KL–TH–91/9
DIAS-STP-91-27

A (2+1)-dimensional model with instanton and sphaleron solutions

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Abstract: We present a (2+1)–dimensional Skyrme–like model with a symmetry–breaking potential, which in $\mathbb{R}_3$ has charge –n instanton solutions, and in the static limit in $\mathbb{R}_2$ a sphaleron solution.

PACS: 03.65.Sq, 11.10.Lm, 11.15.Kc

While the quantum tunnelling between topologically distinct vacua of the Weinberg–Salam (gauge–Higgs) field theory is known to be negligible[1)], it is possible that at sufficiently high temperatures transitions may occur essentially classically via sphaleron field configurations, leading to an appreciable violation of baryon–number conservation. This mechanism was first suggested by Manton[2)] and was further developed in ref. [3].

Using the sphaleron field configuration of the Weinberg–Salam model, which was previously known as the DHN solution[4)], the estimation[5)] of the baryon–number violation of electroweak theory can be a task of considerable complexity in the quantum theory. For this reason, much attention has been devoted to carrying out this programme employing simplified toy models in lower (than physical) dimensions[6) 7) 8)]. Notable among these models are those in 1+1 dimensions, where the sphaleron in question is a constant static solution on S', of the ϕ^4–model and the sine–Gordon model respectively[7) 8)]. In the latter example[8)], an extended version of the O(3) sigma model in 2 dimensions has been proposed as the corresponding dynamical system in 1+1 dimensions. In both these models,[7) 8)] as also in the original DHN solution on $\mathbb{R}_3$, the sphaleron is an unstable field configuration with finite energy. The energy is the d–dimensional integral of the static field configuration, namely d=3 for the DHN case, and d=1 for the toy models of refs. [7] and [8]. The sphaleron field's energy is then regarded as the energy–barrier between the topologically distinct vacua of the non–static theory. In all these models, the topological charges characterizing the distinct vacua are defined by the usual topological invariant. In the Weinberg–Salam theory, this is taken to be the integral of the Chern–Pontryagin density on $\mathbb{R}_4$, while in the O(3) model of ref. [8], the topological charge is the winding number of the order–parameter field defined on $\mathbb{R}_2$. In both cases, the dynamical models on d+1 dimensions, supporting stable instanton field configurations, differ from the dynamical models on d–dimensions, which supports unstable sphaleron field configurations.

The purpose of the present note is to propose a new model in 2+1 dimensions, which supports stable instanton field configurations on $\mathbb{R}_3$, and in the static limit supports unstable sphaleron field configurations on $\mathbb{R}_2$. As such, it is an intermediate example with d=2, between the DHN case[4] with d=3 and the soliton cases[7,8] with d=1. This toy-model aspect though is not the main reason for proposing it. Its most important property is, that unlike the d=1 and d=3 examples discussed above, the instanton and sphaleron field configurations are supported as solutions by one and the same model.

To help us arrive at our model, we shall first note a common feature of both the DHN and the extended–O(3)–model sphalerons. In each case, respectively in d=3 and d=1, the scaling properties of the models are consistent with there being finite energy solutions. Such solutions could be topologically stable if there were topological inequalities supplying lower bounds to the energy integrals. In turn, such topological inequalities can be found only for specific field–multiplets defining the dynamical coordinates. Specifically, for the SU(2) Yang–Mill–Higgs model on $\mathbb{R}_3$, such a topological charge (the monopole charge) can be defined if the Higgs field is in the adjoint representation of SU(2), and, for the soliton model in one dimension, such a topological charge (the kink–number) can be defined if the field variable consists of one real scalar quantity. The (unstable) sphaleron solutions on the other hand do not occur in the two models just described. Instead in the DHN case[4], the Higgs field is an isospinor and consists of four real components as opposed to the three of an adjoint representation Higgs, and in the extended–O(3) model case[8], the order–parameter has two real components as opposed to the single component of the scalar field of the soliton model. In each case (d=3 and 1), the additional component of the dynamical field variable serves to parametrize the noncontractible orbit through the instability point.

In the light of these observations, we proceed to consider the model of ref [9] on $\mathbb{R}_2$, i,j = 1,2,

$$\hat{\mathcal{L}}_0 = \frac{1}{2}(i\,\partial_{[i}\varphi\partial_{j]}\varphi^*)^2 + f(\eta^2-|\varphi|^2, |\partial_i\varphi|^2) + V(\eta^2-|\varphi|^2), \qquad (1)$$

where φ is a complex scalar field and η^2 is the (absolute) scale. V is a symmetry breaking

potential, and f is a symbolic function representing the quadratic kinetic term $|\partial_i\varphi|^2$. $\hat{\mathcal{L}}$ is regarded as the static limit of a Lagrangian $\mathcal{L}$ in 2+1 dimensions.

It was shown in ref. [9] that subject to the asymptotic condition

$$|\varphi|^2 \xrightarrow[|\vec{x}| \to \infty]{} \eta^2 \qquad (2)$$

the volume integral of (1) is minimized by topologically stable field configurations, by virtue of the topological inequality

$$\int \hat{\mathcal{L}}_0\, d^2x \geq 2i\epsilon_{ij} \int \sqrt{V}\, \partial_i\varphi\partial_j\varphi^*\, d^2x. \qquad (3)$$

Following our above descriptions of the d=3 and d=1 sphalerons, we modify the model (1) by augmenting the dynamical coordinate φ with an additional component ϕ_3. Thus, in place of $\varphi = \phi_1 + i\phi_2$, our new field variable is $\Phi = \vec{\phi}\cdot\vec{\sigma}$ in terms of the Pauli spin matrices $\vec{\sigma}$. This yields

$$\mathcal{L}_0 = -\frac{1}{2}\,\mathrm{tr}\,\Phi_{ij}^2 + f(\eta^2-\Phi^2, \Phi_i^2) + V(\eta^2-\Phi^2), \qquad (4)$$

where we use the notation $\Phi_i := \partial_i\Phi$ and $\Phi_{ij} := [\Phi_i, \Phi_j]$. Again f is a symbolic function representing the, now non Abelian, quadratic kinetic term Φ_i^2. One should note that the scaling properties of the integral of (4) over $\mathbb{R}_2$, are still consistent with the existence of finite energy solutions, but now we have lost the topological inequality (3). This is so because, the corresponding topological charge density $\epsilon_{ij}\,\mathrm{tr}\,\sqrt{V}\,\partial_i\Phi\partial_j\Phi$ can be seen not to be a total divergence, in contrast with the density on the right hand side of (3) defined in terms of the complex field φ. As a consequence, we would expect any finite energy solutions to the equations of motion that may be found, to be unstable. But this is precisely what would be expected of a sphaleron field, expecially if we remember that the source of this new instability is the additional component of the multiplet Φ, over and above the number of degrees of freedom of the old field φ in (1). We adopt (4) therefore as the static version of a candidate for a (2+1)–dimensional model with instanton <u>and</u> sphaleron solutions, and proceed to verify these properties. For technical reasons, we consider the instanton properties first.

<u>Instantons:</u> It is useful to specialize the Lagrangian (4), considered on $\mathbb{R}_3$, to analyze the stability of the instanton solutions. This problem was considered in some detail, and analyzed in ref. [10]. To avoid the ubiquity of models afforded by the symbolic functions f and V in (4), we specialize to some specific choices of these functions.

To start with, according to the virial theorem or scaling argument, it is necessary to keep only the first and second, or, the first and third terms in

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,\Phi_{\mu\nu}^2 + f(\eta^2-\Phi^2,\ \Phi_\mu^2) + V(\eta^2-\Phi^2), \tag{5}$$

to enable finite action solutions on $\mathbb{R}_3$. Here $\mu = 1,2,3$, labels the coordinate x_μ of $\mathbb{R}_3$. However, as explained in detail in refs. [10,11], in the absence of the second term f, topological stability would dictate the inclusion of an additional sextic kinetic term $\Phi_{\mu\nu\rho}^2$ which we wish to avoid here. We therefore must retain the second term f in (5). Topological stability does not demand the presence of third term V. Nevertheless, we shall retain V, in anticipation of a similar scaling argument, for the static Lagrangian $\mathcal{L}_0$ of (4), in $\mathbb{R}_2$.

Retaining both f and V in (5), we opt to specialize (5) to the simplest sub–model arising from the direct descent from the 8–dimensional conformally invariant generalized Yang–Mills system[11]. The distinguishing feature of this model, other than its relative simplicity, is that it involves no dimensional constants apart from the constant η setting the scale of the field Φ. Our choice is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,\Phi_{\mu\nu}^2 + \frac{1}{2}\,\mathrm{tr}\,\{S,\Phi_\mu\}^2 + \mathrm{tr}\,S^4 \tag{6}$$

where $S := \eta^2 - \Phi^2$ and { , } means anticommutation. We stress that our choices for f and V in (6) are not unique.

The topological stability of the instanton is then a consequence of the inequality

$$\mathrm{tr}\left[i\Phi_{\mu\nu} - \frac{1}{\sqrt{2}}\,\epsilon_{\mu\nu\rho}\{S,\Phi_\rho\}\right]^2 \geq 0. \tag{7}$$

Adding the positive–definite term $2\,\mathrm{tr}\,S^4$ to the left–hand–side of (7) without disturbing the inequality, and expanding (7), we have

$$\mathcal{L} \geq 2\sqrt{2}\,i\,\epsilon_{\mu\nu\rho}\,\mathrm{tr}\,\{S,\Phi_\mu\}\Phi_\nu\Phi_\rho, \tag{8}$$

the right–hand–side of which can be shown to be a total divergence[10] [11], whose integral, subject to the asymptotic condition

$$\mathrm{tr}\,\Phi^2 \xrightarrow[|\vec{x}| \to \infty]{} \eta^2, \tag{9}$$

guarantees a non–zero lower bound for the action which is proportional to a winding number n. Thus the model (6) is endowed with a stable instanton field configuration in $\mathbb{R}_3$.

Since the instanton field configurations of this (and other) models(s) on $\mathbb{R}_3$ were discussed in some detail in ref. [10], we suffice here by recalling that these instantons correspond to topologically distinct vacua characterized by a winding number n, which in this case is the topological charge given (up to normalisation) by the integral of the right–hand–side of (8). The n–dependence of these field configurations is given[10] by

$$\phi_1 = \phi(R)\sin\theta\cos n\varphi$$
$$\phi_2 = \phi(R)\sin\theta\sin n\varphi$$
$$\phi_3 = \phi(R)\cos\theta, \qquad (10)$$

where $R = \sqrt{x_\mu x_\mu}$, θ and φ are the polar and azymuthal angles in 3 dimensions, and ϕ_μ defines $\Phi = \phi_\mu \sigma_\mu$. θ and φ parametrize both the field and the space $S^2 \subset R_3$.

Sphalerons: The static version of (6), defined on $\mathbb{R}_2$,

$$\mathcal{L}_0 = -\frac{1}{2}\,\mathrm{tr}\,\Phi_{ij}^2 + \frac{1}{2}\,\mathrm{tr}\,\{S,\Phi_i\}^2 + \mathrm{tr}\,S^4 \qquad (11)$$

will now be shown to have a sphaleron solution. First we recall that according to the scaling argument, the equations of motion for (11) can have finite energy solutions irrespective of the absence/presence of the second term quadratic in Φ_i. We also note that now, we have no topological inequality analogous to (8), so that the finite energy configurations are nontopological.

We consider the following Ansatz for the (unstable) sphaleron field configurations

$$\Phi = \sigma_1 \eta f(r)\sin\mu\cos\theta + \sigma_2 \eta f(r)\sin\mu\sin\theta + \sigma_3 \eta g(r)\cos\mu, \qquad (12)$$

where $r^2 = x_i x_i$ $(i = 1,2)$, and θ is the azymuthal angle, while μ is a constant which we expect will parametrize the noncontractible path between topologically distinct vacua, and so the instability of the energy functional. Before proceeding to demonstrate this instability, we must check the consistency of this Ansatz. This involves the verification that the Euler–Lagrange equation of the system (11) on $\mathbb{R}_2$

$$\partial_i\,[\Phi_j,\Phi_{ij}] + \frac{1}{2}\,\partial_i\,\{\{S,\Phi_i\},S\} = 2\{\Phi,S^3\} - \{\Phi,\{\Phi_i\{\Phi_i,S\}\}\} \qquad (13)$$

for the field configuration (12), are solved by the Euler–Lagrange equations for the one–dimensional subsystem with Lagrangian $L[f,g]$, defined by $S = \int \mathcal{L}\, r dr d\Theta \equiv 2\pi \int L dr$, or

$$L\,[f(r),f'(r);g(r),g'(r)] \equiv 2\pi\,\mathcal{L}[f,f';g,g'], \qquad (14)$$

in terms of the coordinates f,g and their "velocities" $f' \equiv df/dr$ and g'. This is a very straightforward if tedious task, and we limit ourselves to stating that indeed the Euler–Lagrange equations arising from the variations of f(r) and g(r), respectively, for (14), solve the equations (13) for the field configuration (12). These equations are rather lengthy expressions, and are not recorded here, but we make a pertinent comment: that if we set $f(r) = g(r)$ in the Ansatz (12), the consistency of this Ansatz is lost. We shall return to the detailed discussion of this inconsistency elsewhere.

The existence[12] of the sphaleron field configuration (12) then follows from the positive definiteness of the energy integral

$$\begin{aligned} E[f,g,\mu] = 4\pi \int_0^\infty \{ & 4\eta^4 \sin^2\mu \frac{f^2}{r}(g'^2\cos^2\mu + f'^2\sin^2\mu) \\ & + 2\eta^6 r\,[1-(g^2\cos^2\mu + f^2\sin^2\mu)]^2\,[(g'^2\cos^2\mu + f'^2\sin^2\mu) + \frac{f^2}{r^2}\sin^2\mu] \\ & + \eta^8 r\,[1-(g^2\cos^2\mu + f^2\sin^2\mu)]^4 \}\,dr. \end{aligned} \qquad (15)$$

The all–important property of instability is manifest, parametrised by the μ–dependence of the integrand in (15). The actual sphaleron is the (unstable) field configuration at the top of the barrier separating the two distinct vacua, for which the value of (15) is a maximum. This occurs for $\mu = \frac{\pi}{2}$, and by varying μ between 0 and π, in the two directions away from the sphaleron value of $\frac{\pi}{2}$, the value of E can be lowered.

Topological charges: We have shown above that the (2+1)–dimensional model given by the Lagrangian (6) is endowed with charge–n instanton solutions in $\mathbb{R}_3$, and its static version (10) with a sphaleron solution in $\mathbb{R}_2$. As the latter is expected to be the energy barrier given by the static fields, between the *topologically distinct vacua of the same* model in $\mathbb{R}_3$, it remains for us to demonstrate this property by verifying that the (topological) charge integral

$$q \equiv \int dt d^2x \rho = 2\sqrt{2}\, i\epsilon_{\mu\nu\rho} \int_{t=-\infty}^{+\infty} \int_{\mathbb{R}_2} \mathrm{tr}\, S\, \Phi_\mu \Phi_\nu \Phi_\rho \, dt\, d^2x, \tag{16}$$

(cf. eq. (8)) for a (2+1)–dimensional field configuration including the sphaleron field (12), can be evaluated as a surface integral whose value is controlled by the topological properties of the field Φ, in $\mathbb{R}_2$. To this end, we follow the procedure first suggested in refs. [2,3], and employed in ref. [13]. This involves adopting a field configuration $\Phi(\vec{x},t)$ given by (12), where the functions f and g depend on the radial variable r of $\mathbb{R}_2$, but where the coordinate μ is taken to be a function of t, $\mu = \mu(t)$. Writing $d\mu/dt \equiv \dot{\mu}$, the integral (16)

$$q = \int dt d^2x \rho = \frac{4}{\sqrt{2}} i\, \epsilon_{ij}\, \mathrm{tr} \int\int dt\, d^2x\, S(\Phi_t \Phi_i \Phi_j + \Phi_i \Phi_t \Phi_j + \Phi_i \Phi_j \Phi_t), \tag{17}$$

can then be expressed as

$$q = q_0 + q_1 \tag{18}$$

$$q_0 = 2\pi \int r dr \int dt\, \dot{\mu} \sin\mu \frac{f}{r} [gf' + (fg' - gf') \cos^2\mu] \tag{18a}$$

$$q_1 = 2\pi \int r dr \int dt\, \dot{\mu} \sin\mu \frac{f}{r} [(g^2 - f^2) \cos^2\mu + f] [(fg' - gf') \cos^2\mu + gf'] \tag{18b}$$

Now allowing $\mu(t)$ to vary between 0 and π as t varies from $-\infty$ to $+\infty$, we can perform the integrals (18a,b) as integrals with respect to $\cos\mu$, between the limits $\cos\mu(t = \pm\infty) = \pm 1$. The result is

$$q_0 = 4\pi \int_0^\infty h_0(r)\, dr \tag{19a}$$

$$q_1 = 4\pi \int_0^\infty h_1(r)\, dr, \tag{19b}$$

where both integrals can be evaluated simply by using the topologically meaningful boundary values of f and g, by virtue of the fact that the functions h_0 and h_1 are given as the derivatives

$$h_0 = \frac{1}{3} \frac{d}{dr} (gf^2) \tag{20a}$$

$$h_1 = \frac{1}{15} \frac{d}{dr} [gf^2(g^2 + 2f^2)]. \tag{20b}$$

The integrals (19a,b) are then immediately evaluated using the asymptotic conditions

$$g(\infty) = f(\infty) = 1 \tag{21}$$

which is consistent with the finite–energy condition

$$\mathrm{tr}\, \Phi^2 \xrightarrow[r \to \infty]{} \eta^2, \tag{22}$$

for the field (12), analogous to the finite–action condition (9), for the field (10). The boundary condition at the origin of r is

$$f(0) = 0, \tag{23}$$

which is also the necessary condition for the singlevaluedness of the field (12). This defines, up to normalization, the topological charge of the (2+1)–dimensional model (6), which has charge –n instanton solutions (10) in $\mathbb{R}_3$, and in the static limit a sphaleron solution (12) in $\mathbb{R}_2$. Thus one can associate a finite value of the instanton (topological) charge with the (nontopological) sphaleron.

This work was supported in part by GKSS (Geesthacht/Germany) and EOLAS (Dublin/Ireland).

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