E}$ and ρ, and one checks by induction on n, that ω is generated by $\mathbb{E}, \rho$. The converse and remaining

statements will be shown using dilation theory in sect.4 ∎

In spite of this equivalence each of the two representations has its own merits. For discussing states on the chain we found the formalism involving the map $\mathbb{E}$ far more useful. For example, the computation of correlation functions and their cluster properties, for which the valence bond picture suggests a rather involved diagrammatic technique **[5]**, is reduced to determining the spectrum of a finite matrix. On the other hand, the main virtue of the valence bond structure is that it generalizes immediately to graphs other than the one dimensional chain: an algebra $\mathcal{A}_i$ is then associated to any vertex "i" of the graph, and an algebra $\mathcal{B}_{ij}$ to each directed edge. The two basic kinds of completely positive maps are then a map $\mathbb{F}_i$ taking each $\mathcal{A}_i$ into the tensor product of the outgoing edge algebras, and "contractions" $\mathcal{B}_{ij} \otimes \mathcal{B}_{ji} \to \mathbb{C}$. With each set Λ of vertices one associates the "observable algebra" $\mathcal{A}_\Lambda = \bigotimes_{i\in\Lambda} \mathcal{A}_i$ and the "outgoing edge algebra" $\mathcal{B}_{\partial\Lambda} = \bigotimes_{i\in\Lambda, j\notin\Lambda} \mathcal{B}_{ij}$. Clearly, every state on $\mathcal{B}_{\partial\Lambda}$ is transformed via the $\mathbb{F}_i$ and the contractions into a state on $\mathcal{A}_\Lambda$. In order to get an explicit definition of a state on the infinite system out of this scheme one either has to show the existence of a unique limit state, independent of the choice of states for $\mathcal{B}_{\partial\Lambda}$ (which serve as "boundary conditions"), or one has to find and verify appropriate compatibility conditions for these states. Both problems appear to be highly non-trivial. Some results about a special case have been obtained in **[41]**. Some general answers can be given in the case of the Bethe lattice (Cayley tree) **[31]**.

3. Ergodic decompositions

It is known that the extreme points of the set $\mathcal{T}$ of translation invariant states, called "ergodic" states, are characterized by the decay of their correlation functions. For a C*-finitely correlated state the correlation functions can be given explicitly, and we shall now utilize this to obtain the ergodic decomposition of any C*-finitely correlated state. The behaviour of correlation functions of any finitely correlated state ω is determined by the map $\hat{\mathbb{E}} := \mathbb{E}_{\mathbb{1}} : \mathcal{B} \to \mathcal{B}$ through the equation

$$\omega(A_n \otimes \underbrace{\mathbb{1} \cdots \otimes \mathbb{1}}_{m-1} \otimes A_{n+m}) = (\rho \circ \mathbb{E}_{A_n}) \circ \hat{\mathbb{E}}^{(m-1)}(\mathbb{E}_{A_{n+m}}(\mathbb{1}_{\mathcal{B}})) \quad . \tag{3.1}$$

Thus determining the m-dependence of all these functions reduces to a standard task from linear algebra, namely computing all powers of the matrix $\hat{\mathbb{E}}$, e.g. by diagonalization.

3.1 Proposition. *Let ω be an C*-finitely correlated state on $\mathcal{A}_{\mathbb{Z}}$. Then the following are equivalent:*

(1) ω is extremal in the convex set $\mathcal{F}$ of C-finitely correlated states.*

(2) ω is ergodic, i.e. extremal in the convex set $\mathcal{T}$ of translation invariant states.

(3) ω is the C-finitely correlated state generated by some $(\mathbb{E}, \rho, e)$ such that $e = \mathbb{1}$ is the only eigenvector of $\hat{\mathbb{E}}$ with eigenvalue one.*

Proof : (2) $\Rightarrow$ (1) is trivial.

(3) $\Rightarrow$ (2): Consider the Jordan decomposition of $\hat{\mathbb{E}}$, i.e. $\hat{\mathbb{E}} = \sum_\lambda(\lambda P_\lambda + R_\lambda)$, where the sum runs over all eigenvalues, $P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_\lambda$, and R_λ is nilpotent with $P_\lambda R_{\lambda'} = R_{\lambda'} P_\lambda = \delta_{\lambda\lambda'} R_\lambda$. Since $\|\hat{\mathbb{E}}\| \leq 1$ we have $R_\lambda = 0$ for λ with $|\lambda| = 1$. (Otherwise there would be a vector $B \in \mathcal{B}$ such that $R_\lambda B \neq 0$ and $R_\lambda^2 B = 0$, making the sequence $\hat{\mathbb{E}}^n(B) = \lambda^n B + n\lambda^{n-1} R_\lambda B$ unbounded). Therefore we may find for every $\epsilon > 0$ convex combination coefficients $\mu_n, n \in \mathbb{N}$, such that $\|P_1 - \sum_n \mu_n \hat{\mathbb{E}}^n\| \leq \epsilon$. Since by assumption P_1 is one-dimensional, this implies the clustering condition **[16,4.3.10,4.3.11]** uniformly for all correlation functions. Hence ω is ergodic.

(1) $\Rightarrow$ (3): According to Lemma 2.5 we can choose a representation of ω with $e = \mathbb{1}$. Consider the cone $\Gamma = \{e \in \mathcal{B} \mid e \geq 0,\ \hat{\mathbb{E}}e = e\}$. Then for each $e \in \Gamma$ let ω_e be the C*-finitely correlated state generated by $\rho, \mathbb{E}, e$ We claim that e is extremal in Γ, iff there is no $e' \in \Gamma$ such that $\operatorname{supp} e' < \operatorname{supp} e$. In fact, if $\mu e \geq e' \neq 0$ and e' not proportional to e, then also $e > e'' = e - \alpha e'$ for all $\alpha \geq 0$, and by choosing the largest α consistent with $e'' \geq 0$, we obtain a non-zero $e'' \in \Gamma$, which is also dominated by e, and satisfies $\operatorname{supp} e'' < \operatorname{supp} e$. Conversely, $\operatorname{supp} e' < \operatorname{supp} e$ implies $e' \leq \mu e$ for some μ and e' not proportional to e.

If $\omega = \omega_{\mathbb{1}}$ is extremal, all states ω_e are equal as convex components of ω. Hence by taking $e \in \Gamma$ extremal, we may choose a representation of ω for which the cone Γ reduces to the single ray $\mathbb{R}^+\mathbb{1}$, i.e. $\mathbb{1}$ is the unique eigenvector with eigenvalue one. Since the adjoint of $\hat{\mathbb{E}}$ has the same spectrum, this also implies that ρ is the unique left eigenvector of $\hat{\mathbb{E}}$.

■

If 1 is a simple eigenvalue of $\hat{\mathbb{E}}$, then the same is true for the adjoint of $\hat{\mathbb{E}}$. Hence in (3) we could have demanded alternatively that up to a scalar ρ is the only element of $\mathcal{B}^*$ with $\rho \circ \hat{\mathbb{E}} = \rho$. Therefore, in the ergodic case ρ is determined by $\hat{\mathbb{E}}$, i.e. we need fewer independent data to characterize ω.

3.2 Corollary. *$\mathcal{F}$ is a face in $\mathcal{T}$, i.e. in any convex combination $\omega = \sum_i \lambda_i \omega_i$ with $\lambda_i > 0$, $\omega_i \in \mathcal{T}$, and $\omega \in \mathcal{F}$, we must have $\omega_i \in \mathcal{F}$ for all i. Moreover, all ω_i can be generated from the same $\mathbb{E}$ with different ρ, e. Every C*-finitely correlated state has a unique decomposition of this kind, such that each ω_i is also ergodic.*

Proof : It is clear from the proof of Proposition 3.1 that we may decompose ω into states ω_e generated by the same $\mathcal{B}, \mathbb{E}$, which are extremal in $\mathcal{F}$, and hence also extremal in $\mathcal{T}$. Since $\mathcal{T}$ is a simplex **[16,4.3.11]**, such a decomposition of ω is unique. Thus the ω_e span the face in $\mathcal{T}$ generated by ω.

■

The condition that the eigenvalue 1 of $\mathbb{E}$ is non-degenerate, does not exclude oscillatory behaviour of the correlation functions, which would result from further eigenvalues of modulus 1. In the Perron-Frobenius theory of Classical Markov chains the set of such eigenvalues, called the "peripheral spectrum" of $\hat{\mathbb{E}}$, is shown to be a group under multiplication. For finite dimensional $\mathcal{B}$, this implies that all such eigenvalues are roots of unity, so that almost periodic behaviour of correlation functions is excluded. The Proposition below carries this result over to the quantum case. Examples of C*-finitely correlated states, for which $\hat{\mathbb{E}}$ has roots of unity as eigenvalues, are provided by the construction in the proof of 2.6.(4): in that case, the spectra of $\hat{\tilde{\mathbb{E}}} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ and $\hat{\mathbb{E}} : \mathcal{B} \to \mathcal{B}$ for the completely positive maps $\tilde{\mathbb{E}}$ and $\mathbb{E}$ generating $\tilde{\omega}$ and ω, respectively, are related by

$$\operatorname{spec}(\hat{\tilde{\mathbb{E}}}) = \{\lambda \in \mathbb{C} \mid \lambda^p \in \operatorname{spec}(\hat{\mathbb{E}})\} \quad .$$

In particular, the p^{th} roots of unity are in the spectrum of $\hat{\tilde{\mathbb{E}}}$. The converse of this construction can be described as the breaking of translational symmetry, or the detection of Néel order in the state ω. We are then given a C*-finitely correlated state and ask whether this state can be represented as a convex combination of p-periodic states. The following Proposition shows how this symmetry breaking can be detected from the C*-finitely correlated representation of a state. We shall call a p-periodic state C*-finitely correlated, if it is C*-finitely correlated as a state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$.

3.3 Proposition. *Let ω be an ergodic C*-finitely correlated state, and choose ω to be generated by $(\mathbb{E},\rho,e)$ such that $e=\mathbb{1}_{\mathcal{B}}$ is the only fixed point of $\hat{\mathbb{E}}$, and $\mathcal{B}$ is generated as a C*-algebra by $\{\mathbb{E}_{A_1}\circ\cdots\circ\mathbb{E}_{A_n}(\mathbb{1})\}$. Then ρ is faithful, and there is a $p\in\mathbb{N}$ such that*

$$\{\lambda\in\operatorname{spec}(\hat{\mathbb{E}})\mid|\lambda|=1\}=\{\exp(\tfrac{2\pi i}{p}n)\mid n=0,\ldots p-1\}\quad .$$

Each of these eigenvalues is simple, and the corresponding eigenvector can be taken to be a unitary in $\mathcal{B}$. Moreover, $\mathcal{B}$ is a direct sum $\mathcal{B}=\bigoplus_{r=1}^{p}\mathcal{B}_r$, and $\mathbb{E}(\mathcal{A}\otimes\mathcal{B}_r)\subset\mathcal{B}_{r-1}$ with $\mathcal{B}_0\equiv\mathcal{B}_p$. ω has a unique representation as the average of p p-periodic states, which are translates of each other. These components are again C-finitely correlated.*

Proof : It is clear from 3.1.(3) that we can choose a representation as described. Since 1 is a non-degenerate eigenvalue of $\mathbb{E}$, there is a unique state ρ with $\rho\circ\hat{\mathbb{E}}=\rho$. Let s be the support projection of ρ. Then $\{x\in\mathcal{B}\mid xs=0\}$ is an invariant subspace of $\hat{\mathbb{E}}$, since $xs=0\Rightarrow\rho(\hat{\mathbb{E}}(x^*x))=\rho(x^*x)=0\Rightarrow s\hat{\mathbb{E}}(x)^*\hat{\mathbb{E}}(x)s\leq s\hat{\mathbb{E}}(x^*x)s=0$. Hence $\hat{\mathbb{E}}$ must have an invariant vector $\tilde{e}$ with $\tilde{e}s=0$, which contradicts the uniqueness of $\mathbb{1}$, unless $s=\mathbb{1}$ and ρ is faithful.

Now consider the positive semi-definite sesquilinear map

$$\beta:\mathcal{B}\times\mathcal{B}\to\mathcal{B}:\quad(x,y)\mapsto\hat{\mathbb{E}}(x^*y)-\hat{\mathbb{E}}(x)^*\hat{\mathbb{E}}(y)\quad .$$

It is easy to see **[33,23]** that $\beta(x,x)=0$ implies $\beta(y,x)=0$ for all $y\in\mathcal{B}$. Now let $u\in\mathcal{B}$ be an eigenvector of $\hat{\mathbb{E}}$, with $\hat{\mathbb{E}}u=e^{i\alpha}u$. Then $\rho(\beta(u,u))=\rho(\hat{\mathbb{E}}(u^*u)-u^*u)=0$ by invariance of ρ. Since ρ is faithful $\beta(u,u)=0$, and hence

$$\hat{\mathbb{E}}(xu)=e^{i\alpha}\hat{\mathbb{E}}(x)u\text{for all }x\in\mathcal{B}.$$

Also, u^*u is invariant under $\hat{\mathbb{E}}$, and hence a multiple of the identity, so that we can take u to be unitary. If $\hat{\mathbb{E}}v=e^{i\gamma}v$, then the above equation gives $\hat{\mathbb{E}}(uv)=e^{i(\alpha+\gamma)}uv$. Hence uv is again an eigenvector of $\hat{\mathbb{E}}$. Since $\hat{\mathbb{E}}u^*=e^{-i\alpha}u^*$ the peripheral spectrum is a (necessarily finite) group under multiplication, i.e. it consists of the p^{th} roots of unity for some $p\in\mathbb{N}$. It was already argued in the proof of Prop.3.1. that peripheral eigenvalues have diagonal Jordan blocks. Moreover, if u_1,u_2 are eigenvectors for the same $e^{i\alpha}$, $u_1^*u_2$ is invariant under $\hat{\mathbb{E}}$, hence it is a multiple of $\mathbb{1}$, and u_1 and u_2 are proportional. This proves that each peripheral eigenvalue is simple.

Let u be the eigenvector with eigenvalue $\lambda=\exp(\frac{2\pi i}{p})$. Then since $u^p=\mathbb{1}$ the spectral resolution of u is of the form $u=\sum_{r=1}^{p}\lambda^rP_r$ with $P_r^*P_{r'}=\delta_{rr'}P_r$ and $\sum_rP_r=\mathbb{1}$. The relation $\hat{\mathbb{E}}(xu^r)=\lambda^r\hat{\mathbb{E}}(x)u^r$ then becomes $\hat{\mathbb{E}}(xP_r)=\hat{\mathbb{E}}(x)P_{r-1}$. Now let $\mathcal{B}_r=P_r\mathcal{B}P_r$, and let $0\leq B_r\in\mathcal{B}_r$, and $0\leq A\in\mathcal{A}$. Then $\mathbb{E}(A\otimes B_r)\leq\|A\|\hat{\mathbb{E}}(B_r)=\|A\|\hat{\mathbb{E}}(P_rB_rP_r)=\|A\|P_{r-1}\hat{\mathbb{E}}(B_r)P_{r-1}\leq\|A\|\cdot\|B_r\|P_{r-1}$. Thus $\mathbb{E}(A\otimes B_r)\in\mathcal{B}_{r-1}$, and this result extends by linearity and continuity to all of $\mathcal{A}\otimes\mathcal{B}_r$. It follows that the algebra $\tilde{\mathcal{B}}=\bigoplus_r\mathcal{B}_r$ is invariant under all operators $\mathbb{E}_A$, and contains $\mathbb{1}$, so that by our minimality assumption $\tilde{\mathcal{B}}=\mathcal{B}$. Clearly, the

p^{th} iterate of $\mathbb{E}^{(p)}:\mathcal{A}^p\otimes\mathcal{B}\to\mathcal{B}$ takes each of the subalgebras $\mathcal{B}_r$ into itself. The restriction of $\mathbb{E}^p$ to $\mathcal{B}^r$ therefore defines a finitely correlated state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$. It is easy to check that the resulting p states are translates of each other, and that their average is ω. The uniqueness of this decomposition follows from the uniqueness of ergodic decompositions, applied to the chain $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$.

■

Combining 3.2 and 3.3 we can summarize the results of this section as follows:

3.4 Corollary. *Every C*-finitely correlated state has a unique decomposition as a finite convex combination of extremal periodic states. These periodic components are again C*-finitely correlated.*

It should be noted that unlike $\mathcal{T}$, or the set of p-periodic states with fixed p, the set of all periodic states is not *weakly compact (it is dense in the whole state space), so it is not a priori clear that it has an abundance of extreme points. It is Proposition 3.3, which provides a criterion for the impossibility of decomposing a state into other states of larger period. Together with 2.6.(5) we have therefore shown that the *weakly closed convex hull of the extremal periodic states is dense in $\mathcal{T}$. We shall later study a set of C*-finitely correlated states, which are even pure as states on $\mathcal{A}_{\mathbb{Z}}$.

4. Dilation theory and purely generated states

The aim of this section is to reduce general C*-finitely correlated states to a particularly simple form, which will then be studied in more detail in the following sections. As a motivation, consider a C*-finitely correlated state generated by $(\mathbb{E}, \rho, \mathbb{1}_{\mathcal{B}})$, and suppose that $\mathbb{E}$ can be decomposed into a finite sum $\mathbb{E} = \sum_{x\in X} \mathbb{E}_x$ such that each $\mathbb{E}_x : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ is completely positive. Then with $\mathbb{E}_{x,A}(B) \equiv \mathbb{E}_x(A \otimes B)$ we can define for all $i < j \in \mathbb{Z}$, and $x_i, \ldots x_j \in X$ a positive linear functional $\omega_{i,j}[x_i, \ldots x_j]$ on $\mathcal{A}_{\mathbb{Z}}$, such that for $n, m > 0$

$$\omega_{i,j}[x_i, \ldots x_j]\Big((A_{i-n} \otimes \cdots A_{i-1}) \otimes (A_i \otimes \cdots A_j) \otimes (A_{j+1} \otimes \cdots A_{j+m})\Big)$$
$$= \rho \circ (\mathbb{E}_{A_{i-n}} \circ \ldots \circ \mathbb{E}_{A_{i-1}}) \circ (\mathbb{E}_{x_i, A_i} \circ \cdots \circ \mathbb{E}_{x_j, A_j}) \circ (\mathbb{E}_{A_{j+1}} \circ \ldots \circ \mathbb{E}_{A_{j+m}})(\mathbb{1}).$$

Clearly, the sum of these functional over all choices of $x_i, \ldots x_j$ is ω. The normalization factors of these functionals define a cylinder measure $\mathbb{P}$ on the set $X^{\mathbb{Z}}$ of "paths" of a process over discrete "time" $\mathbb{Z}$ with state space X, i.e. with

$$Z(x_i, \ldots x_j) = \{\xi \in X^{\mathbb{Z}} \mid \xi_t = x_t \text{ for } t = i, \ldots j\}$$

we have

$$\omega_{i,j}[x_i, \ldots x_j](\mathbb{1}) = \mathbb{P}(Z(x_i, \ldots x_j)) \quad .$$

By increasing the interval $\{i, \ldots, j\}$ we obtain finer and finer decompositions of the state ω. Using the theory of liftings **[37]** one can show that one can assign to each path $\xi \in X^{\mathbb{Z}}$ a state $\Omega[\xi]$ on $\mathcal{A}_{\mathbb{Z}}$, such that $\xi \mapsto \Omega[\xi](A)$ is cylinder measurable for each $A \in \mathcal{A}_{\mathbb{Z}}$, and

$$\omega_{i,j}[x_i, \ldots x_j](A) = \int_{\xi \in Z(x_i, \ldots x_j)} \mathbb{P}(d\xi)\Omega[\xi](A) \quad .$$

In particular, $\omega = \int \mathbb{P}(d\xi)\Omega[\xi]$.

We can view the above construction as the introduction of a new set of "observables" to the system: in the refined description we can compute probabilities for the variables of the stochastic process $(\xi_i)_{i\in\mathbb{Z}}$ in addition to those of the original chain. A more straightforward way to introduce this refinement is to simply enlarge the one-site algebra $\mathcal{A}$, i.e. to use instead of $\mathcal{A}$ the algebra $\tilde{\mathcal{A}} := \mathcal{A} \otimes \mathcal{C}(X)$. A C*-finitely correlated state $\tilde{\omega}$ on the chain $\tilde{\mathcal{A}}_{\mathbb{Z}}$ is then generated by the completely positive map

$$\tilde{\mathbb{E}} : \tilde{\mathcal{A}} \otimes \mathcal{B} \to \mathcal{B} \; : ((A \otimes f) \otimes B) \mapsto \sum_{x\in X} f(x)\mathbb{E}_x(A \otimes B) \quad ,$$

and the same state ρ. Since $\tilde{\mathcal{A}}_{\mathbb{Z}} = (\mathcal{A} \otimes \mathcal{C}(X))_{\mathbb{Z}} \equiv \mathcal{A}_{\mathbb{Z}} \otimes \mathcal{C}(X)_{\mathbb{Z}} \equiv \mathcal{A}_{\mathbb{Z}} \otimes \mathcal{C}(X^{\mathbb{Z}})$, the restiction of $\tilde{\omega}$ to $\mathcal{C}(X^{\mathbb{Z}})$ defines a probability measure on $X^{\mathbb{Z}}$, which is just

the $\mathbb{P}$ defined above. The integral decomposition of ω now simply becomes the direct integral decomposition of a state on a C*-algebra of the form $\hat{\mathcal{A}} \otimes \mathcal{C}(\hat{X})$ for a compact space $\hat{X}$.

We shall now study the relation between decompositions of $\mathbb{E}$ and possible enlargements of the one-site algebra more systematically. The principal tool for this is the Stinespring dilation for completely positive maps **[52]**. We state it here in a form appropriate for our purpose.

4.1 Lemma. *Let $\mathcal{A}$ be a C*-algebra with unit, and let $\mathbb{E} : \mathcal{A} \otimes \mathcal{M}_k \to \mathcal{M}_k$ be a unit preserving completely positive map. Then there is a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, and an isometry $V : \mathbb{C}^k \to \mathcal{H} \otimes \mathbb{C}^k$ such that $\mathbb{E}(A \otimes B) = V^*(\pi(A) \otimes B)V$, and $\{\pi(A) \otimes BV\varphi \mid A \in \mathcal{A}, B \in \mathcal{M}_k, \varphi \in \mathbb{C}^k\}$ spans a dense subspace of $\mathcal{H} \otimes \mathbb{C}^k$. $\mathcal{H}$, π, and V are unique up to unitary transformation.*

Moreover, there is a one-to-one correspondence between linear operators $\mathbb{F} : \mathcal{A} \otimes \mathcal{M}_k \to \mathcal{M}_k$ such that both $\mathbb{F}$ and $\mathbb{E} - \mathbb{F}$ are completely positive, and operators $F \in \mathcal{B}(\mathcal{H})$ with $0 \le F \le \mathbb{1}$ commuting with $\pi(\mathcal{A})$. This correspondence is determined by the relation

$$\mathbb{F}(A \otimes B) = V^*(F \otimes \mathbb{1})(\pi(A) \otimes B)V \quad .$$

Thus all possible decompositions $\mathbb{E} = \sum_{x\in X} \mathbb{E}_x$ are parametrized by partitions of unity in the commutant $\pi(\mathcal{A})' \subset \mathcal{B}(\mathcal{H})$, i.e. the decompositions $\mathbb{1} = \sum_{x\in X} F_x$ with $F_x \ge 0$ commuting with $\pi(\mathcal{A})$. The case of indecomposable $\mathbb{E}$ will play a special rôle:

4.2 Definition. *A completely positive map is called* **pure**, *if it cannot be written as the sum of two completely positive maps, which are not proportional to itself.*
A C-finitely correlated state ω on $\mathcal{A}_{\mathbb{Z}}$ is called* **purely generated**, *if it is generated by a pure map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$.*

Pure states in the usual sense are pure maps from an algebra into the one-dimensional algebra $\mathbb{C}$ in the sense of this definition. We note that, unlike for states, the pure unit preserving completely positive maps are in general only a small subclass of the extremal unit preserving completely positive maps.

4.3 Proposition. *Let $\mathcal{A}$ be a finite dimensional C*-algebra, $\mathcal{B} = \mathcal{M}_k$, and let ω be the state generated by $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, and ρ, and assume that ρ and the one-site restriction of ω are faithful on $\mathcal{A}$.*

(1) Then ω is purely generated if and only if there are $d, k \in \mathbb{N}$ such that up to isomorphisms $\mathcal{A} = \mathcal{M}_d$, $\mathcal{B} = \mathcal{M}_k$, and $\mathbb{E}(A \otimes B) = V^*(A \otimes B)V$ for some isometry $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$.

(2) *If $\mathcal{B} = \mathcal{M}_k$, then there is a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ on a finite dimensional Hilbert space, and a state $\tilde{\omega}$ on the chain $\mathcal{B}(\mathcal{H})_{\mathbb{Z}}$ generated by a pure map $\tilde{\mathbb{E}} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B} \to \mathcal{B}$ such that $\tilde{\mathbb{E}}(\mathbb{1} \otimes B) = \mathbb{E}(\mathbb{1} \otimes B)$ for all $B \in \mathcal{B}$, and*

$$\omega(A_i \otimes \cdots A_{i+n}) = \tilde{\omega}(\pi(A_i) \otimes \cdots \pi(A_{i+n})) \quad .$$

Proof : (1) It is clear from 4.1 that a map $\mathbb{E}$ of the given form is pure. If $\mathbb{F} : \bigoplus_i \mathcal{A}_i \to \mathcal{B}$ is a completely positive map, it has a direct sum decomposition $\mathbb{F} = \bigoplus_i \mathbb{F}_i$. Hence such a map can be pure only if just one of the components is non- zero. The pure map $\mathbb{E}$ generating ω is therefore supported by only one direct summand $\tilde{\mathcal{A}} \otimes \tilde{\mathcal{B}} \subset \mathcal{A} \otimes \mathcal{B}$, where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are irreducible summands of $\mathcal{A}$ and $\mathcal{B}$.

Since $A \mapsto \rho(\mathbb{E}(A \otimes \mathbb{1}))$ and $B \mapsto \rho(\mathbb{E}(\mathbb{1} \otimes B)) = \rho(B)$ vanish unless $A \in \tilde{\mathcal{A}}$, $B \in \tilde{\mathcal{B}}$, the faithfulness of these states implies that $\mathcal{A} = \tilde{\mathcal{A}}$ and $\mathcal{B} = \tilde{\mathcal{B}}$. Hence $\mathcal{A} = \mathcal{M}_d$ and $\mathcal{B} = \mathcal{M}_k$ for some $d, k \in \mathbb{N}$. The dilation theorem 4.1 then produces a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, which must be irreducible by purity of $\mathbb{E}$. Hence π is an isomorphism, and we may identify $\mathcal{A}$ with $\mathcal{B}(\mathcal{H})$.

(2) Apply Lemma 4.1 to $\mathbb{E}$. Since $A \mapsto \rho(V^*(\pi(A) \otimes \mathbb{1})V)$ is assumed to be faithful, π must also be faithful. Define $\tilde{\mathbb{E}} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_k \to \mathcal{M}_k \;: (A \otimes B) \mapsto V^*(A \otimes B)V$. Then the equations stated in the Proposition are evident, and the purity of $\tilde{\mathbb{E}}$ can be seen directly from the form of its Stinespring dilation.

■

Note that by Lemma 2.5 the condition $\mathcal{B} = \mathcal{M}_k$ is not a restriction on the state ω. Since π in this Proposition is an isomorphism, we may therefore say that every C*-finitely correlated state arises from a purely generated state by restriction to a subalgebra of the one-site algebra. In classical probability, when observable algebras are thought of as being generated by random variables, a subalgebra consists of some set of functions of the basic random variables. A more general relation between two classical systems would allow for stochasticity also in these functions. In the non-commutative setting this would mean the replacement of the representation π by a general completely positive map. The following Proposition makes use of this idea. It underlines even more the rôle played by the map $\hat{\mathbb{E}}$: with each completely positive unit preserving map $\hat{\mathbb{E}} : \mathcal{B} \to \mathcal{B}$ we associate a "universal" spin chain with finitely correlated state $\tilde{\omega}$, of which all states generated from completely positive maps with the same $\hat{\mathbb{E}}$ are "stochastic functions".

4.4 Proposition. *Let $\mathcal{B} = \mathcal{M}_k$, and let $\hat{\mathbb{E}} : \mathcal{B} \to \mathcal{B}$ be a unit preserving completely positive map. Call a completely positive map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ compatible with $\hat{\mathbb{E}}$, if $\mathbb{E}(\mathbb{1}_{\mathcal{A}} \otimes B) = \hat{\mathbb{E}}(B)$ for all $B \in \mathcal{B}$. Then there is a C*-algebra $\mathcal{A}_0$, and a pure map $\mathbb{E}_0 : \mathcal{A}_0 \otimes \mathcal{B} \to \mathcal{B}$ compatible with $\hat{\mathbb{E}}$ such that every other map compatible with $\hat{\mathbb{E}}$ is of the form*

$$\mathbb{E}(A \otimes B) = \mathbb{E}_0(\mathbb{F}(A) \otimes B)$$

for some unit preserving completely positive map $\mathbb{F} : \mathcal{A} \to \mathcal{A}_0$. The states generated by $\mathbb{E}$ and $\mathbb{E}_0$ are then related by

$$\omega(A_i \otimes \cdots A_{i+n}) = \omega_0(\mathbb{F}(A_i) \otimes \cdots \mathbb{F}(A_{i+n})) \quad .$$

Proof : Consider the dilation of any map $\mathbb{E}$ compatible with $\hat{\mathbb{E}}$, i.e. a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and an isometry $V : \mathcal{K} \equiv \mathbb{C}^k \to \mathcal{H} \otimes \mathcal{K}$ as in 4.1. Consider the subspace of $\mathcal{H} \otimes \mathcal{K}$ generated by all vectors of the form $(\mathbb{1}_{\mathcal{A}} \otimes B)V\varphi$ for arbitrary $B \in \mathcal{B}$, and $\varphi \in \mathcal{K}$. Since all matrices $B \in \mathcal{B}$ are allowed here, it is clear that this subspace is of the form $\mathcal{H}_0 \otimes \mathcal{K}$ with $\mathcal{H}_0 \subset \mathcal{H}$. We denote the injection of $\mathcal{H}_0$ into $\mathcal{H}$ by J_0, and define $\mathcal{A}_0 = \mathcal{B}(\mathcal{H}_0)$, and $\mathbb{F} : \mathcal{A} \to \mathcal{A}_0 \;: A \mapsto J_0^*\pi(A)J_0$, $V_0 := (J_0 \otimes \mathbb{1}) \circ V : \mathcal{K} \to \mathcal{H}_0 \otimes \mathcal{K}$, and $\mathbb{E}_0(X) = V_0^*XV$. Then the stated relations between $\mathbb{E}$ and $\mathbb{E}_0$, and between ω and ω_0 are obviously true, and ω_0 is purely generated. What is left to check is merely that $\mathbb{E}_0$ is canonically associated with $\hat{\mathbb{E}}$, and does not depend on the map $\mathbb{E}$ from which it was constructed. Since $\hat{\mathbb{E}}(B) = V_0^*(\mathbb{1} \otimes B)V_0$, and $(\mathbb{1} \otimes \mathcal{B})V\mathcal{K}$ spans $\mathcal{H}_0 \otimes \mathcal{K}$ by construction, this follows readily from the uniqueness statement in 4.1, applied with one-dimensional algebra $\mathcal{A}$.

■

Since Proposition 3.1.(3) gives a criterion for the ergodicity of ω, which depends only on $\hat{\mathbb{E}}$, it is clear that the purely generated states $\tilde{\omega}, \omega_0$ associated with ω by Propositions 4.3 and 4.4 will be ergodic, whenever ω is. Similarly, if translation symmetry is not broken in ω, i.e. if the peripheral spectrum of $\hat{\mathbb{E}}$ consists only of the simple eigenvalue 1, the same will be the case for $\tilde{\omega}, \omega_0$. We therefore arrive to the following procedure for studying a general C*-finitely correlated state: by applying 3.4 we first decompose the state into its unique extremal periodic components, which are C*-finitely correlated states with the additional property that $\hat{\mathbb{E}}^n$ converges to $\hat{\mathbb{E}}^{\infty}(B) = \rho(B)\mathbb{1}_{\mathcal{B}}$ exponentially fast. Then, by applying 4.3 or 4.4 we associate with each component a purely generated state with the same property. Thus purely generated states with strictly contracting $\hat{\mathbb{E}}$ are the basic building blocks for all C*-finitely correlated states, and will be studied in detail in the following two sections.

Using Lemma 4.1 we can now give a simple proof of the remainder of 2.7.

Proof of 2.7: It remains to be shown that every C*-finitely correlated state admits a valence bond representation with the special properties listed in the Proposition. Let ω be the state generated by $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, $\rho : \mathcal{B} \to \mathbb{C}$ and $\mathbb{1} \in \mathcal{B}$. By Lemma 2.5 we may assume that $\mathcal{B} = \mathcal{B}(\mathcal{K})$ is the algebra of operators on a k-dimensional Hilbert space $\mathcal{K}$, and that ρ is faithful. We can therefore write $\rho(B) = \sum_\alpha \rho_\alpha \langle \chi_\alpha, \; B\chi_\alpha \rangle$ for some orthonormal basis $\{\chi_\alpha\}_{\alpha=1}^k \subset \mathcal{K}$. Let $\mathcal{H}, V$ and π be as described in Lemma 4.1. We shall then define the objects in the valence bond construction as follows: $\overline{\mathcal{B}} = \mathcal{B}(\overline{\mathcal{K}})$ will be the algebra of operators on the conjugate Hilbert space $\overline{\mathcal{K}}$, i.e. on a space of the same dimension k as $\mathcal{K}$, which is connected with $\mathcal{K}$ via some anti-unitary operator $\chi \mapsto \overline{\chi}$. The state $\Phi : \overline{\mathcal{B}} \otimes \mathcal{B} \to \mathbb{C}$ will be pure, and its restriction to $\mathcal{B}$ will be just the faithful state ρ. We set

$$\Phi(X) = \langle \varphi,\ X\varphi\rangle \quad \text{with} \quad \varphi = \sum_\alpha \sqrt{\rho_\alpha}\, \overline{\chi_\alpha} \otimes \chi_\alpha \quad \in \overline{\mathcal{K}} \otimes \mathcal{K} \quad .$$

(Thus $B \mapsto \mathbb{1}_{\overline{\mathcal{B}}} \otimes B$ is just the GNS-representation of $\mathcal{B}, \rho$ with cyclic vector φ). The map $\mathbb{F}$ is defined in terms of its Stinespring dilation $\hat{V} : \overline{\mathcal{K}} \otimes \mathcal{K} \to \mathcal{H}$ by

$$\mathbb{F}(A) = \hat{V}^* \pi(A) \hat{V} \quad \text{with} \quad \langle \psi, \hat{V}\ \chi \otimes \overline{\chi'}\rangle = \langle \psi \otimes (\rho^{-1/2}\chi'),\ V\chi\rangle \quad .$$

In order to complete the proof we have to check the compatibility conditions for $\mathbb{F}$ and Φ, which ensure that $\mathbb{F}, \Phi$ define a valence bond state, as well as the two equations used in the proof of the trivial direction to show that this valence bond state coincides with ω. One of the latter equations, namely $\rho(B) = \Phi(\mathbb{1}_{\overline{\mathcal{B}}} \otimes B)$ has already been noted above. We check equation $\mathbb{E}(A \otimes B) = (\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(A) \otimes B)$ by taking matrix elements, using a basis $\{\psi_\mu\}_{\mu=1}^d \subset \mathcal{H}$:

$$\begin{aligned}
&\langle \chi,\ (\mathrm{id}_{\mathcal{B}} \otimes \Phi)(\mathbb{F}(A) \otimes B)\chi'\rangle \\
&\quad = \sum_{\alpha,\beta} \sqrt{\rho_\alpha \rho_\beta}\langle \chi \otimes \overline{\chi_\alpha} \otimes \chi_\alpha,\ (\mathbb{F}(A) \otimes B)\chi' \otimes \overline{\chi_\beta} \otimes \chi_\beta\rangle \\
&\quad = \sum_{\alpha,\beta} \sqrt{\rho_\alpha \rho_\beta}\langle \hat{V}\chi \otimes \overline{\chi_\alpha}, \pi(A)\hat{V}\chi' \otimes \overline{\chi_\beta}\rangle\langle \chi_\alpha,\ B\chi_\beta\rangle \\
&\quad = \sum_{\alpha,\beta,\mu,\nu} \sqrt{\rho_\alpha \rho_\beta}\langle \hat{V}\chi \otimes \overline{\chi_\alpha},\ \psi_\mu\rangle\langle \psi_\mu,\ \pi(A)\psi_\nu\rangle\langle \psi_\nu,\ \hat{V}\chi' \otimes \overline{\chi_\beta}\rangle\langle \chi_\alpha, B\chi_\beta\rangle \\
&\quad = \sum_{\alpha,\beta,\mu,\nu} \langle V\chi,\ \psi_\mu \otimes \chi_\alpha\rangle\langle \psi_\mu \otimes \chi_\alpha,\ (\pi(A) \otimes B)\psi_\nu \otimes \chi_\beta\rangle\langle \psi_\nu \otimes \chi_\beta,\ V\chi'\rangle \\
&\quad = \langle V\chi,\ (\pi(A) \otimes B)V\chi'\rangle = \langle \chi, \mathbb{E}(A \otimes B)\chi'\rangle \quad .
\end{aligned}$$

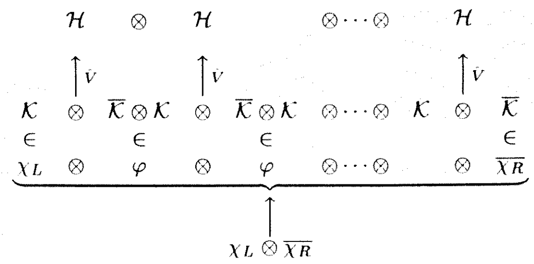
This immediately implies the compatibility condition for $\mathbb{1}_{\mathcal{B}}$. To demonstrate the other condition we proceed similarly:

$$\begin{aligned}
&\langle \chi, (\Phi \otimes \mathrm{id}_{\overline{\mathcal{B}}})(\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(\mathbb{1}))\,\chi'\rangle \\
&\quad = \sum_{\alpha,\beta} \sqrt{\rho_\alpha \rho_\beta}\langle \overline{\chi_\alpha} \otimes \chi_\alpha \otimes \chi,\ (\mathbb{1}_{\overline{\mathcal{B}}} \otimes \mathbb{F}(\mathbb{1}))\,\overline{\chi_\beta} \otimes \chi_\beta \otimes \chi'\rangle \\
&\quad = \sum_\alpha \rho_\alpha\langle \chi_\alpha \otimes \chi,\ \mathbb{F}(\mathbb{1})\ \chi_\alpha \otimes \chi'\rangle \\
&\quad = \sum_{\alpha,\mu,\nu} \rho_\alpha\langle \hat{V}\chi_\alpha \otimes \chi, \psi_\mu\rangle\langle \psi_\mu, \mathbb{1}_{\overline{\mathcal{B}}}\psi_\nu\rangle\langle \psi_\mu, \hat{V}\chi_\alpha \otimes \chi'\rangle \\
&\quad = \sum_{\alpha,\mu} \rho_\alpha\langle V\chi_\alpha, \psi_\mu \otimes \rho^{-1/2}\overline{\chi}\rangle\langle \psi_\mu \otimes \rho^{-1/2}\overline{\chi'},\ V\chi_\alpha\rangle \\
&\quad = \mathrm{tr}\big(\rho\mathbb{E}(\mathbb{1} \otimes (\rho^{-1/2}|\overline{\chi}\rangle\langle\overline{\chi'}|\rho^{-1/2}))\big) \\
&\quad = \mathrm{tr}(\rho\rho^{-1/2}|\overline{\chi}\rangle\langle\overline{\chi'}|\rho^{-1/2}) \\
&\quad = \langle \overline{\chi'}, \overline{\chi}\rangle = \langle \chi,\ \chi'\rangle
\end{aligned}$$

by the compatibility condition for $\mathbb{E}$ and ρ.

■

It is clear from this proof that for purely generated states, i.e. when the representation π is irreducible, the completely positive map $\mathbb{F}$ will also be pure. The scheme for defining valence bond states can then be transformed into a scheme for maps between Hilbert spaces, with $\mathbb{F}$ replaced by $\hat{V}$, Φ replaced by the map $\lambda \in \mathbb{C} \mapsto \lambda \cdot \varphi \in \overline{\mathcal{K}} \otimes \mathcal{K}$, and all arrows are reversed.



The map $\Gamma_n : \mathcal{K} \otimes \overline{\mathcal{K}} \to \mathcal{H}^{\otimes n}$ depicted in this diagram will play an important part in the next two sections. In the literature **[7,18,17,19,20,21]** valence bond states have usually been discussed in terms of the vectors $\Gamma_n(\chi_L \otimes \overline{\chi_R})$. This approach has the disadvantage that it yields a state on the infinite chain only in the limit $n \to \infty$. This limit need not exist, i.e. there may be different accumulation points of the sequence of n-particle states, depending on the choice of $\chi_{L,n}$, and $\chi_{R,n}$. In contrast, we can work with an explicit expression for (the n-site restriction) of the state ω from the beginning, and even in the non-ergodic situation, we have an explicit parametrization of the translation invariant limit points by the 1-eigenspace of $\hat{\mathbb{E}}$.

5. Ground state property of purely generated states

In this section we shall begin a more detailed study of the states, which were identified as the basic building blocks for all C*-finitely correlated states in section 4, namely the purely generated states which cannot be further decomposed into periodic states. By Proposition 4.3.(2) we can therefore take $\mathcal{A} = \mathcal{M}_d$ and $\mathcal{B} = \mathcal{M}_k$ as the algebras of $d \times d$- and $k \times k$-matrices, and take them to be represented on Hilbert spaces $\mathcal{H}, \mathcal{K}$ of dimensions d, k, respectively. Moreover, the pure map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ is of the form $\mathbb{E}(A \otimes B) = V^*(A \otimes B)V$ for some isometry $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$. The property that translation symmetry is not broken, or, equivalently, that $\hat{\mathbb{E}}$ has trivial peripheral spectrum, can be expressed as

$$\lim_{n\to\infty} \hat{\mathbb{E}}^n(B) \equiv \hat{\mathbb{E}}^\infty(B) = \mathrm{tr}(\rho B) \cdot \mathbb{1}_{\mathcal{B}} \tag{5.1}$$

for all $B \in \mathcal{B}$, where ρ is the non-singular density matrix invariant under $\hat{\mathbb{E}}$. (Here we abuse notation, writing $\rho(B) = \mathrm{tr}(\rho B)$.) Note that $\mathbb{E}$, and hence ρ are both determined by V, so the state ω is completely specified by this isometry.

Before entering into the study of this manifold states, it may be useful to give a rough estimate of its dimension. For fixed d, k we have to study the set of isometries $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$. Starting from the given isometry V_0 we get all others in the form $V = UV_0U'$ with unitaries $U \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^k)$, and $U' \in \mathcal{M}_k$. The transformation $V = (\mathbb{1} \otimes U'^*)V_0U'$ corresponds to a change of basis in $\mathbb{C}^k$, hence does not change ω. Thus we only have to consider $V = UV_0$. Then $U_1V_0 = U_2V_0$ iff $U_1^*U_2V_0 = V_0$, i.e. if the projection $V_0V_0^*$ reduces $U_1^*U_2$, and $U_1^*U_2$ is determined by an arbitrary unitary operator in the complement of $V_0\mathbb{C}^k$. Since the unitary group in $\mathbb{C}^d \otimes \mathbb{C}^k$ is a manifold of dimension d^2k^2, and the unitaries U yielding the same isometry are parametrized by the unitary group on a $(dk - k)$-dimensional space, we find a manifold of isometries of dimension $k^2(2d-1)$. From this we have to subtract one, since isometries differing by a phase yield the same $\mathbb{E}$. For example, the state of [5], which is also studied in sect.7.4 below (d=3,k=2), is therefore embedded into a $4 \cdot 5 - 1 = 19$ dimensional manifold.

It will be convenient to choose bases $\{\psi_\mu\}_{\mu=1}^d \subset \mathcal{H}$ and $\{\chi_\alpha\}_{\alpha=1}^k \subset \mathcal{K}$. This determines matrices $v(\mu) \in \mathcal{B}$ such that

$$V\chi = \sum_\mu \psi_\mu \otimes (v(\mu)^*\chi) \quad . \tag{5.2}$$

In a more basis-free spirit we could also define a linear map $\tilde{v} : \mathcal{H} \to \mathcal{B}$ by $\langle \chi, \tilde{v}(\psi)\chi' \rangle = \langle V\chi, \psi \otimes \chi' \rangle$, so that $v(\mu) = \tilde{v}(\psi_\mu)$. However, some of the equations become more transparent in a fixed basis. The following are obtained by considering the general matrix element $\langle \chi, \mathbb{E}(A \otimes B)\chi' \rangle$.

$$\mathbb{E}_A(B) \equiv \mathbb{E}(A \otimes B) = \sum_{\mu,\nu} \langle \psi_\mu, A\psi_\nu \rangle v(\mu) B v(\nu)^* \tag{5.3.a}$$

$$\sum_\mu v(\mu)v(\mu)^* = \mathbb{E}(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1} \tag{5.3.b}$$

$$\sum_\mu v(\mu)^* \rho v(\mu) = \rho \tag{5.3.c}$$

$$\hat{\mathbb{E}}(B) = \sum_\mu v(\mu) B v(\mu)^* \tag{5.3.d}$$

The advantage of writing V and $\mathbb{E}$ in this form is that these formulas are easily generalized to longer segments of the chain. We merely have to iterate (5.3.a). This gives

$$\begin{aligned} \mathbb{E}^{(n)}_{A_1 \otimes \cdots A_n} &= \mathbb{E}_{A_1} \circ \cdots \circ \mathbb{E}_{A_n}(B) \\ &= \sum_{\substack{\mu_1,\ldots\mu_n \\ \nu_1,\ldots\nu_n}} \langle \psi_{\mu_1} \otimes \cdots \psi_{\mu_n}, A_1 \otimes \cdots A_n \psi_{\nu_1} \otimes \cdots \psi_{\nu_n} \rangle \\ &\qquad \times v(\mu_1) \cdots v(\mu_n) B v(\nu_n)^* \cdots v(\nu_1)^* \end{aligned} \tag{5.4}$$

This formula has exactly the same structure as (5.3.a), with $\{\psi_\mu\}_{\mu=1}^d$ replaced by the corresponding product basis $\{\psi_{\mu_1,\ldots\mu_n} = \psi_{\mu_1} \otimes \cdots \psi_{\mu_n}\} \subset \mathcal{H}^{\otimes n}$, and $v(\mu_1, \ldots \mu_n) = v(\mu_1) \cdots v(\mu_n) \in \mathcal{A}$.

Using this notation, we can give a more useful expression for the map $\Gamma_n : \mathcal{K} \otimes \overline{\mathcal{K}} \to \mathcal{H}^{\otimes n}$ described at the end of section 4. For the purposes of this section it will be better to use the natural identification of $\mathcal{K} \otimes \overline{\mathcal{K}}$ with $\mathcal{M}_k \equiv \mathcal{B}$. Then Γ_n becomes a map $\Gamma_n : \mathcal{B} \to \mathcal{H}^{\otimes n}$, with

$$\Gamma_n(B) = \sum_{\mu_1,\ldots\mu_n} \psi_{\mu_1} \otimes \cdots \psi_{\mu_n} \mathrm{tr}\Big(B\, v(\mu_n)^* \cdots v(\mu_1)^*\Big) \quad . \tag{5.5}$$

We shall only use this definition in the sequel and leave it to the reader to check that this indeed coincides with the map introduced in section 4. Note that equation (5.5) can be written simply as the corresponding expression for $n = 1$, when μ_1 is replaced by the tuple $(\mu_1, \cdots \mu_n)$. Therefore it suffices in the proof of some algebraic relations involving Γ_n to take $n = 1$. In section 4 the range of Γ_n was described as the set of valence bond vectors associated with the state ω. We shall denote this range by $\mathcal{G}_n = \Gamma_n(\mathcal{B}) \subset \mathcal{H}^{\otimes n}$, and the corresponding orthogonal projection by G_n.

This suggests that the n-step restriction of ω will be supported by G_n. Here we prove a more detailed result, giving an alternative formula for ω in terms of Γ_n, and a fixed density matrix W_∞ on $\mathcal{B}$, where $\mathcal{B}$ is considered as Hilbert space with the inner product

$$\langle A, B \rangle_\rho := \mathrm{tr}(\rho A^* B) \quad . \tag{5.6}$$

Since W_∞ is given by an invertible linear transformation, the formula below also shows that the support of the n-step restriction is, in fact, equal to G_n. The Lemma also gives a formula for matrix elements between valence bond states, which will be useful below.

5.1 Lemma.

(1) For all $A \in \mathcal{A}^{\otimes n}$

$$\omega(A) = \mathrm{tr}(\Gamma_n W_\infty \Gamma_n{}^* \cdot A) \quad , \tag{5.7}$$

where $W_\infty : \mathcal{B} \to \mathcal{B}$ is the density matrix on $(\mathcal{B}, \langle \cdot, \cdot \rangle_\rho)$ with $W_\infty(B) = \rho B \rho$.

(2) For all $A \in \mathcal{A}^{\otimes n}$, and $B, C \in \mathcal{B}$

$$\langle \Gamma_n(B),\ A\,\Gamma_n(C)\rangle = \sum_{\alpha,\beta} \langle \chi_\alpha,\ \mathbb{E}_A^{(n)}(B^* |\chi_\alpha\rangle\langle\chi_\beta| C)\chi_\beta\rangle \quad . \tag{5.8}$$

Proof : We need to prove only the case $n = 1$, from which the general case follows by substituting n-tuples for μ, ν. From equation5.3 we get

$$\begin{aligned}\omega(A) &= \sum_{\mu,\nu} \langle \psi_\mu,\ A\psi_\nu\rangle \mathrm{tr}(\rho v(\mu) v(\nu)^*) \\ &= \sum_{\mu,\nu,\alpha,\beta} \langle \psi_\mu,\ A\psi_\nu\rangle\langle \chi_\alpha, \sqrt{\rho}\, v(\mu)\chi_\beta\rangle\langle\chi_\beta, v(\nu)^* \sqrt{\rho}\,\chi_\alpha\rangle \\ &= \sum_{\alpha,\beta} \langle \Gamma_1(B_{\alpha\beta}),\ A\,\Gamma_1(B_{\alpha\beta})\rangle \quad ,\end{aligned}$$

where $B_{\alpha\beta} = |\sqrt{\rho}\,\chi_\alpha\rangle\langle\chi_\beta|$. Hence equation5.7 holds with W_∞ determined from

$$\begin{aligned}\langle C,\ W_\infty C'\rangle_\rho &= \sum_{\alpha,\beta} \langle B_{\alpha\beta},\ C'\rangle_\rho \langle C,\ B_{\alpha\beta}\rangle_\rho \\ &= \sum_{\alpha,\beta} \mathrm{tr}(\rho|\chi_\beta\rangle\langle\chi_\alpha|\sqrt{\rho}\, C')\mathrm{tr}(\rho C^* \sqrt{\rho}\,|\chi_\alpha\rangle\langle\chi_\beta|) \\ &= \mathrm{tr}(\rho C^* \sqrt{\rho}\sqrt{\rho}\, C'\rho) = \langle C,\ \rho C'\rho\rangle_\rho \quad .\end{aligned}$$

This proves (1). For part (2) we write out the traces in the definition of Γ with respect to the basis $\{\chi_\alpha\}$:

$$\begin{aligned}\langle \Gamma_1(B),\ A\,\Gamma_1(C)\rangle &= \sum_{\alpha\beta,\mu\nu} \langle\psi_\mu,\ A\psi_\nu\rangle\langle\chi_\alpha,\ v(\mu)B^*\chi_\alpha\rangle\langle\chi_\beta,\ Cv(\nu)^*\chi_\beta\rangle \\ &= \sum_{\alpha\beta} \langle \chi_\alpha,\ \mathbb{E}_A(B^*|\chi_\alpha\rangle\langle\chi_\beta|C)\chi_\beta\rangle\end{aligned}$$

■

For large n scalar products involving valence bond vectors can be evaluated by using the strict contraction property equation5.1. The following Lemma gives two basic estimates of this kind.

5.2 Lemma. *Let λ be such that $|\lambda_i| < \lambda < 1$ for all eigenvalues λ_i of $\hat{\mathbb{E}}$ different from 1. Then there is a constant c such that for all n:*

$$a(n) := \mathrm{tr}(\rho^{-1})\|\hat{\mathbb{E}}^n - \hat{\mathbb{E}}^\infty\| \le c\lambda^n \quad .$$

Moreover, we have the following estimates:

(1) For all $B, C \in \mathcal{B}$:

$$|\langle \Gamma_n(B), \Gamma_n(C)\rangle \ - \ \langle B, C\rangle_\rho| \ \le \ a(n)\ \|B\|_\rho \cdot \|C\|_\rho \quad . \tag{5.9}$$

(2) For $A \in \mathcal{A}^{\otimes m}$, $\ell, r \in \mathbb{N}$, and $B, C \in \mathcal{B}$:

$$\begin{aligned}&|\langle \Gamma_{\ell+m+r}(B),\ (\mathbb{1}_\ell \otimes A \otimes \mathbb{1}_r)\Gamma_{\ell+m+r}(C)\rangle - \omega(A)\langle B, C\rangle_\rho| \\ &\quad \le (a(\ell) + a(r))\,\|A\| \cdot \|B\|_\rho \cdot \|C\|_\rho \quad .\end{aligned} \tag{5.10}$$

Proof : (1) Applying equation5.8 with $A = \mathbb{1}$ we get

$$\langle \Gamma_n(B), \Gamma_n(C)\rangle = \sum_{\alpha,\beta} \langle \chi_\alpha, \hat{\mathbb{E}}^n(B^*|\chi_\alpha\rangle\langle\chi_\beta|C)\chi_\beta\rangle \quad .$$

Replacing $\hat{\mathbb{E}}^n$ by $\hat{\mathbb{E}}^\infty$ in the last expression we obtain

$$\sum_{\alpha,\beta} \langle\chi_\alpha, \chi_\beta\rangle \mathrm{tr}(\rho B^* |\chi_\alpha\rangle\langle\chi_\beta|C) = \mathrm{tr}(\rho B^* C) = \langle B, C\rangle_\rho \quad .$$

The difference is less than

$$\begin{aligned}&\sum_{\alpha,\beta} \|\chi_\alpha\| \cdot \|\hat{\mathbb{E}}^n - \hat{\mathbb{E}}^\infty\| \cdot \|B^*\chi_\alpha\| \cdot \|C^*\chi_\beta\| \cdot \|\chi_\beta\| \\ &\le \|\hat{\mathbb{E}}^n - \hat{\mathbb{E}}^\infty\| (\sum_\alpha \|B\chi_\alpha\|)(\sum_\beta \|C\chi_\beta\|) \\ &\le \|\hat{\mathbb{E}}^n - \hat{\mathbb{E}}^\infty\| \cdot \mathrm{tr}(\rho^{-1})\|B\|_\rho \cdot \|C\|_\rho \quad ,\end{aligned}$$

where at the last inequality we have used a special basis $\{\chi_\alpha\}$ with $\rho = \sum_\alpha \rho_\alpha |\chi_\alpha\rangle\langle\chi_\alpha|$ to obtain

$$\begin{aligned}(\sum_\alpha \|B\chi_\alpha\|)^2 &= (\sum_\alpha \|B\chi_\alpha\| \rho_\alpha^{1/2} \cdot \rho_\alpha^{-1/2})^2 \\ &\le (\sum_\alpha \rho_\alpha \langle\chi_\alpha, B^* B\chi_\alpha\rangle) \sum_\alpha \rho_\alpha^{-1} = \|B\|_\rho^2 \mathrm{tr}(\rho^{-1}) \quad .\end{aligned}$$

(2) Again by 5.8 we have

$$\langle \Gamma_{\ell+m+r}(B), (\mathbb{1}_\ell \otimes A \otimes \mathbb{1}_r)\Gamma_{\ell+m+r}(C)\rangle = \sum_{\alpha,\beta} \langle \chi_\alpha, \hat{\mathbb{E}}^\ell \mathbb{E}_A^{(m)} \hat{\mathbb{E}}^r (B^*|\chi_\alpha\rangle\langle\chi_\beta|C)\chi_\beta\rangle \quad .$$

Writing the product of the three $\mathbb{E}$-operators as

$$\hat{\mathbb{E}}^\infty \mathbb{E}_A^{(m)} \hat{\mathbb{E}}^\infty + \hat{\mathbb{E}}^\ell \mathbb{E}_A^{(m)} (\hat{\mathbb{E}}^r - \hat{\mathbb{E}}^\infty) + (\hat{\mathbb{E}}^\ell - \hat{\mathbb{E}}^\infty)\mathbb{E}_A^{(m)} \hat{\mathbb{E}}^\infty \quad ,$$

we obtain the leading term

$$\sum_\alpha \mathrm{tr}(\rho \hat{\mathbb{E}}_A^{(m)}(\mathbb{1}))\mathrm{tr}(\rho B^* |\chi_\alpha\rangle\langle\chi_\alpha|C) = \omega(A)\langle B, C\rangle_\rho$$

and two remainder terms, which are estimated exactly as in (1).

■

As a consequence of Lemma 5.2.(1), the maps Γ_n are injective for all sufficiently large n. However, the bound given does not exclude that this property holds for sporadically for some small n, but fails for some larger n' before becoming valid *universally*. The following Lemma excludes this possibility by showing a quantity to be monotone, which vanishes iff Γ_n is not injective.

5.3 Lemma. *The quantity*

$$a_-(n) = \inf \operatorname{spec}(\Gamma_n{}^*\Gamma_n) := \inf_{B\neq 0} \frac{\|\Gamma_n(B)\|^2}{\|B\|_\rho^2} \geq 1 - a(n)$$

is non-decreasing in n.

Proof : Since $(\|\Gamma_n(B)\|^2 - \|B\|_\rho^2) \geq -a(n)\|B\|_\rho^2$, it is clear that $a_-(n) \geq 1 - a(n)$. The monotonicity of a_- follows from the estimate

$$\begin{aligned}\|\Gamma_{n+1}(B)\|^2 &= \sum_{\mu_{n+1}} \sum_{\mu_1,\ldots\mu_n} |\operatorname{tr}((Bv(\mu_{n+1})^*)v(\mu_n)^* \cdots v(\mu_1)^*)|^2 \\ &\geq \sum_{\mu_{n+1}} a_-(n)\operatorname{tr}(\rho(Bv(\mu_{n+1})^*)^*(Bv(\mu_{n+1})^*)) \\ &= a_-(n)\operatorname{tr}(\rho\sum_\mu v(\mu)B^*Bv(\mu)^*) \\ &= a_-(n)\operatorname{tr}(\rho\hat{\mathbb{E}}(B^*B)) = a_-(n)\operatorname{tr}(\rho B^*B) \quad ,\end{aligned}$$

i.e. $a_-(n+1) \geq a_-(n)$.

■

5.4 Definition *The smallest $\ell \in \mathbb{N}$ such that $\Gamma_\ell : \mathcal{B} \to \mathcal{H}^{\otimes\ell}$ has rank k^2 is called the* **interaction length** *ℓ_0 of the purely generated state ω. A positive operator $h \in \mathcal{A}^{\otimes\ell}$ is called an* **interaction exposing** *ω, if $\ell > \ell_0$, and the kernel of h coincides with $\mathcal{G}_\ell = \Gamma_\ell(\mathcal{B})$. The Hamiltonian of the system is then the formal expression*

$$H = \sum_{i\in\mathbb{Z}} \alpha_i(h) \quad ,$$

where $\alpha_i(h) \in \mathcal{A}_{\{i,i+1,\ldots,i+\ell-1\}}$ is the i^{th} translate of h.

The reason for this terminology is that $\omega(h)$ represents the energy density of the Hamiltonian. By Lemma 5.1 the ℓ-step density matrix of ω has support in $\mathcal{G}_\ell$, so $\omega(h) = 0$, realizes the smallest possible energy density, and ω is a ground state in this sense. This is analogous to a state on a C*-algebra being contained in the set $\{\varphi \mid \varphi(H) = 0\}$ for some positive element H, which is usually called the face "exposed" by H. The "typical" interaction length of purely generated states can be obtained by a simple counting of dimensions: in the space of $k^2 \times d^n$-matrices the matrices of maximal rank form an open set. Therefore, we expect Γ_n to be

non-singular as soon as $k^2 \leq d^n$, i.e. we expect ℓ_0 to be the least integer with $\ell_0 \geq 2\log k/\log d$.

It is clear that if h exposes ω, the ω-expectations of the "finite size Hamiltonians"

$$H_{\{n+1,\ldots n+m\}} = \sum_{i=0}^{m-\ell} \alpha_{n+i}(h) \quad \in \mathcal{A}_{\{n+1,\ldots,n+m\}} \tag{5.11}$$

for $m > \ell$ also vanish. The kernel of $H_{\{1,\ldots m\}} \in \mathcal{A}^{\otimes m}$ is clearly equal to the intersection of the kernels of the positive operators h_k. On the other hand, since $\omega(H_{\{1,\ldots m\}}) = 0$, the support $\mathcal{G}_m$ of the m-step density matrix must be contained in the kernel of $H_{\{1,\ldots m\}}$. The following Lemma asserts that these two spaces are, in fact, equal. Hence if $h \in \mathcal{A}^{\otimes\ell}$ exposes ω, then so does $H_{\{1,\ldots m\}} \in \mathcal{A}^{\otimes m}$.

5.5 Lemma. *For all $m \geq \ell > \ell_0$*

$$\mathcal{G}_m = \bigcap_{i=0}^{m-\ell} \mathcal{H}^{\otimes i} \otimes \mathcal{G}_\ell \otimes \mathcal{H}^{\otimes(m-\ell-i)} \quad .$$

Proof : Proceeding by induction over m, beginning with the trivial statement for $m = \ell$, we have to show that $\mathcal{G}_{\ell+1} = \mathcal{G}_\ell \otimes \mathcal{H} \cap \mathcal{H} \otimes \mathcal{G}_\ell$, provided that $\ell > \ell_0$. The latter condition means that $\Gamma_{\ell-1} : \mathcal{B} \to \mathcal{G}_{\ell-1}$ is injective, i.e. that $\operatorname{tr}(Bv(\mu_1)\cdots v(\mu_{\ell-1})) = 0$ for all $(\ell-1)$-tuples $(\mu_1,\ldots\mu_{\ell-1})$ implies $B = 0$. Now the vectors $\Phi = \sum \Phi(\mu_1,\ldots\mu_{\ell+1})\psi_{\mu_1} \otimes \cdots \psi_{\mu_{\ell+1}}$ in $\mathcal{G}_\ell \cap \mathcal{H}$ are precisely those with

$$\Phi(\mu_1,\cdots\mu_{\ell+1}) = \operatorname{tr}(B(\mu_{\ell+1})v(\mu_\ell)^* \cdots v(\mu_1)^*) \quad ,$$

with $B(\mu_{\ell+1})$ an arbitrary $\mu_{\ell+1}$-dependent matrix, which is uniquely determined by Φ because Γ_ℓ is injective. The condition $\Phi \in \mathcal{H} \otimes \mathcal{G}_\ell$ can be expressed similarly with a μ_1-dependent matrix $C(\mu_1)$. Then $\Phi \in \mathcal{G}_\ell \otimes \mathcal{H} \cap \mathcal{H} \otimes \mathcal{G}_\ell$ iff

$$\begin{aligned}0 &= \operatorname{tr}(B(\mu_{\ell+1})v(\mu_\ell)^* \cdots v(\mu_1)^*) - \operatorname{tr}(C(\mu_1)v(\mu_{\ell+1})^*v(\mu_\ell)^* \cdots v(\mu_2)^*) \\ &= \operatorname{tr}(\{v(\mu_1)^*B(\mu_{\ell+1}) - C(\mu_1)v(\mu_{\ell+1})^*\}v(\mu_\ell)^* \cdots v(\mu_2)^*) \quad .\end{aligned}$$

Since this relation holds for all $(\ell-1)$-tuples $(\mu_\ell,\ldots\mu_2)$, the expression in braces must vanish for all $\mu_{\ell+1},\mu_1$. Hence using (5.3.b):

$$B(\mu) = \sum_\nu v(\nu)v(\nu)^*B(\mu) = \sum_\nu v(\nu)C(\nu)v(\mu)^* = Dv(\mu)^* \quad ,$$

with $D = \sum_\nu v(\nu)C(\nu)$. Hence $\Phi = \Gamma_{\ell+1}(D) \in \mathcal{G}_{\ell+1}$. The converse inclusion is trivial, since for given D we can take $B(\mu) = Dv(\mu)^*$ and $C(\mu) = v(\mu)^*D$.

■

This Lemma points out an interesting feature of the structure we investigate here. Given an arbitrary subspace $\mathcal{G}_\ell \subset \mathcal{H}^{\otimes\ell}$ we could take the intersection in the statement of the Lemma as a definition of a subspace $\mathcal{G}_m \subset \mathcal{H}^{\otimes m}$. Then $\mathcal{G}_m$ is the kernel of any positive $h \in \mathcal{A}^{\otimes\ell}$ with kernel $\mathcal{G}_\ell$. Obviously, the definition of "exposing interactions" depends only on these spaces, and one might try to set up a general theory of such interactions and their ground states. The problem with this is that for a generic subspace $\mathcal{G}_\ell \subset \mathcal{H}^{\otimes \ell}$ the intersection $\mathcal{G}_m$ simply becomes empty for large m. In fact, for n generic subspaces R_i of a vector space R the inequality

$$\dim(\bigcap_i R_i) \geq \sum_i \dim R_i - (n-1)\dim R$$

is an equality, whenever the right hand side is non-zero. Therefore, a naive estimate of the above intersection would be

$$\dim \mathcal{G}_m \approx (m-\ell+1)d^m \left(d^{-\ell}\dim \mathcal{G}_\ell - \frac{m-\ell}{m-\ell+1}\right) \quad ,$$

and this certainly becomes negative for large m. Thus the spaces $\mathcal{G}_\ell = \Gamma_\ell(\mathcal{B})$ are special in that these intersections stay non-empty, and it is precisely this property, which makes the existence of "exposed" states possible.

With the help of Lemma 5.5 we can now give a concise characterization of the different interactions exposing ω. Since ω, considered as a state on the chain $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$ satisfies the general assumptions of this section, we may also look for interactions $h' \in (\mathcal{A}^{\otimes p})^{\otimes \ell'} \equiv \mathcal{A}^{\otimes p\ell'}$ exposing ω. All these interactions are equivalent in the following strong sense.

5.6 Lemma. *Let $h \in \mathcal{A}^{\otimes \ell}$ be an interaction exposing ω, and let $p, \ell' \in \mathbb{N}$ with $p\ell' > \ell_0$. For $h' \in \mathcal{A}^{\otimes p\ell'}$, and $m \in \mathbb{N}$ let*

$$H'_{\{1,\dots pm\}} = \sum_{i=0}^{m-\ell'} \alpha_{pi}(h') \quad .$$

Suppose that h' is an interaction exposing ω considered as a state on $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$. Then there are constants $C_\pm$ such that for $pm \geq \ell + p - 1$ and $m \geq \ell'$

$$C_- \; H_{\{1,\dots pm\}} \leq H'_{\{1,\dots pm\}} \leq C_+ \; H_{\{1,\dots pm\}}$$

Proof : Let m_0 be the smallest m with $m \geq \ell'$ and $m \geq (\ell + p - 1)/p$. Then the Hamiltonians $H_0 \equiv H_{\{1,\dots pm_0\}}$ and $H'_0 \equiv H'_{\{1,\dots pm_0\}}$ are both defined in $\mathcal{A}_{\{1,\dots pm_0\}}$. Since both interactions expose ω, both H_0 and H'_0 have the same kernel, namely $\mathcal{G}_{pm_0}$. Hence

$$H_0 \leq \quad \|H_0\| \, (\mathbb{1} - \mathrm{G}_{pm_0}) \quad \leq \|H_0\| \, (\eta')^{-1} \; H'_0 \quad ,$$

where η' is the smallest non-zero eigenvalue of H'_0. Similarly, we obtain the estimate $H'_0 \leq \eta^{-1}\|H'_0\|$. Now for $m \geq m_0$ we have

$$H_{\{1,\dots pm\}} \leq \sum_{i=0}^{m-m_0} \alpha_{pi}(H_0) \leq m_0 \, H_{\{1,\dots pm\}}$$

$$H'_{\{1,\dots pm\}} \leq \sum_{i=0}^{m-m_0} \alpha_{pi}(H'_0) \leq (m_0 - \ell' + 1) \, H'_{\{1,\dots pm\}} \quad .$$

This estimate follows simply by inserting the definitions of H_0, H'_0, and counting how often each translate $\alpha_i(h), \alpha_{pi}(h')$ occurs in the sum in the middle. Combining the estimates we find the inequality stated in the Lemma with $C_+ = \eta^{-1}\|H'_0\|m_0/\eta$, and $C_- = \eta'\|H_0\|^{-1}(m_0 - \ell/ + 1)^{-1}$.

■

5.7 Theorem. *Let $h \in \mathcal{A}^{\otimes \ell}$ be an interaction exposing ω. Then ω is the unique state on $\mathcal{A}_{\mathbb{Z}}$ such that*

$$\omega(\alpha_i(h)) = 0$$

for all $i \in \mathbb{Z}$.

Proof : Let $\tilde\omega$ be a state with $\tilde\omega(h_i) \equiv 0$. Then for all $i \in \mathbb{Z}$ and $n \geq \ell$ the density matrix $W_{\{i+1,\dots i+n\}}$ of $\tilde\omega \mid \mathcal{A}_{\{i+1,\dots i+n\}}$ is supported by the subspace $\bigcap_{s=0} \mathcal{H}^{\otimes s} \otimes \mathcal{G}_\ell \otimes \mathcal{H}^{\otimes(n-s-\ell)}$. Hence by Lemma 5.5 $W_{\{i+1,\dots i+n\}}$ is supported by $\mathcal{G}_n$ for all i, n. Thus we have a representation $W_{\{i+1,\dots i+n\}} = \sum_s |\Gamma_n(B_s)\rangle\langle\Gamma_n(B_s)|$ with $B_s \in \mathcal{B}$, and $\sum_s \|\Gamma_n(B_s)\|^2 = 1$. For $A \in \mathcal{A}_{\{j+1,\dots j+m\}}$ we apply this representation and Lemma 5.2.(2) with sufficiently small j and sufficiently large m to obtain

$$\begin{aligned}
|\tilde\omega(A) - \omega(A)| &= |\sum_s \{\langle\Gamma_n(B_s), A\Gamma_n(B_s)\rangle - \omega(A)\langle\Gamma_n(B_s), \Gamma_n(B_s)\rangle\}| \\
&\leq (a(i-j) + a(n+i-j-m))\|A\| \sum_s \|B_s\|_\rho^2 \\
&\leq (a(i-j) + a(n+i-j-m))a_-(n)^{-1} \; \|A\| \quad .
\end{aligned}$$

For small j and large n the bracket can be made arbitrarily small.

■

We close this section with a collection of properties of ω, which are immediate consequences of the foregoing.

5.8 Proposition. *ω is a pure state on $\mathcal{A}_{\mathbb{Z}}$ with zero entropy density. The non-zero eigenvalues of the density matrix of $\omega \mid \mathcal{A}^{\otimes n}$ converge to the numbers $\{\rho_\alpha \cdot \rho_\beta\}_{\alpha,\beta=1}^k$, where $\{\rho_\alpha\}_{\alpha=1}^k$ are the eigenvalues of ρ. The limiting absolute entropy of ω is twice the entropy of ρ.*

Proof : Any convex component $\tilde\omega \leq \lambda\omega$ of ω satisfies the condition of the theorem, and is hence equal to ω. Since the n-step density matrix is supported by $\mathcal{G}_n$, which has dimension k^2 for large n, its entropy is bounded by $2\ln(k)$, so the entropy per site vanishes as $n \to \infty$. By Lemma 5.1.(1) the n-step density matrix is $\Gamma_n W_\infty \Gamma_n{}^*$, and since Γ_n becomes an isometry in the limit, we merely have to compute the eigenvalues of $W_\infty(B) = \rho B \rho$. The eigenvectors of W_∞ are $B = |\chi_\alpha\rangle\langle\chi_\beta|$, where $\{\chi_\alpha\}$ is an eigenbasis of ρ, so the eigenvalues are $\rho_\alpha\rho_\beta$. The limiting entropy of $\omega \mid \mathcal{A}^{\otimes n}$ is the entropy of $W_\infty \cong \rho \otimes \rho$.

■

For C*-finitely correlated states, which are not purely generated, the entropy density does not vanish in general. One may even find interactions with a (non-unique) C*-finitely correlated ground state having positive entropy density **[30]**. We do not know whether a C*-finitely correlated state with vanishing entropy density is necessarily purely generated.

6. The ground state energy gap

There are two natural ways of looking at the infinite sum in the formal Hamiltonian $H = \sum_{n\in\mathbb{Z}}\alpha_m(h)$. The first is to discuss only energy densities, i.e. the expectations of the individual terms in this sum. For example, in the last section we considered states, in which each term had zero expectation so that we never had to consider the convergence of the sum. Another natural approach is to consider the Hamiltonian not as an observable, but as the generator of the dynamical automorphism group $t \mapsto \tau_t \in \mathrm{Aut}\mathcal{A}_{\mathbb{Z}}$. More precisely, the generator of this group is the closure of $X \mapsto i[H,X]$, defined on strictly local operators X, and for such X only a finite number of terms in the Hamiltonian contributes to the commutator. The notion of "ground state" corresponding to the latter view of the Hamiltonian is the inequality

$$\omega(X^*[H,X]) \geq 0 \quad \text{for all local } X \in \mathcal{A}_{\mathbb{Z}} \tag{6.1}$$

This is equivalent to the positivity of the Hamiltonian H_ω, which is defined in the GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ of the state ω by

$$\pi_\omega(\tau_t(X))\Omega_\omega = e^{itH_\omega}\pi_\omega(X)\Omega_\omega \tag{6.2}$$

When $h \in \mathcal{A}^{\otimes\ell}$ is an interaction exposing ω in the sense of Definition 5.4, we have for $X \in \mathcal{A}_{\{n,\ldots n+m\}}$:

$$\omega(X^*[H,X]) = \omega(X^*[H_{\{n-\ell,\ldots n+m+\ell\}},X]) = \omega(X^*H_{\{n-\ell,\ldots n+m+\ell\}}X) \quad , \tag{6.3}$$

since for all $n \in \mathbb{Z}$ we have $\omega(X^*X\alpha_n(h)) = 0$. Therefore, the positivity of $H_{\{n-\ell,\ldots n+m+\ell\}}$ implies that the ground states considered in the previous section are also ground states in the sense of inequality (6.1). It is known **[16]** that, conversely, inequality (6.1) implies the minimum energy density propΓerty for translation invariant states.

In this section we want to investigate the existence of gaps above the ground state. Again there will be two interrelated notions of "gap". The first is to replace the positivity of $H_{\{n,\ldots n+m\}}$ by the stronger requirement that the first non-zero eigenvalue of this operator is bounded below by a constant $\gamma > 0$, independently of n and $m \geq \ell$. The second notion is to postulate that H_ω has a spectral gap, i.e. that the eigenvalue zero is isolated from the remainder of the spectrum by an interval of length γ. This is equivalent to the inequality

$$\omega(X^*[H,X]) \geq \gamma\{\omega(X^*X) - |\omega(X)|^2\} \tag{6.4}$$

for all local $X \in \mathcal{A}_{\mathbb{Z}}$. Again we can use equation (6.3) to simplify this expression, so that only the finite volume Hamiltonians $H_{\{n,\ldots n+m\}}$ appear.

The following Lemma shows that for the states under consideration a gap in the first sense implies the inequality (6.4).

6.1 Lemma. *Let $h \in \mathcal{A}^{\otimes \ell}$ be an interaction exposing the C^*-finitely correlated state ω. Suppose that for all sufficiently large $m \in \mathbb{N}$ the first non-zero eigenvalue of $H_{\{1,\dots,m\}}$ is larger than $\gamma > 0$. Then inequality (6.4) holds.*

Proof: Let $X \in \mathcal{A}_{\mathbb{Z}}$ be local. Then by translation invariance of H and ω we may assume $X \in \mathcal{A}_{\{1,\dots,m\}}$. Since neither side of inequality (6.4) changes, if we replace X by $X - \omega(X)\mathbb{1}$, we may also assume that $\omega(X) = 0$. Consider for each L the vector

$$\Psi_L = \left(\mathbb{1}_L \otimes X \otimes \mathbb{1}_L \Gamma_{L+m+L}(\mathbb{1})\right) \in \mathcal{H}^{\otimes L} \otimes \mathcal{H}^{\otimes m} \otimes \mathcal{H}^{\otimes L} \quad .$$

We abbreviate by H^L the Hamiltonian $H_{\{1-L,\dots,m+L\}}$ acting in this space, and its ground state projection by G^L. By assumption, $H^L \geq \gamma(\mathbb{1} - \mathrm{G}^L)$ for sufficiently large L. Then by Lemma 5.2.(2) we have $\|\Psi_L\|^2 = \omega(X^*X) + \mathbf{O}(a(L))$, and for an arbitrary vector $\Gamma_{2L+m}(B)$ in the range of G^L we have

$$\begin{aligned}\langle \Gamma_{2L+m}(B),\, \Psi_L\rangle &= \omega(X)\langle B, \mathbb{1}\rangle_\rho + \|B\|_\rho \cdot \mathbf{O}(a(L))\\ &= \|\Gamma_{2L+m}(B)\| \cdot \mathbf{O}(a(L)) \quad .\end{aligned}$$

Hence $\|\mathrm{G}^L \Psi_L\| = \mathbf{O}(a(L))$. Using Lemma 5.2.(2) once more we find

$$\begin{aligned}\omega(X^*[H, X]) &= \omega(X^* H_{\{1-\ell,\dots,m+\ell\}} X) = \langle \Psi_L, H_{\{1-\ell,\dots,m+\ell\}} \Psi_L\rangle - \mathbf{O}(a(L-\ell))\\ &= \langle \Psi_L, H^L \Psi_L\rangle - \mathbf{O}(a(L))\\ &\geq \gamma\langle \Psi_L, (\mathbb{1} - \mathrm{G}^L)\Psi_L\rangle - \mathbf{O}(a(L)) = \gamma\omega(X^*X) - \mathbf{O}(a(L)) \quad .\end{aligned}$$

The result follows by letting $L \to \infty$.

■

It is clear from Lemma 5.6 and equation (6.3) that if one interaction exposing ω has a non-zero gap, then all other such interactions will have the same property. The special interaction, for which we shall prove this property in Theorem 6.4 will be of the form $(\mathbb{1} - \mathrm{G}_{2p})$ for some p. The following Lemma establishes the basic estimate for ground state projections needed in the proof of 6.4.

6.2 Lemma. *For all $\ell, m, r \in \mathbb{N}$, with $m \geq \ell_0$, and $a(m), a_-(m)$ as in Lemma 5.2:*

$$\|(\mathrm{G}_{\ell+m} \otimes \mathbb{1}_r)(\mathbb{1}_\ell \otimes \mathrm{G}_{m+r}) - \mathrm{G}_{\ell+m+r}\| \leq a(m)\frac{1 + a(m)}{a_-(m)} \quad .$$

Proof: Since $\mathrm{G}_{\ell+m+r} \leq (\mathrm{G}_{\ell+m} \otimes \mathbb{1}_r)$, we can write $(\mathrm{G}_{\ell+m} \otimes \mathbb{1}_r)(\mathbb{1}_\ell \otimes \mathrm{G}_{m+r}) - \mathrm{G}_{\ell+m+r} = (\mathrm{G}_{\ell+m} \otimes \mathbb{1}_r - \mathrm{G}_{\ell+m+r})(\mathbb{1}_\ell \otimes \mathrm{G}_{m+r} - \mathrm{G}_{\ell+m+r})$. Therefore, we have to prove the following statement: for any vectors $\Phi \in \mathcal{G}_{\ell+m} \otimes \mathcal{H}^{\otimes r}$ and $\Psi \in \mathcal{H}^{\otimes \ell} \otimes \mathcal{G}_{m+r}$ such that $\Phi, \Psi \perp \mathcal{G}_{\ell+m+r}$, we have $|\langle \Phi, \Psi\rangle| \leq RHS \cdot \|\Phi\| \cdot \|\Psi\|$. We shall write all vectors in components with respect to a basis $\{\psi_\mu\}_{\mu=1}^d \subset \mathcal{H}$, grouping the $(\ell + m + r)$-tuple of indices into three tuples μ^ℓ, μ^m, μ^r of lengths ℓ, m, r, respectively. We use the abbreviation $v(\mu^m) = v(\mu_{\ell+1})v(\mu_{\ell+2}) \cdots v(\mu_{\ell+m})$, and similar ones for $v(\mu^\ell)$ and

$v(\mu^r)$. Then by definition of Γ_n we can write the components of Φ and Ψ in the form

$$\begin{aligned}\Phi(\mu^\ell, \mu^m, \mu^r) &= \mathrm{tr}\left(\hat\Phi(\mu^r) v(\mu^m)^* v(\mu^\ell)^*\right)\\ \Psi(\mu^\ell, \mu^m, \mu^r) &= \mathrm{tr}\left(\hat\Psi(\mu^\ell) v(\mu^r)^* v(\mu^m)^*\right) \quad ,\end{aligned}$$

where $\hat\Phi(\mu^r), \hat\Psi(\mu^\ell) \in \mathcal{M}_k$ for each tuple μ^r or μ^ℓ.

We show first an estimate of $\langle \Phi, \Psi\rangle$, which does not use the orthogonality of these vectors to $\mathcal{G}_{\ell+m+r}$, namely

$$|\langle \Phi, \Psi\rangle - \langle \Delta^\Phi, \Delta_\Psi\rangle_\rho| \leq \frac{a(m)}{a_-(m)}\|\Phi\| \cdot \|\Psi\| \quad , \qquad (\star)$$

where

$$\Delta^\Phi = \sum_{\mu^r} \hat\Phi(\mu^r) \rho v(\mu^r) \rho^{-1} \qquad \text{and} \qquad \Delta_\Psi = \sum_{\mu^\ell} v(\mu^\ell) \hat\Psi(\mu^\ell) \quad .$$

Upon noting that

$$\langle \Phi, \Psi\rangle = \sum_{\mu^\ell, \mu^r} \langle \Gamma_m(v(\mu^\ell)^* \hat\Phi(\mu^r)),\ \Gamma_m(\hat\Psi(\mu^\ell) v(\mu^r)^*)\rangle \quad .$$

we can use Lemma 5.2 to write this as

$$\begin{aligned}\sum_{\mu^\ell, \mu^r} \langle v(\mu^\ell)^* \hat\Phi(\mu^r),\ \hat\Psi(\mu^\ell) v(\mu^r)^*\rangle_\rho &= \sum_{\mu^\ell, \mu^r} \mathrm{tr}\left(\rho\, \hat\Phi(\mu^r)^* v(\mu^\ell) \hat\Psi(\mu^\ell) v(\mu^r)^*\right)\\ &= \sum_{\mu^\ell, \mu^r} \mathrm{tr}\left(\rho\{\hat\Phi(\mu^r) \rho v(\mu^r) \rho^{-1}\}^* \{v(\mu^\ell) \hat\Psi(\mu^\ell)\}\right) = \langle \Delta^\Phi, \Delta_\Psi\rangle_\rho \quad ,\end{aligned}$$

and a remainder, which is bounded by

$$a(m) \sum_{\mu^\ell, \mu^r} \|v(\mu^\ell)^* \hat\Phi(\mu^r)\|_\rho \cdot \|\hat\Psi(\mu^\ell) v(\mu^r)^*\|_\rho \quad .$$

This sum is estimated with the Cauchy-Schwartz inequality, using

$$\begin{aligned}\sum_{\mu^\ell, \mu^r} \|v(\mu^\ell)^* \hat\Phi(\mu^r)\|_\rho^2 &= \sum_{\mu^\ell, \mu^r} \mathrm{tr}\left(\rho \hat\Phi(\mu^r)^* v(\mu^\ell) v(\mu^\ell)^* \hat\Phi(\mu^r)\right)\\ &= \sum_{\mu^r} \mathrm{tr}\left(\rho \hat\Phi(\mu^r)^* \hat\Phi(\mu^r)\right) \quad \leq a_-(\ell + m)^{-1} \sum_{\mu^r} \|\Gamma_{\ell+m}(\hat\Phi(\mu^r))\|^2\\ &= a_-(\ell + m)^{-1} \|\Phi\|^2 \quad \leq a_-(m)^{-1} \|\Phi\|^2 \quad ,\end{aligned}$$

and a similar computation for Ψ, using $\sum_{\mu^r} v(\mu^r)^* \rho v(\mu^r) = \rho$. This yields the error estimate given in $(\star)$.

Equation $(\star)$ takes a particularly simple form if Φ (resp. Ψ) is in the subspace $\mathcal{G}_{\ell+m+r}$, say equal to $\chi = \Gamma_{\ell+m+r}(\hat\chi)$ with $\hat\chi \in \mathcal{M}_k$. This condition is equivalent to the special form $\hat\Phi(\mu^r) = \hat\chi v(\mu^r)^*$ (resp. $\hat\Psi(\mu^\ell) = v(\mu^\ell)^* \hat\chi$). We then have $\Delta^\Phi = \hat\chi$ (resp. $\Delta_\Psi = \hat\chi$), and that the sum $\sum_{\mu^\ell, \mu^r} \|v(\mu^\ell)^* \hat\Phi(\mu^r)\|_\rho^2$ appearing in the error estimate of $(\star)$ is equal to $\|\hat\chi\|_\rho^2$.

If $\Psi \perp \mathcal{G}_{\ell+m+r}$, we then find that for all $\hat\chi \in \mathcal{M}_k$

$$|\langle \hat\chi, \Delta_\Psi\rangle_\rho| \leq a(m) a_-(m)^{-1/2} \|\hat\chi\|_\rho \cdot \|\Psi\| \quad .$$

In other words, $\|\Delta_\Psi\|_p \leq a(m)a_-(m)^{-1/2}\|\Psi\|$. Together with the analogous estimate for $\|\Delta^\Phi\|$ and $(\star)$ we finally obtain

$$\begin{aligned}|\langle\Phi,\Psi\rangle| &\leq |\langle\Delta^\Phi,\Delta_\Psi\rangle| + a(m)/a_-(m)\|\Phi\|\|\Psi\| \\ &\leq \left(a(m)^2/a_-(m) + a(m)/a_-(m)\right)\|\Phi\|\|\Psi\| \quad .\end{aligned}$$

■

In the following Lemma $E \wedge F$ and $E \vee F$ denote the largest lower bound and least upper bound in the lattice of projections, respectively.

6.3 Lemma. *Let E and F be orthogonal projections on a finite dimensional Hilbert space $\mathcal{H}$ then:*

(1) $\|EF - E\wedge F\| = \|(\mathbb{1}-E)(\mathbb{1}-F) - (\mathbb{1}-E)\wedge(\mathbb{1}-F)\|$

(2) $EF + FE \geq -\|EF - E\wedge F\|\,(E+F)$

Proof: Both $E\vee F$ and $E\wedge F$ reduce all the operators that appear in the statement of the Lemma. The inequality (2) is trivially satisfied on $(E\vee F)^\perp$ and on $E\wedge F$ and both $EF - E\wedge F$ and the corresponding expression for the orthogonal complements vanish on $(E\vee F)^\perp\mathcal{H}$ and $(E\wedge F)\mathcal{H}$. We can therefore as well suppose that $E\vee F = \mathbb{1}$ and $E\wedge F = 0$.

(1) Since $\mathcal{H}$ is finite dimensional, we can find unit vectors $\Phi,\Psi \in \mathcal{H}$, for which $\langle\Phi, EF\Psi\rangle = \|EF\| \equiv \eta$ is real and attains its maximum. Clearly, we must have $E\Phi = \Phi$ and $F\Psi = \Psi$. For fixed Ψ, $E\mathcal{H} \ni \varphi \mapsto \langle\varphi, E\Psi\rangle$ attains its maximum only when φ is a positive multiple of $E\Psi$. Hence $E\Psi = \eta\Phi$, and $F\Phi = \eta\Psi$. Consider now the vectors $\Phi' = \eta\Phi - \Psi$ and $\Psi' = \eta\Psi - \Phi$. These satisfy $E\Phi' = F\Psi' = 0$, and $\|\Phi'\|^2 = \|\Psi'\|^2 = 1-\eta^2$. Moreover, $\langle\Phi',\Psi'\rangle = \eta^3 - \eta$. Hence

$$\begin{aligned}\|EF\|\cdot\|\Phi'\|\cdot\|\Psi'\| &= \eta(1-\eta^2) = -\langle\Phi',\Psi'\rangle = \langle\Phi',(\mathbb{1}-E)(\mathbb{1}-F)\Psi'\rangle \\ &\leq \|(\mathbb{1}-E)(\mathbb{1}-F)\|\cdot\|\Phi'\|\cdot\|\Psi'\| \quad ,\end{aligned}$$

and $\|EF\| \leq \|(\mathbb{1}-E)(\mathbb{1}-F)\|$. The reversed inequality follows by exchanging $E \leftrightarrow (\mathbb{1}-E)$ and $F \leftrightarrow (\mathbb{1}-F)$.

(2) Since $E\vee F = \mathbb{1}$ and $E\wedge F = 0$, any vector in $\mathcal{H}$ can be written uniquely as $\varphi + \psi$ with $E\varphi = \varphi$, $F\psi = \psi$. Consider the eigenvalue equation

$$(E+F)(\varphi+\psi) = (1-\alpha)(\varphi+\psi) \quad .$$

Then by uniqueness of the decomposition we must have $E(\varphi+\psi) = (1-\alpha)\varphi$, i.e. $E\psi = -\alpha\varphi$, and, similarly, $F\varphi = -\alpha\psi$. Taking the inner product of the first equation with ψ and of the second with φ, we get $\langle\varphi,\psi\rangle = -\alpha\|\varphi\|^2 = \langle\psi,\varphi\rangle = -\alpha\|\psi\|^2$. Hence $\alpha\|\varphi\|\cdot\|\psi\| = -\langle\varphi,\psi\rangle = -\langle\varphi, EF\psi\rangle \leq \|EF\|\cdot\|\varphi\|\cdot\|\psi\|$. Thus $\alpha \leq \|EF\|$, and $(E+F) \geq (1-\|EF\|)\mathbb{1}$. Squaring the last inequality we get $EF + FE = (E+F-\mathbb{1})(E+F) \geq -\|EF\|(E+F)$.

Combining the two parts of the proof, it is clear that the eigenvector of $E+F$ with smallest eigenvalue is $\Phi + \Psi$.

■

6.4 Theorem. *Let $h \in \mathcal{A}^{\otimes\ell}$ be an interaction exposing a purely generated C^*-finitely correlated state ω. Then the gap-inequality (6.4) is satisfied for some strictly positive γ.*

Proof : By Lemma 5.6 and equation (6.3) it is clear that we can pick some $p \geq \ell_0$, consider ω as a state on the chain $(\mathcal{A}^{\otimes p})_{\mathbb{Z}}$ and prove the theorem for the special nearest neighbour interaction $h \equiv (\mathbb{1} - \mathrm{G}_{2p}) \in (\mathcal{A}^{\otimes p})^{\otimes 2}$. By Lemma 6.1 this can be achieved by showing that for all $m \in \mathbb{N}$

$$H_{\{1,\dots mp\}} = \sum_{i=0}^{m-2}\alpha_{1+pi}(\mathbb{1} - \mathrm{G}_{2p}) \geq \gamma(\mathbb{1} - \mathrm{G}_{mp}) \quad .$$

Since G_{mp} is the ground state projection of $H_{\{1,\dots mp\}}$, the two sides of this inequality commute, and by the functional calculus the inequality is equivalent to $(H_{\{1,\dots mp\}})^2 \geq \gamma H_{\{1,\dots mp\}}$. We thus have to find a lower bound for the sum of the $(m-1)^2$ terms defining $(H_{\{1,\dots mp\}})^2$. The sum of the diagonal terms in this square just reproduces $H_{\{1,\dots mp\}}$. Since $\alpha_{ip+1}(h)$ and $\alpha_{jp+1}(h)$ commute for $|i-j|>1$, we can bound the sum of all such cross terms by zero. To the projections $E \equiv \alpha_{ip+1}(h)$ and $F = \alpha_{jp+1}(h)$ with $|i-j| = 1$ we apply Lemmas 6.3.(2), 6.3.(1), 5.5, and 6.2, obtaining

$$\begin{aligned}\|EF+FE\| &\geq -\|EF - E\wedge F\|\,(E+F) \\ &= -\|(\mathrm{G}_{2p}\otimes\mathbb{1}_p)(\mathbb{1}_p\otimes\mathrm{G}_{2p}) - (\mathrm{G}_{2p}\otimes\mathbb{1}_p)\wedge(\mathbb{1}_p\otimes\mathrm{G}_{2p})\|(E+F) \\ &\geq -a(p)\frac{1+a(p)}{a_-(p)}(E+F) \equiv -\epsilon_p(E+F) \quad .\end{aligned}$$

Note that the coefficient ϵ_p in this bound can be made arbitrarily small, by choosing p large enough. Since each $\alpha_{ip+1}(h)$ occurs in at most two of these cross terms, we find that $(H_{\{1,\dots mp\}})^2 \geq (1-2\epsilon_p)H_{\{1,\dots mp\}}$, i.e. the special interaction chosen in this proof has a gap at least $(1-\epsilon_p)$.

■

7. Applications

7.1. Classical systems

In this section we consider C*-finitely correlated states for which both algebras $\mathcal{A}$ and $\mathcal{B}$ are abelian and finite dimensional. Hence $\mathcal{A} = \mathcal{C}(\Omega)$ is the set of complex valued functions on a finite set Ω, say $\Omega = \{1, \ldots d\}$. Thus as a vector space $\mathcal{A}$ is just $\mathbb{C}^d$, and its hermitian part $\mathbb{R}^d$ is ordered componentwise. The projections $e_i \in \mathcal{A}$ with $e_i(j) = \delta_{ij}$ obviously form a basis of $\mathcal{A}$. Similarly, $\mathcal{B} \equiv \mathcal{C}(\{1, \ldots k\})$ for some $k < \infty$. The map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ is best decomposed into the d operators $\mathbb{E}_i : \mathcal{B} \mapsto \mathbb{E}(e_i \otimes B)$. Since a map from or into an abelian C*-algebra is completely positive iff it is positive **[53]**, this constraint on $\mathbb{E}$ just means that each $\mathbb{E}_i$, written as a $k \times k$-matrix with respect to the canonical basis of $\mathcal{B} \equiv \mathbb{C}^k$ has positive matrix elements. In order to get a C*-finitely correlated state we further need a vector $e \in \mathcal{B} = \mathbb{C}^k$ with positive components (which we can take as $\mathbb{1}$ by Lemma 2.5), and another vector ρ with positive components. With the notations $\langle \cdot, \cdot \rangle$ for the scalar product of $\mathbb{C}^k$, $X^\top$ for the transpose in $\mathcal{M}_k$, and $\hat{\mathbb{E}} = \sum_{i=1}^d \mathbb{E}_i$ these objects have to satisfy $\hat{\mathbb{E}} e = e$, and $\hat{\mathbb{E}}^\top \rho = \rho$.

In probability theory a state on the chain $\mathcal{C}(\Omega)_{\mathbb{Z}}$ is usually called a "stochastic process" with state space Ω, and the state is usually expressed via the Riesz representation theorem as a cylinder measure μ on $\Omega^{\mathbb{Z}}$. In our construction this measure is given by

$$\mu(\{k_n, \ldots k_m\}) = \langle \rho, \mathbb{E}_{k_n} \cdots \mathbb{E}_{k_m} e \rangle \quad ,$$

where $\{k_n, \ldots k_m\}$ denotes the cylinder in $\Omega^{\mathbb{Z}}$ consisting of those configurations of the chain that coincide with $\{k_n, \ldots k_m\}$ at the sites $\{n, n+1, \ldots m\}$. We shall also call μ a C*-finitely correlated measure (in **[27]**these were called "manifestly positive").

It is straightforward to see that any finite state space m-step Markovian measure is a manifestly positive and finitely correlated. In section 4 it was shown that the purely generated C*-finitely correlated states are the basic building blocks for constructing arbitrary C*-finitely correlated states. In order to get a general C*-finitely correlated state we had to allow for larger one-site algebras and product completely positive mappings between chain algebras. In the context of probability theory it is natural to restrict attention to product positive maps between algebras of continuous functions on configuration spaces. We will moreover drop the stochastic character of the mapping by considering product mappings between configuration spaces. Such mappings, which are homomorphisms on the level of functions on configuration space, are usually called 'functions' of a process. We can now state the following result:

7.1 Theorem. *Let μ be a C*-finitely correlated measure on $\Omega^{\mathbb{Z}}$. Then there exists a finite set Ω_1, a Markovian measure μ_1 on $\Omega_1^{\mathbb{Z}}$ and a function $\Phi : \Omega \to \Omega_1$ such that $\mu = \mu_1 \circ \Phi^{\mathbb{Z}}$. Moreover we can choose Ω_1 in such a way that $\#\Omega_1 \leq (\#\Omega)^4$.*

Our next aim is to give an expression for the entropy density of the measure. Such an expression has been obtained by **[15]** and was extensively studied in **[27]**. For technical convenience we assume the rather strong irreducibility condition that all matrix elements of the $\mathbb{E}_k$, $\{k = 1, \ldots d\}$ are strictly positive. This implies that $\hat{\mathbb{E}}$ has trivial peripheral spectrum, and hence that the measure μ has no non-trivial periodic components. Much weaker conditions are discussed in **[27]**. We first introduce a dynamical system for the purpose of describing the structure of the 'conditionings' of the process μ. So let us denote by $\mathcal{B}_e$ the set of positive elements ν in $\mathbb{C}^k$ such that $\langle \nu, e \rangle = 1$. Thus if we take $e = \mathbb{1}_{\mathcal{B}}$, as we may, $\mathcal{B}_e$ is just the state space of $\mathcal{B}$.

An operator T_μ is now defined on the space $\mathcal{C}(\mathcal{B}_e)$ of continuous complex-valued functions on $\mathcal{B}_e$:

$$(T_\mu f)(\nu) = \sum_{a \in \Omega} \langle \nu, \mathbb{E}_a e \rangle \, f(\Gamma_a(\nu)) \qquad f \in \mathcal{C}(\mathcal{B}_e) \quad ,$$

where $\Gamma_a : \mathcal{B}_e \to \mathcal{B}_e$ is defined by

$$\Gamma_a(\nu) = \frac{\mathbb{E}_a^\top \nu}{\langle \nu, \mathbb{E}_a e \rangle} \quad .$$

7.2 Theorem. *With the above notations there exists a unique probability measure $\overline{\varphi}$ on $\mathcal{B}_e$, which is invariant under T_μ. The mean entropy $s(\mu)$ of the measure μ is given by:*

$$s(\mu) = \sum_{a \in \Omega} \int_{\mathcal{B}_e} \overline{\varphi}(d\nu) h_a(\nu) \quad ,$$

where $h_a(\nu) = -\langle \nu, \mathbb{E}_a e \rangle \log \langle \nu, \mathbb{E}_a e \rangle$.

The C*-finitely correlated states described in this section may, of course, be used to generate states on chains $\tilde{\mathcal{A}}_{\mathbb{Z}}$ with non-commutative $\tilde{\mathcal{A}}$ by applying a completely positive map $\mathbb{F} : \mathcal{C}(\Omega) \to \tilde{\mathcal{A}}$ at each site as in Proposition 4.4. These C*-finitely correlated states, which could be called non-classical functions of Markov processes, exhaust only a small subset of the C*-finitely correlated states. In such a state the correlations across any bond will be "classically correlated" in the sense of **[56]**, i.e. the state can be decomposed as an integral over states, in which the right and left halves of the chain are completely uncorrelated. It is easy to see that non-trivial purely generated states, as studied in sections 5 and 6 cannot have this property.

It would be interesting to have examples for states over a classical chain ($\mathcal{A}$ abelian), generated with a non-abelian algebra $\mathcal{B}$. More generally, one might look for finitely correlated states over a classical chain, which are not even C*-finitely correlated. We did not succeed in settling the question whether this is possible.

7.2. Integrable systems

Since C*-finitely correlated states are easy to construct, it is natural to use them as trial states in the ground state variational problem of a given interaction. Here we prove a general result, which illuminates the nature of this variation. It also allows a neat one-line proof of the fact that the ground state of the antiferromagnetic spin-1/2 Heisenberg chain with nearest neighbor interaction $h = \sum_{mu=1}^{3} \sigma^\mu \otimes \sigma^\mu \in \mathcal{M}_2 \otimes \mathcal{M}_2$ and of some of its generalizations **[55,11]** are not C*-finitely correlated.

7.3 Proposition. *Let $h \in (\mathcal{M}_d)^{\otimes \ell}$ be hermitian, and suppose that with respect to some basis $\{\varphi_\mu\}_{\mu=1}^{d}$ the real and imaginary parts of all matrix elements*

$$\langle \varphi_{\mu_1} \otimes \cdots \varphi_{\mu_n},\ h\varphi_{\nu_1} \otimes \cdots \varphi_{\nu_n} \rangle$$

are in some subfield $\mathbb{F} \subset \mathbb{R}$. Suppose that $h_{\min} \equiv \inf\{\omega(h) \mid \omega \in \mathcal{T}\}$ is attained at a C-finitely correlated state. Then $h_{\min}$ is algebraic over $\mathbb{F}$.*

Proof : We may suppose that the minimizing state ω is generated by $\mathbb{E} : \mathcal{M}_d \otimes \mathcal{M}_k \to \mathcal{M}_k$ and $\rho : \mathcal{M}_k \to \mathbb{C}$. In particular, this state has minimal energy density among all states generated by different maps $\mathbb{E}, \rho$ acting on the same spaces. We have to show that minimizing the energy functional over this set leads to an algebraic minimal value.

Since $\{\pi(\mathcal{M}_d) \otimes \mathcal{M}_k V \mathbb{C}^k\}$ is total in the Stinespring dilation space $\mathcal{H} \otimes \mathbb{C}^k$ of $\mathbb{E}$, $\mathcal{H}$ has at most dimension $d^2k^2 \cdot k < \infty$. We may therefore fix a sufficiently large dimensional space $\mathcal{H}$ and a representation $\pi : \mathcal{M}_d \to \mathcal{H}$, and the map V of Proposition 4.3 and a matrix $R \in \mathcal{M}_k$, with $\rho(B) = \mathrm{tr}(BR^*R)$ to parametrize all C*-finitely correlated states generated in $\mathcal{M}_k$.

Using this parametrization there are no positivity constraints, but only the constraints $\mathbb{E}(\mathbb{1}) = \mathbb{1}$, and $\rho(\mathbb{E}(\mathbb{1} \otimes B) = \rho(B)$, which are a set of polynomial identities with integer coefficients in the (real and imaginary parts) of the matrix elements of V and R. The energy functional

$$(V, R) \longmapsto \mathrm{tr}\Big(R\mathbb{E}^{(\ell)}(h \otimes \mathbb{1}_{\mathcal{M}_k})R^*\Big)$$

is a polynomial of degree 2 in R and degree 2ℓ in V, with coefficients in $\mathbb{F}$. Since the constraints force V and R to lie in given compact sets, minimizers of the constrained variational problem exist. Introducing as additional variables the Lagrange multipliers λ_i for the constraints, we obtain a system of polynomial equations for the minimizing (V, R, λ). We can cut down the set of minimizing (V, R, λ) by further

arbitrary polynomial conditions (with coefficients in $\mathbb{F}$), until we have one isolated solution of a system of algebraic equations, which represents a minimizer. We can separate this solution from possible further solutions of the same system (which might not minimize the energy) by some polynomial inequalities. The resulting system of polynomial equations and inequalities thus has a unique solution in the real field. By Tarski's Theorem **[40,sect.5.6]** we can find a set of integer polynomial conditions on the coefficients of all these polynomials, which decides the existence of solutions of the system for any real closed field. Since there is a solution in real variables, this condition is satisfied for the given coefficients. Hence there must also be a solution in the real closed extension of $\mathbb{F}$, i.e. the unique solution is algebraic over $\mathbb{F}$. Therefore also the value of the energy functional must be algebraic. ■

Recently the exact ground state energy density has been computed for a class of models generalizing the usual spin 1/2 Heisenberg antiferromagnet **[55,11]**. These models are spin-J chains with isotropic nearest neighbor Hamiltonians and the matrix elements of the interaction are algebraic numbers.

7.4 Corollary. *The ground state of the spin-$\frac{1}{2}$ Heisenberg antiferromagnet and of its generalizations* **[55,11]** *to higher half-integer spins is not finitely correlated.*

Proof : The known exact values **[36,55,11]** of the ground state energies for these models are not algebraic. ■

7.3. Gauge invariant states

It is clear that under suitable covariance conditions $\mathbb{E}$ and ρ will generate a state, which is invariant under the action of some compact symmetry group on $\mathcal{A}$. Here we shall briefly look at some groups, which are naturally associated with each C*-finitely correlated state. For simplicity we shall assume throughout this subsection that ω is ergodic, and represented such that the eigenvalue 1 of $\hat{\mathbb{E}}$ is simple.

7.5 Definition. *Let ω be an ergodic C*-finitely correlated state generated by $\mathbb{E}$ and ρ. Then we shall call the set of $U \in \mathcal{A}$ with $\|U\| \leq 1$, for which there is some non-zero $B \in \mathcal{B}$ with*

$$\mathbb{E}(U \otimes B) = B \quad ,$$

the **gauge group** *G of ω.*

Of course, we shall have to show that G is indeed a group. This is established in the following result, together with some further properties.

7.6 Proposition. *Let $U \in G$. Then the the equation $\mathbb{E}(U \otimes B) = B$ determines B up to a scalar factor. U is unitary, and B can be taken to be unitary. We shall denote one unitary choice of B by $\lambda(U)$. For all $X \in \mathcal{A} \otimes \mathcal{B}$ and $C \in \mathcal{B}$ we have*

$$\mathbb{E}(X\,U \otimes \lambda(U)) = \mathbb{E}(X)\lambda(U) \quad ,$$

$$\hat{\mathbb{E}}(\lambda(U)^* C\lambda(U)) = \lambda(U)^*\hat{\mathbb{E}}(C)\lambda(U), \quad \text{and} \quad \rho(\lambda(U)^* C\lambda(U)) = \rho(C) \quad .$$

Moreover, G is a subgroup of the unitary group of $\mathcal{A}$, and λ is a representation of G up to a phase.

Suppose that $\mathcal{B} \cong \mathcal{M}_k \equiv \mathcal{B}(\mathcal{K})$, and let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ be the Stinespring dilation of $\mathbb{E}$, as in Proposition 4.1. Then there is a unitary representation $\lambda' : G \to \pi(\mathcal{A})' \subset \mathcal{B}(H)$ such that for $U \in G$

$$V\lambda(U) = (\pi(U)\lambda'(U) \otimes \lambda(U))V \quad .$$

Conversely, if for some unitary $U \in \mathcal{A}$ we can find unitaries $\lambda'(U) \in \pi(\mathcal{A})'$ and $\lambda(U) \in \mathcal{B}$ satisfying this equation, we have $U \in G$.

Proof : We use the same technique as in the proof of 3.3. Consider the positive semidefinite sesquilinear map $\beta : (\mathcal{A} \otimes \mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}) \to \mathcal{B}$ given by

$$\beta(X, Y) = \mathbb{E}(X^*Y) - \mathbb{E}(X)^*\mathbb{E}(Y) \quad .$$

Then if $\mathbb{E}(U \otimes B) = B$ and $\|U\| \leq 1$, we have

$$0 \leq \beta(U \otimes B, U \otimes B) = \mathbb{E}(U^*U \otimes B^*B) - B^*B \leq \mathbb{E}(\mathbb{1} \otimes B^*B) - B^*B \quad .$$

Since ρ is invariant under $\hat{\mathbb{E}}$, the ρ-expectation of the right hand side vanishes, and by faithfulness of ρ we have $\beta(U \otimes B, U \otimes B) = 0$. Since the eigenvalue 1 of $\hat{\mathbb{E}}$

is simple, $\hat{\mathbb{E}}(B^*B) = B^*B$ implies that B^*B is a multiple of the identity, so B is proportional to a unitary $\lambda(U)$. Moreover, $\beta(X, U \otimes B) = 0 = \mathbb{E}(X^*U \otimes B) - \mathbb{E}(X)^*B$ for all X, which proves that $\mathbb{E}(X\,U \otimes B) = \mathbb{E}(X)B$. It is clear that the set Π of pairs (U, B) satisfying this equation for all X is a group (the existence of an inverse follows by putting $X = X\,U^* \otimes B^*$ in this equation). In particular, if $(U, B_1), (U, B_2) \in \Pi$ we also have $(\mathbb{1}, B_1^*B_2) \in \Pi$. But $(\mathbb{1}, B) \in \Pi$ implies $\hat{\mathbb{E}}(B) = B$, and hence that B is a multiple of $\mathbb{1}$. This shows the uniqueness of B up to a factor. Now

$$\begin{aligned}\hat{\mathbb{E}}(B^*CB) &= \mathbb{E}((U \otimes B)^*(\mathbb{1} \otimes C)(U \otimes B)) \\ &= B^*\mathbb{E}(\mathbb{1} \otimes C)B = B^*\hat{\mathbb{E}}(C)B \quad .\end{aligned}$$

Consequently, the unique invariant state of $\hat{\mathbb{E}}$ must also be invariant under unitary transformation by B.

Consider now the dilation π, V. The set $\{(\pi(A) \otimes B)V\varphi\}$ spans $\mathcal{H} \otimes \mathcal{K}$ by definition of the Stinespring dilation. On this set we define

$$\tilde{U}(\pi(A) \otimes B)V\varphi = (\pi(A) \otimes B)(\pi(U)^* \otimes \lambda(U)^*)V\lambda(U)\varphi \quad .$$

That this definition is unambiguous, and specifies a unitary operator $\tilde{U}$ both follows from the computation of its matrix elements between $\Phi = (\pi(A) \otimes B)V\varphi$ and $\Phi' = (\pi(A') \otimes B')V\varphi'$

$$\begin{aligned}\langle\tilde{U}\Phi, \tilde{U}\Phi'\rangle &= \langle V\lambda(U)\varphi,\ \Big(\pi(UA^*A'U^*) \otimes (\lambda(U)B^*B'\lambda(U)^*)\Big)V\lambda(U)\varphi'\rangle \\ &= \langle\lambda(U)\varphi,\ \mathbb{E}\Big((UA^*A'U^*) \otimes (\lambda(U)B^*B'\lambda(U)^*)\Big)\lambda(U)\varphi'\rangle \\ &= \langle\varphi,\ \mathbb{E}((A^*A') \otimes (B^*B'))\varphi'\rangle \\ &= \langle\Phi, \Phi'\rangle \quad .\end{aligned}$$

From its definition it is clear that $\tilde{U}$ commutes with $\mathbb{1} \otimes \mathcal{B}$, hence is of the form $\tilde{U} = \lambda'(U) \otimes \mathbb{1}_{\mathcal{B}}$, and commutes with $\pi(\mathcal{A}) \otimes \mathbb{1}$, so $\lambda'(U)$ commutes with $\pi(\mathcal{A})$.

The equation $V\lambda(U) = \pi(U)\lambda'(U) \otimes \lambda(U)V$ implies that $\mathbb{E}(X(U \otimes \lambda(U))) = \mathbb{E}(X)\lambda(U)$, and hence that $U \in G$. ∎

Definition 7.5 does not only depend on the state ω, but also on the particular representation as a C*-finitely correlated state in terms of $\mathbb{E}$ and ρ. The following gives a simple criterion for deciding for some $U \in \mathcal{A}$, whether they belong to G, without looking at the C*-finitely correlated representation.

7.7 Proposition. *Let G_0 denote the set of all $U \in \mathcal{A}$ with $\|U\| \leq 1$, and such that*

$$\liminf_n \sup_{A_-, A_+} |\omega(A_- \otimes U^{\otimes n} \otimes A_+)| > 0 \quad ,$$

where the supremum is over $N > 0$, and all $A_- \in \mathcal{A}_{\{-N,\dots 0\}}$ and $A_+ \in \mathcal{A}_{\{n+1,\dots n+N\}}$ with norm less than one. Let G_1 denote the set of all unitaries $U \in \mathcal{A}$ such that for all $N > 0$, and $X \in \mathcal{A}^{\otimes N}$:

$$\omega(U^{\otimes N} X (U^*)^{\otimes N}) = \omega(X) \quad .$$

Then $G_0 \subset \{e^{i\alpha} \mid \alpha \in \mathbb{R}\} \cdot G \subset G_1$, and G_0 generates a normal subgroup $\tilde{G}_0 \subset G_1$.

Proof : By the 2-positivity of $\mathbb{E}$ we have for all $A \in \mathcal{A}$, $b \in \mathcal{B}$:

$$\begin{aligned}\|\mathbb{E}_A B\|_\rho^2 &= \mathrm{tr}\rho\mathbb{E}(A \otimes B)^*\mathbb{E}(A \otimes B) \leq \mathrm{tr}\rho\mathbb{E}(A^*A \otimes B^*B) \\ &\leq \|A\|^2 \mathrm{tr}\rho\hat{\mathbb{E}}(B^*B) = \|A\|^2 \mathrm{tr}\rho B^*B = \|A\|^2\|B\|_\rho^2 \quad .\end{aligned}$$

Hence $\|\mathbb{E}_A\| \leq \|A\|$ as an operator on $(\mathcal{B}, \langle\cdot,\cdot\rangle_\rho)$. Using the iterates $\mathbb{E}^{(N)}$ of $\mathbb{E}$ (see equation 2.2) we can write

$$\omega(A_- \otimes U^{\otimes n} \otimes A_+) = \langle \mathbb{E}_{A_-}^{(N+1)}\mathbb{1},\ \mathbb{E}_U^n \mathbb{E}_{A_+}^{(N)}\mathbb{1}\rangle_\rho \equiv \langle \Phi_-,\ \mathbb{E}_U^n \Phi_+\rangle \quad .$$

Since $\|\Phi_\pm\|_\rho \leq \|A_\pm\| \leq 1$ is uniformly bounded, this matrix element of $\mathbb{E}_U^n$ can stay away from zero only if the spectral radius of $\mathbb{E}_U$ is at least one. Since $\|\mathbb{E}_U\| \leq 1$, and $\mathcal{B}$ is finite dimensional, this means that $\mathbb{E}_U$ must have an eigenvalue of modulus one, say $\mathbb{E}_U(B) = \mathbb{E}(U \otimes B) = e^{i\alpha}B$. Hence $e^{-i\alpha}U \in G$. For $U \in G$, we get $\mathbb{E}^{(n)}(X(U^{\otimes n} \otimes \lambda(U))) = \mathbb{E}^{(n)}(X)\lambda(U)$ by induction on n, which together with the $\lambda(U)$-invariance of ρ implies that $U \in G_1$.

If $U \in G_0$ and $U' \in G_1$ we have $U'UU'^* \in G_0$, since we can simply transform $A_\pm$ in the definition of G_0 by a tensor power of U'. Hence G_0 generates a normal subgroup.

■

Using the dilation theory in Proposition 7.6 it is quite easy to construct states with a given gauge group. For simplicity, let us take $\mathcal{A} = \mathcal{M}_d$, so the gauge group will be some subgroup of the unitary group U(d) in d dimensions. We can then pick any two representations λ, λ' of G subject to the sole constraint that there exists a non-zero intertwining operator (and hence an intertwining isometry) $V : \mathcal{K} \to \mathcal{H}\otimes\mathcal{K}'\otimes\mathcal{K}$ satisfying

$$V\lambda(U) = \Big(U \otimes \lambda'(U) \otimes \lambda(U)\Big)V$$

for all $U \in G$. Set $\mathcal{B} = \mathcal{B}(\mathcal{K})$, $\mathbb{E}(A \otimes B) = V^*(A \otimes \mathbb{1}_{\mathcal{K}'} \otimes B)V$, $e = \mathbb{1} \in \mathcal{B} \equiv \mathcal{B}(\mathcal{K})$, and choose an $\hat{\mathbb{E}}$-invariant state ρ on $\mathcal{B}(\mathcal{K})$, e.g. the normalized trace. Then $(\mathbb{E}, \rho, e)$ generates a state ω, whose gauge group in the above sense contains G. A map $\mathbb{E}$ of this form is pure if $\mathcal{K}'$ is one dimensional. Thus, in order to construct purely generated gauge invariant states by this formula, we only have to pick the representation λ. Note that the intertwining relation does not automatically imply that $\hat{\mathbb{E}}$ has only one fixed vector, so this condition has to be checked by hand. If it is satisfied, however, the theory of sections 5 and 6 applies. In the simplest example of this structure the gauge group is SU(2), $\mathcal{B} = \mathcal{M}_2$, and $\mathcal{A} = \mathcal{M}_3$ contains the spin-1

representation of SU(2). The state thus obtained is the one studied in **[5,Sect.2]**. A generalization of this example will be studied in the next section.

In the case of purely generated states we can strengthen Proposition 7.7 as follows. By G_1° we denote the connected component of the identity in G_1, considered with the topology inherited from $\mathcal{A}$. Recall that $\tilde{G}_0$ denotes the group generated by G_0.

7.8 Proposition. *Let $\mathcal{A} = \mathcal{M}_d$, and let ω be an ergodic purely generated C*-finitely correlated state of $\mathcal{A}_\mathbb{Z}$. Then $G_1^\circ \subset \tilde{G}_0$. In particular, if G_1 is connected we have $\tilde{G}_0 = G = G_1$.*

Proof : By Theorem 5.7 ω is the unique ground state of an interaction $h \in \mathcal{A}^{\otimes\ell}$ exposing ω. By averaging $U^{\otimes\ell}hU^{\otimes\ell*}$ over $U \in G_1$ we may also assume that h commutes with $U^{\otimes\ell}$ for $U \in G_1$. Now by Theorem 6.4 there is a positive γ such that for all $m \in \mathbb{N}$ and $X \in \mathcal{A}_{\{1,\dots m\}}$:

$$\omega(X^*H_{\{1-\ell,\dots 1+m+\ell\}}X) \geq \gamma\{\omega(X^*X) - |\omega(X)|^2\} \quad .$$

Let $U \in G_1$, and apply this inequality to $X = U^{\otimes m}$. Inserting $H_{\{1-\ell,\dots 1+m+\ell\}} = \sum_{i=1-\ell}^{m+1}\alpha_i(h)$ into the left hand side we find three types of terms: The majority (for large m) is equal to

$$\omega(U^{\otimes\ell}h(U^*)^{\otimes\ell}) = \omega(h) = 0 \quad .$$

The other terms, coming from the boundaries, are of the forms

$$\omega((U^{\otimes k} \otimes \mathbb{1}^{\otimes\ell-k})h(U^{\otimes k} \otimes \mathbb{1}^{\otimes\ell-k})^*) \quad \text{and} \quad \omega((\mathbb{1}^{\otimes\ell-k} \otimes U^{\otimes k})h(\mathbb{1}^{\otimes\ell-k} \otimes U^{\otimes k})) \quad .$$

Let $\Delta(U)$ denote the sum of the 2ℓ boundary terms. Then $\Delta(\mathbb{1}) = 0$, and by continuity of Δ there is a neighbourhood $\mathcal{N}$ of the identity in G_1 such that for $U \in \mathcal{N}$ we have

$$|\omega(U^{\otimes m})|^2 \geq 1 - \gamma^{-1}\Delta(U) > 0$$

for all sufficiently large m. Hence $\mathcal{N} \subset \tilde{G}_0$, and the result follows by taking products of elements in $\mathcal{N}$.

■

7.4. Integer spin chains

In this section we apply the scheme outlined at the end of section 7.3 to the simplest case: the construction of SU(2)-invariant purely generated C*-finitely correlated states on spin chains. Thus $\mathcal{A} = \mathcal{M}_{2J+1}$, where J is the value of the spin at each site. For reasons that will become apparent immediately, we shall assume that J is an integer. The algebra $\mathcal{B}$ will be chosen as $\mathcal{M}_{2j+1}$ for some not necessarily integer $j \in \frac{1}{2}\mathbb{N}$, satisfying $j \geq J/2$. These are precisely the constraints on j and J for an intertwining operator

$$V : \mathbb{C}^{2j+1} \to \mathbb{C}^{2J+1} \otimes \mathbb{C}^{2j+1} \quad \text{with} \quad (\mathcal{D}_g^{(J)} \otimes \mathcal{D}_g^{(j)})V = V\mathcal{D}_g^{(j)}$$

to exist. In this case V is unique up to a scalar factor, and we can, and will choose this facto so that V is an isometry. Then $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$, given by $\mathbb{E}(X) = V^*XV$ is completely positive and unit preserving. Since V is an intertwiner, $\mathbb{E}$ is obviously covariant in the sense that

$$\mathcal{D}_g^{(j)}\mathbb{E}(X)\mathcal{D}_g^{(j)*} = \mathbb{E}((\mathcal{D}_g^{(J)} \otimes \mathcal{D}_g^{(j)})X(\mathcal{D}_g^{(J)} \otimes \mathcal{D}_g^{(j)})^*) \quad .$$

Let us denote by τ the normalized trace on $\mathcal{B}$, which is the only rotation invariant state on that algebra. Since $\hat{\mathbb{E}}$ obviously maps rotation invariant into rotation invariant states, it is clear that $\tau \circ \hat{\mathbb{E}} = \tau$. Consequently, $(\mathbb{E}, \tau, \mathbb{1})$ generate a C*-finitely correlated state ω_j, which is SU(2)-invariant in the sense that $\omega_j\left(\mathcal{D}_g^{(J)\otimes\infty} A(\mathcal{D}_g^{(J)\otimes\infty})^*\right) = \omega_j(A)$. Note also that $\mathbb{E}$ is pure, so ω_j is purely generated, and since the eigenvalue 1 of $\hat{\mathbb{E}}$ is non-degenerate (see below) the whole theory of sections 5 and 6 applies.

7.9 Proposition. *Any correlation function $n \in \mathbb{N} \mapsto c(n) \equiv \omega_j(X_1\alpha_n(X_2))$ with $X_1 \in \mathcal{A}_{\mathbb{N}}$, $X_2 \in \mathcal{A}_{\mathbb{Z}\setminus\mathbb{N}}$ is of the form $c(n) = \sum_{k=0}^{2j} a_k\lambda_k^n$ for some constants a_k, where λ_k is the k^{th} eigenvalue of $\hat{\mathbb{E}}$. λ_k is $(2k+1)$-fold degenerate, and equal to*

$$\lambda_k = (-1)^k(2j+1)\begin{Bmatrix} j & j & J \\ j & j & k \end{Bmatrix} \qquad k = 0, \ldots 2j \quad ,$$

where the symbol between braces is a Wigner 6j-symbol using the conventions of [25].

Proof : In order to get at the behavior of the correlation functions we must diagonalize $\hat{\mathbb{E}}$. There is a natural identification of the $k \times k$ matrices $\mathcal{M}_k$ with $\mathbb{C}^k \otimes \mathbb{C}^k$: the rank 1 operator $|\psi\rangle\langle\varphi|$ is mapped onto $\psi \otimes \overline{\varphi}$ where $\varphi \mapsto \overline{\varphi}$ is a complex conjugation on $\mathbb{C}^k$. This is in fact a unitary transformation if we equip $\mathcal{M}_k$ with the Hilbert-Schmidt inner product $\langle A, B\rangle \equiv \operatorname{Tr} A^*B$. The representation $g \in \mathrm{SU}(2) \mapsto \alpha_g^{(k)}(\cdot) \equiv \mathcal{D}_g^{(k)} \cdot \mathcal{D}_g^{(k)*}$ in the automorphisms of $\mathcal{M}_k$ is transported by this unitary transformation into the representation $g \in \mathrm{SU}(2) \mapsto \mathcal{D}_g^{(k)} \otimes \overline{\mathcal{D}_g^{(k)}}$ where

$\overline{\mathcal{D}_g^{(k)}}\varphi \equiv \overline{\mathcal{D}_g^{(k)}\overline{\varphi}}$ but, as there is up to unitary equivalence only one irreducible spin k representation of SU(2), $\mathcal{D}^{(k)}$ and $\overline{\mathcal{D}^{(k)}}$ are unitarily equivalent.

As we have to consider decompositions of tensor representations of SU(2) we recall the usual conventions [25]:

- $\{|\, k, m\rangle \mid m = -k, -k+1, \ldots k\}$ denotes the standard basis of $\mathbb{C}^{2k+1}$ which corresponds to the spin k representation of SU(2): $|\, k, m\rangle$ is the normalized eigenvector of the z-component of the spin corresponding to the eigenvalue m and the successive $|\, k, m\rangle$ are obtained by applying the lowering operator to the highest spin vector $|\, k, k\rangle$ and normalizing with a positive factor.
- $|\, (k_1, k_2)k_3, m_3\rangle$ denotes the $|\, k_3, m_3\rangle$ vector in the spin k_3 subrepresentation of $\mathcal{D}_1^{(k)} \otimes \mathcal{D}_2^{(k)}$. The overall phase in each $\mathcal{D}_3^{(k)}$ subrepresentation is fixed by requiring that $|\, k_1, k_1\rangle \otimes |\, k_2, k_3 - k_1\rangle$ appears with a positive coefficient in $|\, (k_1, k_2)k_3, k_3\rangle$.

As V intertwines $\mathcal{D}^{(j)}$ and $\mathcal{D}^{(J)} \otimes \mathcal{D}^{(j)}$ we have with the notations of above that the matrix elements of V are precisely the Clebsch-Gordon coefficients:

$$\begin{aligned}\langle J, m_1 \mid \otimes \langle j, m_2 \mid V \mid j, m_1 + m_2\rangle &= \langle J, m_1 \mid \otimes \langle j, m_2 \mid (J, j)j, m_1 + m_2\rangle \\ &\equiv \langle J, m_1, j, m_2 \mid (J, j)j, m_1 + m_2\rangle\end{aligned}$$

As $\hat{\mathbb{E}} \circ \alpha_g^{(j)} = \alpha_g^{(j)} \circ \hat{\mathbb{E}}$, $\hat{\mathbb{E}}$ will be constant on each of the subspaces of $\mathcal{M}_{2j+1}$ that carries an irreducible subrepresentation of $\alpha_g^{(j)}$. Using the identifications of above the spectrum of $\hat{\mathbb{E}}$ consists of eigenvalues $\{\lambda_k \mid k = 0, 1, \ldots 2j\}$, and the multiplicity of λ_k is $2k+1$ which is the dimension of the spin k irreducible representation of $\alpha_g^{(j)}$. In order to compute the values of the λ_k it is useful to make the following explicit choice for the complex conjugation:

$$\overline{|\, k, m\rangle} \equiv (-1)^{k-m} \mid k, -m\rangle \qquad m = -k, -k+1, \ldots k$$

With this choice $\overline{\mathcal{D}_g^{(k)}} = \mathcal{D}_g^{(k)}$. As $\mathbb{1} \in \mathcal{M}_{2J+1}$ carries the spin 0 subrepresentation of $\alpha_g^{(J)}$ and has Hilbert-Schmidt norm $\sqrt{1+2J}$ it can be identified with $\sqrt{2J+1}$ $|\, (J, J)0, 0\rangle$ also the spin k subspace of $\mathcal{M}_{2j+1}$ is generated by $\{|\, (j, j)k, m\rangle \mid m = -k, -k+1, \ldots k\}$. It is now straightforward to write down the eigenvalue equation for $\hat{\mathbb{E}}$ and to compute the λ_k using the conventions of [25]. As the eigenvalue 1 is non-degenerate ω_j is pure.

■

We can now proceed to construct interactions exposing these states. Restricting, for simplicity, to the case $j \leq J \leq 2j$, it is not difficult to see that the range of Γ_2 has its maximal value $k^2 = (2j+1)^2$. Hence by Definition 5.4 the interaction length of all these states is 2, and we know that we can find exposing interactions in $\mathcal{A}^{\otimes 3}$, i.e. an exposing next-nearest neighbour interaction. When $j < J$, $\mathcal{G}_2$ is a proper subspace of $\mathcal{H}^{\otimes 2}$. This subspace is easily described in terms of the representation theory. Given two representations $\mathcal{D}^{(s_i)}$, $i = 1, 2$, let us denote by $\mathcal{R}^2_{s_1,s_2}$ the subspace of $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}$ carrying representations with spin less than or equal to 2, and, similarly, denote by $\mathcal{R}^s_{s_1,s_2,s_3}$ the subspace of $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1} \otimes \mathbb{C}^{2s_3+1}$

with spin $\leq s$. Then since $V^{(2)} \equiv (\mathbb{1}_{\mathcal{H}} \otimes V)V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{K}$ intertwines $\mathcal{D}^{(j)}$ with $\mathcal{D}^{(J)} \otimes \mathcal{D}^{(J)} \otimes \mathcal{D}^{(j)}$, it is clear that $V^{(2)}\mathcal{K} \subset \mathcal{R}^{2j}_{JJ}$. Similarly, $V^{(3)}\mathcal{K} \subset \mathcal{R}^{2j}_{JJJ}$. Thus if P^s_2 denotes the projection on $\mathcal{H} \otimes \mathcal{H}$ onto the subspace carrying the spin-s representation, we have $\omega_j(\alpha_i(k^j)) \equiv 0$ with $k^j \in \mathcal{A}^{\otimes 2}$ given by

$$k^j = \sum_{s=2j+1}^{2J} P^s_2 \quad .$$

Note that k^j cannot be an exposing interaction for $j > J/2$, since also $\omega_{J/2}(\alpha_i(k^j)) \equiv 0$, contradicting the uniqueness theorem 5.7. However, for the smallest possible value $j = J/2$, $h = k^j$ is indeed an interaction exposing ω_j. This reduction from a next-nearest neighbour to a nearest neighbour interaction follows from the following Lemma (inserting $s_i = s_{ij} = J$), which is a direct application of the technique used in **[9,41]**.

7.10 Proposition. *Let* $s_1, s_2, s_3, s_{12}, s_{23} \in \frac{1}{2}\mathbb{N}$. *Let* $(s_{12} - |s_1 - s_2|), (s_{23} - |s_2 - s_3|) \in \mathbb{N}$. *Then*

$$\mathcal{R}^{s_{12}}_{s_1,s_2} \otimes \mathbb{C}^{2s_3+1} \cap \mathbb{C}^{2s_1+1} \otimes \mathcal{R}^{s_{23}}_{s_2,s_3} \subset \mathcal{R}^{s_{123}}_{s_1,s_2,s_3},$$

provided that $s_{12} + s_{23} - s_2 \leq s_{123}$.

Proof : It is most convenient to realize $\mathbb{C}^{2s+1}$ as the space of complex polynomials in two variables u and v, which are homogeneous of degree $2s$. The elements of $\mathbb{C}^{2s_1+1} \otimes \mathbb{C}^{2s_2+1}$ thus become polynomials in four variables u_1, v_1, u_2, v_2, and so on for higher tensor products. Then $\psi \in \mathcal{R}^j_{s_1,s_2}$ iff the polynomial ψ can be factorized as

$$\psi(u_1, v_1, u_2, v_2) = (u_1 v_2 - v_1 u_2)^{s_1+s_2-j}\varphi(u_1, v_1; u_2, v_2),$$

for a polynomial φ, which is homogeneous of degree $s_1 - s_2 + j$ in the variables (u_1, v_1), and of degree $s_2 - s_1 + j$ in the second set of variables. For a discussion of this structure see **[35, p.369 ff]**. Consider now a polynomials in six variables, which is in the intersection described in the Proposition, that is a polynomial with two factorizations

$$\begin{aligned}\psi(u_1, v_1; u_2, v_2; u_3, v_3) &= (u_1 v_2 - v_1 u_2)^{s_1+s_2-s_{12}}\varphi(u_1, v_1; u_2, v_2; u_3, v_3)\\ &= (u_2 v_3 - v_2 u_3)^{s_2+s_3-s_{23}}\chi(u_1, v_1; u_2, v_2; u_3, v_3) \quad ,\end{aligned}$$

with polynomials φ, χ. Clearly the factors $(u_1 v_2 - v_1 u_2)$ can not be further factorized into polynomials. Hence by the prime factorization theorem for many variable-polynomials **[40,Sect.2.16]** we find that there must be a polynomial $\tilde{\psi}$ such that

$$\psi(u_1, \cdots v_3) = (u_1 v_2 - v_1 u_2)^{s_1+s_2-s_{12}}(u_2 v_3 - v_2 u_3)^{s_2+s_3-s_{23}}\tilde{\psi}(u_1, \cdots v_3) \quad .$$

Clearly, $\tilde{\psi}$ is homogeneous of total degree $2(s_1 + s_2 + s_3 - (s_1 + s_2 - s_{12}) - (s_2 + s_3 - s_{23})) = 2(s_{12} + s_{23} - s_2)$. Consider now a simultaneous transformation of each variable pair by an SU(2)-transformation $(u_i, v_i) \mapsto (au_i + bv_i, -b^* u_i + a^* v_i)$ with $aa^* + bb^* = 1$. Since the factor multiplying $\tilde{\psi}$ is invariant under such transformations, this degree is also the homogeneous power, with which a, a^*, b, b^* appear in the

transformed polynomial. That is to say, ψ is supported by the subspace of spins less than $s_{12} + s_{23} - s_2$.

■

The simplest example of this situation occurs when $J = 1$ and $j = 1/2$. In this case the nearest neighbor interaction h is precisely the projection onto the spin 2 subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$, which can explicitly been written in terms of the generators $\vec{S}$ of the spin 1 representation of SU(2) as:

$$h = \frac{1}{3} + \frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 \quad .$$

For examples of half-integer spin models we refer to **[30]**.

Appendix: Matrix order and conditions for positivity

The concept of matrix order originated in the theory of operator algebras **[10,23,24,26,50]**. As a starting point one might take the observation that the order structure of a C*-algebra almost determines the algebraic structure, in the sense that an order isomorphism beween C*-algebras can be split in a certain sense into a homomorphism and an antihomomorphism. Antihomomorphisms like the transpose map on a matrix algebra behave strangely also in that the tensor product of such a map with the identity map of another algebra fails to be positive. However, if one imposes on (iso-)morphisms the requirement of "complete positivity", i.e. the stability of positivity under tensoring with identity maps, then "order isomorphism" implies algebraic isomorphism. A matrix ordering of a vector space is just the "enhanced order structure",corresponding to this more restrictive notion of order isomorphism. The reason this strucure appears in the present context is that an ordered linear subspace or quotient of a C*-algebra automatically inherits a matrix ordering from the algebra, but, unless it is a subalgebra, it carries no canonical product operation. We now proceed with the formal definitions.

For any complex vector space $\mathcal{B}$, we shall denote by $\mathcal{M}_n(\mathcal{B})$ the space of $n \times n$-matrices with entries in $\mathcal{B}$. We shall also identify this space with $\mathcal{M}_n \otimes \mathcal{B}$, where we have written $\mathcal{M}_n$ for $\mathcal{M}_n(\mathbb{C})$. $\mathcal{M}_{n,m}$ will denote the space of complex $n \times m$ matrices $V = (V_{ij})_{i=1}^{n}{}_{j=1}^{m}$, and for any $B \in \mathcal{M}_n(\mathcal{B})$, $V \in \mathcal{M}_{n,m}$ we define $V^*BV \in \mathcal{M}_m(\mathcal{B})$ by

$$(V^*BV)_{jj'} = \sum_{i,i'} \overline{V_{ij}} B_{ii'} V_{i'j'} .$$

When $\mathcal{B}$ has an antilinear involution $B \mapsto B^*$, an **ordering** of $\mathcal{B}$ is defined by a proper generating cone $\mathcal{B}_+ \subset \mathcal{B}_h = \{B \in \mathcal{B} \mid B = B^*\}$, i.e. $\mathcal{B}_+$ is closed under addition and multiplication with positive scalars, $\mathcal{B}_+ \cap (-\mathcal{B}_+) = \{0\}$, and $\mathcal{B}_+$ generates $\mathcal{B}$ as a vector space. $\mathcal{M}_n(\mathcal{B})$ will then always be taken with the involution $(B^*)_{ij} = (B_{ji})^*$. A **matrix ordered space** $\mathcal{B}$ is by definition a complex vector space with involution, such that every $\mathcal{M}_n(\mathcal{B})$ is ordered by a proper generating cone $\mathcal{M}_n(\mathcal{B})_+ \subset \mathcal{M}_n(\mathcal{B})_h$, and these cones have the property that for all $n, m \in \mathbb{N}, B \in \mathcal{M}_n(\mathcal{B})_+$, $V \in \mathcal{M}_{nm}$ we have $V^*BV \in \mathcal{M}_m(\mathcal{B})_+$. A linear map $\mathbb{F} : \mathcal{A} \to \mathcal{B}$ between matrix ordered spaces is called **completely positive**, if the maps $\mathbb{F}_n = \mathrm{id}_{\mathcal{M}_n} \otimes \mathbb{F}$, i.e. the maps defined by $(\mathbb{F}_n A)_{ij} = \mathbb{F}(A_{ij})$, $i, j = 1, \cdots n$, are positive for all n.

If $\mathcal{B}$ is matrix ordered, and $\mathcal{A}$ is a finite dimensional C*-algebra, then $\mathcal{A} \otimes \mathcal{B}$ is matrix ordered in a canonical way: since $\mathcal{A} = \bigoplus_\alpha \mathcal{M}_{n_\alpha}$, for some finite set of

(possibly equal) numbers $n_\alpha \in \mathbb{N}$, we can set

$$\begin{aligned} \mathcal{M}_n(\mathcal{A} \otimes \mathcal{B})_+ &= \bigoplus_\alpha (\mathcal{M}_n \otimes \mathcal{M}_{n_\alpha} \otimes \mathcal{B})_+ \\ &= \bigoplus_\alpha (\mathcal{M}_n \cdot n_\alpha(\mathcal{B}))_+ \ . \end{aligned}$$

Therefore, it makes sense to demand in Proposition 2.3 that $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ is completely positive.

It is evident that the composition of completely positive maps is completely positive. Moreover, if $\mathbb{F} : \mathcal{B}_1 \to \mathcal{B}_2$ is completely positive, and $\mathcal{A}$ is a finite dimensional C*-algebra, the map $\mathrm{id}_{\mathcal{A}} \otimes \mathbb{F} : \mathcal{A} \otimes \mathcal{B}_1 \to \mathcal{A} \otimes \mathcal{B}_2$ is completely positive. Note that this is all that is needed for the argument given after Proposition 2.3, which shows that complete positivity of $\mathbb{E}$ is indeed sufficient to ensure positivity of the state generated by $\mathbb{E}$ and positive elements $e \in \mathcal{B}, \rho \in \mathcal{B}^*$.

The second direction of Proposition 2.3 is now contained in the following Lemma:

A.1 Lemma. *Let $\mathcal{A}$ be a finite dimensional C*-algebra, and let ω be a finitely correlated state on $\mathcal{A}_{\mathbb{Z}}$. Let $\mathcal{B}$ denote the unique minimal space characterized in Proposition 2.1. Then $\mathcal{B}$ can be matrix ordered such that $\mathbb{E}$ is completely positive.*

Proof: Clearly, $\mathcal{B}$ inherits an involution from $\mathcal{A}_\sharp$ by setting $[\hat{A}]^* = [\hat{A}^*]$. We define $B \in \mathcal{M}_n(B)$ to be positive, if there is some $\hat{A} \in \mathcal{M}_n(\mathcal{A}_\sharp)_+$, such that $B_{ij} = [\hat{A}_{ij}]$. Clearly, this defines a generating cone in $\mathcal{M}_n(B)_h$. It is also proper, because if both $B \in \mathcal{M}_n(B)_+$ and $-B \in \mathcal{M}_n(B)_+$, we have $A, A' \in \mathcal{M}_n(\mathcal{A}_\sharp)$ such that for all $0 \le X \in \mathcal{A}_{\{n|n\le 0\}}$ the $n \times n$-matrix $\omega(X \otimes A_{ij}) = \Phi_X(B_{ij}) = \omega(X \otimes A'_{ij})$ is both positive and negative semidefinite. Thus $\Phi_X(B_{ij})$ vanishes for all positive X, hence for all $X \in \mathcal{A}_{\{n|n\le 0\}}$, hence $B = 0$. The compatibility of these cones for different n follows directly from the corresponding property of $\mathcal{A}_\sharp$.

Now let $\mathbb{E}(A \otimes [\hat{A}]) = [A \otimes \hat{A}]$ as in Lemma 1.1. and let $\mathcal{A} = \bigoplus_\alpha \mathcal{M}_{n_\alpha}$ as above. Then by definition $X = \bigoplus_\alpha X^\alpha \in \mathcal{M}_n(\mathcal{A} \otimes \mathcal{B}) = \bigoplus_\alpha \mathcal{M}_n(\mathcal{M}_{n_\alpha}(\mathcal{B}))$ is positive iff for each α there is a positive $\hat{X}^\alpha \in \mathcal{M}_n(\mathcal{M}_{n_\alpha}(\mathcal{A}_\sharp))$ such that for $i, j = 1, \ldots n$, $\mu, \nu = 1, \ldots n_\alpha$ we have $(X^\alpha_{ij})_{\mu\nu} = [(\hat{X}^\alpha_{ij})_{\mu\nu}]$. Hence $\bigoplus_\alpha \hat{X}^\alpha \in \mathcal{M}_n(\mathcal{A} \otimes \mathcal{A}_\sharp)$ is also positive, and so is its equivalence class $[\hat{X}] = \mathbb{E}(X)$.

■

Note that the matrix order for $\mathcal{B}$ is defined completely in terms of ω. This has an important consequence: if there is some automorphism α of $\mathcal{A}$, such that ω is invariant under sitewise application of α, formally $\omega \circ \alpha^\infty$, then $\beta([A]) = [\alpha^\infty(A)]$ defines an invertible linear map on $\mathcal{B}$. Obviously, $\mathbb{E}(\alpha(A) \otimes \beta(B)) = \alpha(\mathbb{E}(A \otimes B)$.

And by simply transforming every step in the construction with α or β, we find that β is even completely positive. Clearly, this is would be a very useful fact for the discussion of gauge groups, as in section 7.3, were it not for the untractability of the theory of group representations on general matrix ordered spaces.

We remark that some of the results stated in the paper for C*-finitely correlated states can be proven for general finitely correlated states as well. Among these are Proposition 2.6, and a variant of Proposition 3.1. However, Proposition 3.3 explicitly uses the product in $\mathcal{B}$, and all of section 4,5,6 would be very difficult to generalize, since there seems to be no dilation theory for completely positive maps between general matrix ordered spaces.

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