

# ON THE ORIGIN OF THE UV-IR MIXING IN NONCOMMUTATIVE MATRIX GEOMETRY

Sachindeo Vaidya<sup>†1</sup> and Badis Ydri<sup>\*2</sup>

<sup>†</sup>*Department of Physics,  
University of California, Davis, CA 95616, USA.*

<sup>\*</sup>*School of Theoretical Physics,  
Dublin Institute for Advanced Studies, Dublin, Ireland.*

## Abstract

Scalar field theories with quartic interaction are quantized on fuzzy  $S^2$  and fuzzy  $S^2 \times S^2$  to obtain the 2- and 4-point correlation functions at one-loop. Different continuum limits of these noncommutative matrix spheres are then taken to recover the quantum noncommutative field theories on the noncommutative planes  $\mathbb{R}^2$  and  $\mathbb{R}^4$  respectively. The canonical limit of large stereographic projection leads to the usual theory on the noncommutative plane with the well-known singular UV-IR mixing. A new planar limit of the fuzzy sphere is defined in which the noncommutativity parameter  $\theta$ , beside acting as a short distance cut-off, acts also as a conventional cut-off  $\Lambda = \frac{2}{\theta}$  in the momentum space. This noncommutative theory is characterized by absence of UV-IR mixing. The new scaling is implemented through the use of an intermediate scale that demarcates the boundary between commutative and noncommutative regimes of the scalar theory. We also comment on the continuum limit of the 4-point function.

## 1 Introduction and Results

Noncommutative manifolds derive their interest not only from the fact that they make their appearance in string theory (see for eg [1] for a review of noncommutative geometry in string theory), but also because they can potentially lead to natural ultra-violet regularization of quantum field theories. The notion of noncommutativity suggests a “graininess” for spacetime, and hence can have interesting implications for models of quantum gravity.

Theoretical research has usually focused on either “flat” noncommutative spaces like  $\mathbb{R}^{2n}$  or noncommutative tori  $T^{2n}$ , or “curved” spaces that can be obtained as co-adjoint orbits of Lie groups. In the latter category, attention has mostly focused on using compact groups leading to noncommutative versions of  $\mathbb{CP}^n$  [2–4], which are described by finite-dimensional matrices and one or more size moduli: for example, the fuzzy sphere is described by  $N \times N$  matrices and its radius  $R$ . (We use descriptions like “flat” or “curved compact” only in a loose sense here).

Considerable attention has thus been devoted in trying to understand properties of simple theories written on noncommutative manifolds. In this endeavor, attention has most often been devoted to theories on noncommutative  $\mathbb{R}^{2n}$  and  $T^{2n}$  in the case of flat spaces, and the curved space  $S_F^2$  (the fuzzy sphere). Theories on the noncommutative flat spaces generally possess infinite number of degrees of freedom in contrast to those on “compact” spaces like  $S_F^2$ . In either case, a key property of noncommutative theories that is different from ordinary ones is the nature of the rule for multiplying two functions. For example, the star-product on  $\mathbb{R}^{2n}$  (involving the noncommutativity parameter  $\theta$ ) is used for noncommutative theories, while ordinary theories use the usual point-wise multiplication. On the other hand, functions on curved compact noncommutative spaces are simply finite-dimensional matrices, and are multiplied by the usual matrix multiplication. This makes theories on “curved” noncommutative spaces easier to study numerically (although it must also be mentioned here that the torus with rational noncommutativity can also be studied using finite-dimensional matrices [6]).

Working on curved compact spaces also allows us to study the flattening limit, which is when we take matrix size as well as the length moduli to infinity. For example, the fuzzy sphere  $S_F^2$  can be flattened to give us the noncommutative plane. In this limit, we expect to reproduce the behavior of the theory on the flat manifold. Surprisingly, this limit can be crafted in a variety of ways.

A simple way to understand this is as follows. All dimensionful quantities can be expressed in terms of “radius moduli”, i.e. the length scale that defines the size of the compact space. Continuum limit usually corresponds to taking the size of the matrices to infinity, while flattening corresponds to taking large radii. However, there is a large family of scales available to us in this flattening limit. In other words, there are many ways of getting a relevant length dimension quantity on the non-compact space. We could scale both  $R$  and  $N$  to infinity keeping  $R/N^\alpha$  fixed, where  $\alpha$  is some number. This corresponds to a length scale on the plane, and all quantities in the quantum field theory (QFT) on the plane can be measured with respect to this scale. A priori, one would suspect that different values of  $\alpha$  can lead to theories that behave dramatically differently.

As we will argue here, this variety in the choice of scaling gives us a refined probe to understand the nature of noncommutativity more clearly. In particular, we will show with two different scaling limits how this works. One corresponds to “strongly” noncommutative theories, possessing singular properties like UV-IR mixing that makes it impossible to write down corresponding low-energy Wilsonian actions. The other corresponds to “weak” noncommutativity in a sense that we will make precise. Briefly, these weakly noncommutative theories are defined on a noncommutative plane, but do not exhibit UV-IR mixing. In some sense, these theories mark the edge between noncommutativity and commutativity.

The standard method of investigating perturbative properties of a scalar QFT is by introducing an ultra-violet cut-off (see for example [20]). Instead of working with arbitrarily high energies, one works with the partition function of this cut-off theory, and attempts to study quantities that depend only weakly on the UV cut-off. However, applying this technique to noncommutative theories is problematic [5]: taking the limit of small external momentum does not commute with taking the limit of infinite cut-off. This problem is commonly known as UV-IR mixing.

QFTs on noncommutative curved spaces allow us to implement a finer version of the above procedure. In addition to the natural UV cut-off (characterized by  $1/N$  where  $N$  is the matrix size), we can introduce an intermediate scale  $1/j$  characterized by an integer  $j < N$ . It is the interplay between  $j$ ,  $N \rightarrow \infty$  and  $R \rightarrow \infty$  that we will exploit to understand the “edge” between commutativity and noncommutativity.

In this article, we make concrete this set of ideas by applying them to  $S_F^2$  and  $S_F^2 \times S_F^2$ . The former is characterized by  $(2l+1) \times (2l+1)$  matrices and radius  $R$ , the latter by two copies of the same matrix algebra and two radii  $R_1$  and  $R_2$ . Flattening these spaces by taking  $l$  and  $R_i$  to infinity (in a prescribed manner) gives us noncommutative  $\mathbb{R}^2$  and  $\mathbb{R}^4$  respectively.

In particular, we will study two such scalings here. For example for  $S_F^2$ , we keep  $\theta' = R/\sqrt{l}$  fixed in the first case, and keep  $\theta = R/l$  fixed in the second, as we take  $l$  and  $R$  to infinity. The former gives us the usual theory on the noncommutative plane, which at the one-loop level reproduces the singularities of UV-IR mixing. The latter is a new limit, and corresponds to keeping the UV cut-off fixed in terms of the noncommutative parameter  $\theta$ .

A short version explaining the new scaling limit appeared in [7].

The fuzzy sphere is described by three matrices  $x_i^F = \theta L_i$  where  $L_i$ ’s are the generators of  $SU(2)$  for the spin  $l$  representation and  $\theta$  has dimension of length. The radius  $R$  of the sphere is related to  $\theta$  and  $l$  as  $R^2 = \theta^2 l(l+1)$ . The usual action for a matrix model on  $S_F^2$  is

$$S = \frac{R^2}{2l+1} \text{Tr} \left( \frac{[L_i, \Phi]^\dagger [L_i, \Phi]}{R^2} + m^2 \Phi^2 + V[\Phi] \right), \quad (1.1)$$

and has the right continuum limit as  $l \rightarrow \infty$ . Because of the noncommutative nature of  $S_F^2$ , there is a natural ultra-violet (UV) cut-off: the maximum energy  $\Lambda_{max}^2$  is  $= 2l(2l+1)/R^2$ . To get the theory on a noncommutative plane, the usual strategy is to restrict to (say) the north pole, define the noncommutative coordinates as  $x_a^{NC} = x_a^F$ , ( $a = 1, 2$ ), and then take both  $l$  and  $R$  to infinity in a precisely specified manner. For example, a com-

<sup>1</sup>vaidya@dirac.ucdavis.edu

<sup>2</sup>ydri@usuge.stp.dias.ie

monly used limit requires us to hold  $\theta' = R/\sqrt{l}$  fixed as both  $R$  and  $l$  increase, which gives us a noncommutative plane with [10, 11]

$$[x_1^{NC}, x_2^{NC}] = -i\theta'^2. \quad (1.2)$$

It is easy to see that in this limit,  $\Lambda_{max}$  diverges, while  $\theta$  tends to zero. This is the analogous to the standard stereographic projection.

A second scaling limit which is of interest to us here is one in which  $R$  and  $l$  become large with noncommutativity parameter given now by  $\theta = R/l$  kept fixed. The above noncommutativity relation becomes simply  $[x_1^{NC}, x_2^{NC}] = -iR\theta$  which means that  $x_a^{NC}$ 's are now strongly noncommuting coordinates ( $R \rightarrow \infty$ ) and hence nonplanar amplitudes are expected to simply drop out in accordance with [5]. This can also be seen from the fact that in this scaling (as is obvious from the relation  $R^2 = \theta^2 l(l+1)$ )  $\Lambda_{max}$  no longer diverges: it is now of order  $1/\theta$ , and there are no momentum modes in the theory larger than this value. Alternatively we will also show that in this limit the noncommutative coordinates can be instead identified as  $X_a^{NC} = x_a^{NC}/\sqrt{l}$  with noncommutative structure

$$[X_1^{NC}, X_2^{NC}] = -i\theta^2. \quad (1.3)$$

While this scaling for obtaining  $\mathbb{R}_\theta^2$  is simply stated, obtaining the corresponding theory with the above criteria is somewhat subtle. Indeed passing from  $[x_1^{NC}, x_2^{NC}] = -iR\theta$  to (1.3) corresponds in the quantum theory to a re-scaling of momenta sending thus the finite cut-off  $\Lambda = \frac{2}{\theta}$  to infinity. In order to bring the cut-off back to a finite value  $\Lambda_x = x\Lambda$ , where  $x$  is an arbitrary positive real number, we modify the Laplacian on the fuzzy sphere  $\Delta = [L_i, [L_i, \cdot]]$  so that to project out modes of momentum greater than a certain value  $j$  given by  $j = \lfloor \frac{2\sqrt{l}}{x} \rfloor$ . In other words, the theory on the noncommutative plane  $\mathbb{R}_\theta^2$  with UV cut-off  $\theta^{-1}$  is obtained by flattening not the full theory on the fuzzy sphere but only a “low energy” sector. One can argue that only for when  $\Lambda_x = \Lambda$  that the canonical UV-IR singularities become smoothen out. At this value we have  $j = \lfloor 2\sqrt{l} \rfloor$  which marks somehow the boundary between commutative and noncommutative field theories.

The generalization to noncommutative  $\mathbb{R}^4$  is obvious. We work on  $S_F^2 \times S_F^2$  and then take the scaling limit with  $\theta$  fixed, which is the case of most interest in this article. By analogy with (1.1), the scalar theory with quartic self-interaction on  $S_F^2 \times S_F^2$  is

$$S = \frac{R_a^2}{2l_a + 1} \frac{R_b^2}{2l_b + 1} Tr_a Tr_b \left( \frac{[L_i^{(a)}, \Phi]^\dagger [L_i^{(a)}, \Phi]}{R_a^2} + \frac{[L_i^{(b)}, \Phi]^\dagger [L_i^{(b)}, \Phi]}{R_b^2} + \mu_i^2 \Phi^2 + \frac{\lambda_4}{4!} \Phi^4 \right), \quad (1.4)$$

where  $a$  and  $b$  label the first and the second sphere respectively, and  $L_i^{(a,b)}$ 's are the generators of rotation in spin  $l_{a,b}$ -dimensional representation of  $SU(2)$ , and  $\Phi$  is a  $(2l_a + 1) \times (2l_a + 1) \otimes (2l_b + 1) \times (2l_b + 1)$  hermitian matrix. As  $l_a, l_b$  go to infinity, we recover the scalar theory on an ordinary  $S^2 \times S^2$ .

Our strategy for obtaining the theory on noncommutative  $\mathbb{R}^4$  is straightforward: as discussed in [8, 14], we expand  $\Phi$  of action (1.4) in terms of  $SU(2)$  polarization tensors (for definition and various properties of polarization tensors, see for example [17]). Using standard perturbation theory and a conventional renormalization procedure, we calculate the two- and four-point correlation functions, and then we scale  $R, l \rightarrow \infty$  with  $\theta$  fixed. Actually (and as we just have said), implementation of the new scaling is somewhat subtle, in that we will need to work not with the full theory on  $S_F^2 \times S_F^2$  but with a suitably defined low-energy sector. This low-energy sector is selected by projecting out the high energy modes in an appropriate manner using projection operators, and thus working with a modified Laplacian:

$$\Delta_j = \Delta + \frac{1}{\epsilon} (1 - P_j), \quad j = \lfloor 2\sqrt{l} \rfloor,$$

where  $P_j$  is the projector on all the modes associated with the eigenvalues  $k = 0, \dots, j$ , and  $\Delta$  is the canonical Laplacian on the full fuzzy sphere  $S_F^2 \times S_F^2$ . The flattening limit (1.3) is thus implemented on the scalar field theory (1.4) as the limit in which we first take  $\epsilon \rightarrow 0$  above, then we proceed with  $R, l \rightarrow \infty$  keeping  $\theta = \frac{R}{l}$  fixed.

An obvious consequence of our scaling procedure is that the correlation functions are not singular functions of external momenta.

There is a nice intuitive explanation for using the modified Laplacian. If the momenta are cut-off at too low a value, the system becomes in the commutative regime, while if the cut-off is too close to  $l$ , the system remains noncommutative. The choice  $\lfloor 2\sqrt{l} \rfloor$  for the cut-off is in some sense the edge between these two situations: there is some noncommutativity in the behaviour, but there is no UV-IR mixing. We will have more to say about this in section 4.

The paper is organized as follows: in section 2 we quantize  $\phi^4$  theory on  $S_F^2 \times S_F^2$  and obtain the one-loop corrections to the 2-point and 4-point functions. We also define in this section the precise meaning of UV-IR mixing on  $S_F^2 \times S_F^2$  and write down the effective action. Section 3 is the central importance, in which we define continuum planar limits of the fuzzy sphere. In particular we show how the singular UV-IR mixing emerges in the canonical limit of large stereographic projection of the spheres onto planes. We also show that in a new continuum flattening limit, a natural momentum space cut-off (inversely proportional to the noncommutativity parameter  $\theta$ ) emerges, and as a consequence the UV-IR mixing is completely absent. Section 3 contains also the computation of the continuum limit of the 4-point function. As it turns out we recover exactly the planar one-loop correction to the 4-point function on noncommutative  $\mathbb{R}^4$ . We conclude in section 4 with some general observations.

## 2 Effective Action on $S_F^2 \times S_F^2$

In this section, we will set up the quantum field theory on  $S_F^2 \times S_F^2$ , making explicit our notation and conventions. These reflect our intent to consider  $S_F^2 \times S_F^2$  as a discrete approximation of noncommutative  $\mathbb{R}^4$ .

Each of the spheres  $(\sum_i x_i^{(a)} x_i^{(a)} = R^{(a)^2}, a = 1, 2)$  is approximated by the algebra  $Mat_{2l_a+1}$  of  $(2l_a + 1) \times (2l_a + 1)$  matrices. The quantization prescription is given as usual, by

$$n_i^{(a)} = \frac{x_i^{(a)}}{R^{(a)}} \rightarrow n_i^{(a)F} = \frac{L_i^{(a)}}{\sqrt{l_a(l_a + 1)}}. \quad (2.1)$$

This prescription follows naturally from the canonical quantization of the symplectic structure on the classical sphere (see for example [19]) by treating it as the co-adjoint orbit  $SU(2)/U(1)$ . The  $L_i^{(a)}$ 's above are the generators of the IRR representation  $l_a$  of  $SU(2)$ : they satisfy  $[L_i^{(a)}, L_j^{(a)}] = i\epsilon_{ijk} L_k^{(a)}$  and  $\sum_{i=1}^3 L_i^{(a)^2} = l_a(l_a + 1)$ . Thus

$$[n_i^{(a)F}, n_j^{(b)F}] = \frac{i}{\sqrt{l(l+1)}} \delta^{ab} \epsilon_{ijk} n_k^{(a)F}. \quad (2.2)$$

Formally,  $S_F^2 \times S_F^2$  is the algebra  $\mathbf{A} = Mat_{2l_1+1} \otimes Mat_{2l_2+1}$  generated by the identity  $\mathbf{1} \otimes \mathbf{1}$  together with  $L_i^{(1)} \otimes \mathbf{1}$  and  $\mathbf{1} \otimes L_i^{(2)}$ . This algebra  $\mathbf{A}$  acts trivially on the  $(2l_1 + 1)(2l_2 + 1)$ -dimensional Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  with an obvious basis  $\{|l_1 m_1\rangle | l_2 m_2\rangle\}$ .

The fuzzy analogue of the continuum derivations  $\mathcal{L}_i^{(a)} = -i\epsilon_{ijk} n_j^{(a)} \partial_k^{(a)}$  are given by the adjoint action: we make the replacement

$$\mathcal{L}_i^{(a)} \rightarrow K_i^{(a)} = L_i^{(a)L} - L_i^{(a)R}. \quad (2.3)$$

The  $L_i^{(a)L}$ 's generate a left  $SO(4)$  (more precisely  $SU(2) \otimes SU(2)$ ) action on the algebra  $\mathbf{A}$  given by  $L_i^{(a)L} M = L_i^{(a)} M$  where  $M \in \mathbf{A}$ . Similarly, the  $L_i^{(a)R}$ 's generate a right action on the algebra, namely  $L_i^{(a)R} M = M L_i^{(a)}$ . Remark that  $K_i^{(a)}$ 's annihilate the identity  $\mathbf{1} \otimes \mathbf{1}$  of the algebra  $\mathbf{A}$  as is required of a derivation.

In fact, it is enough to set  $l_a = l_b = l$  and  $R_a = R_b = R$  as this corresponds in the limit to a noncommutative  $\mathbb{R}^4$  with a Euclidean metric on  $\mathbb{R}^2 \times \mathbb{R}^2$ . The general case simply corresponds to different deformation parameters in the two  $\mathbb{R}^2$  factors, and the extension of all results is thus obvious (see equation (6) of [1]).

In close analogy with the action on continuum  $S^2 \times S^2$ , we put together the above ingredients to write the action on  $S_F^2 \times S_F^2$ :

$$S_l = \frac{R^4}{(2l+1)^2} Tr_{\mathcal{H}} \left[ \frac{1}{R^2} \hat{\Phi} [L_i^{(1)}, [L_i^{(1)}, \hat{\Phi}]] + \frac{1}{R^2} \hat{\Phi} [L_i^{(2)}, [L_i^{(2)}, \hat{\Phi}]] + \mu_l^2 \hat{\Phi}^2 + V(\hat{\Phi}) \right] \equiv S_l^{(0)} + S_l^{int}. \quad (2.4)$$

This action has the correct continuum (i.e.  $l \rightarrow \infty, R$  fixed) limit:

$$\mathcal{S}_\infty = R^4 \int_{S^2} \frac{d\Omega^{(1)}}{4\pi} \frac{d\Omega^{(2)}}{4\pi} \left[ \frac{1}{R^2} \Phi \mathcal{L}_i^{(1)} \mathcal{L}_i^{(1)}(\Phi) + \frac{1}{R^2} \mathcal{L}_i^{(2)} \mathcal{L}_i^{(2)}(\Phi) + \mu_\infty^2 \Phi^2 + V(\Phi) \right]. \quad (2.5)$$

While the technology presented here can be applied to any polynomial potential, we will restrict ourselves to  $V(\Phi) = \frac{\lambda_{4,l}}{4!} \Phi^4$ . We have explicitly introduced factors of  $R$  wherever necessary to sharpen the analogy with flat-space field theories: the integrand  $R^4 d\Omega_1 d\Omega_2$  has canonical dimension of (Length)<sup>4</sup> like  $d^4x$ , the field has dimension (Length)<sup>(-1)</sup>,  $\mu_l$  has (Length)<sup>(-1)</sup> and  $\lambda_{4,l}$  is dimensionless.

Following [8, 14], the fuzzy field  $\hat{\Phi}$  can be expanded in terms of polarization operators [17] as

$$\begin{aligned} \hat{\Phi} &= (2l+1) \sum_{k_1=0}^{2l} \sum_{m_1=-k_1}^{k_1} \sum_{p_1=0}^{2l} \sum_{n_1=-p_1}^{p_1} \phi^{k_1 m_1 p_1 n_1} T_{k_1 m_1}(l) \otimes T_{p_1 n_1}(l) \\ &\equiv (2l+1) \sum_{11} \phi^{11} T_1(l_1) \otimes T_1(l_2) \end{aligned} \quad (2.6)$$

In our shorthand notation  $\phi^{11}$  (for  $\phi^{k_1 m_1 p_1 n_1}$ ), the quantum numbers from the first sphere come with subscript 1 (as in  $(k_1 m_1)$ ), as do those for the second sphere.

The  $T_{km}(l)$  are the polarization tensors which satisfy

$$\begin{aligned} K_\pm^{(a)} T_{k_1 m_1}(l) &= \mp \frac{1}{\sqrt{2}} \sqrt{k_1(k_1+1) - m_1(m_1 \pm 1)} T_{k_1 m_1 \pm 1}(l), \\ K_3^{(a)} T_{k_1 m_1}(l) &= m_1 T_{k_1 m_1}(l), \\ (\vec{K}^{(a)})^2 T_{k_1 m_1}(l) &= k_1(k_1+1) T_{k_1 m_1}(l), \end{aligned}$$

and the identities

$$\text{Tr}_{\mathcal{H}} T_{k_1 m_1}(l) T_{p_1 n_1}(l) = (-1)^{m_1} \delta_{k_1 p_1} \delta_{m_1 + n_1, 0}, \quad T_{k_1 m_1}^\dagger(l) = (-1)^{m_1} T_{k_1 -m_1}(l).$$

The field  $\hat{\Phi}$  has a finite number of degrees of freedom, totaling to  $(2l_1+1)^2(2l_2+1)^2$ .

Our interest is restricted to hermitian fields since they are the analog of real fields in the continuum. Imposing hermiticity  $\hat{\Phi}^\dagger = \hat{\Phi}$ , we obtain the conditions  $\phi^{k_1 m_1 p_1 n_1} = (-1)^{m_1+n_1} \phi^{k_1 -m_1 p_1 -n_1}$ .

Since the field on our fuzzy space has only a finite number of degrees of freedom, the simplest and most obvious route to quantization is via path integrals. The partition function

$$\mathcal{Z} = N \int \mathcal{D}\phi e^{-S_l - S_l^{int}}, \quad \mathcal{D}\phi = \prod_{11} \frac{d\bar{\phi}^{11} d\phi^{11}}{2\pi} \quad (2.7)$$

for the theory yields the (free) propagator

$$\langle \phi^{k_1 m_1 p_1 n_1} \phi^{k_2 m_2 p_2 n_2} \rangle = \frac{(-1)^{m_2+n_2}}{R^2} \frac{\delta^{k_1 k_2} \delta^{m_1, -m_2} \delta^{p_1 p_2} \delta^{n_1, -n_2}}{k_1(k_1+1) + p_1(p_1+1) + R^2 \mu_l^2}. \quad (2.8)$$

The Euclidean “4-momentum” in this setting is given by  $(11) \equiv (k_1, m_1, p_1, n_1)$  with “square”  $(11)^2 = k_1(k_1+1) + p_1(p_1+1)$ . For quartic interactions, the vertex is given by the expression

$$S_l^{int} = \sum_{11} \sum_{22} \sum_{33} \sum_{44} V(11, 22, 33, 44) \phi^{11} \phi^{22} \phi^{33} \phi^{44}, \quad (2.9)$$

with

$$\begin{aligned} V(11, 22, 33, 44) &= R^4 \frac{\lambda_{4,l}}{4!} V_1(1234, km) V_2(1234, pm), \quad \text{where} \\ V_1(1234, km) &= (2l+1) \text{Tr}_{H_1} \left[ T_{k_1 m_1}(l) T_{k_2 m_2}(l) T_{k_3 m_3}(l) T_{k_4 m_4}(l) \right] \end{aligned} \quad (2.10)$$

and similarly for  $V_2(1234, pm)$ .

## 2.1 The 2-Point Function

The energy of each mode  $\phi^{k_1 m_1 p_1 n_1}$  is the square of the fuzzy 4-momentum, namely  $(11)^2 = k_1(k_1+1) + p_1(p_1+1)$ . Since  $m_1 = -k_1, \dots, k_1$  and  $n_1 = -p_1, \dots, p_1$ , there are  $(2k_1+1)(2p_1+1)$  modes with the same energy for each pair of values  $(k_1, p_1)$ , and may thus be thought of as naturally forming an energy shell. Integrating out the high energy modes (with  $(k_1 = 2l_1, p_1 = 2l_2)$  in the path integral implements for us the “shell” approach to renormalization group adapted to fuzzy space field theories [8].

Integrating out only over the high momentum modes  $1_f 1_f \equiv (k = 2l_1, m, p = 2l_2, n)$ , the terms in the action that contribute to the 2-point function at one-loop are given by

$$\begin{aligned} \Delta S_2^{(1)} &= \dots + 4 \sum_{1\bar{1}} \sum_{2\bar{2}} \sum_{3_f 3_{\bar{f}}} \sum_{4_f 4_{\bar{f}}} V(\bar{1}\bar{1}, \bar{2}\bar{2}, 3_f 3_{\bar{f}}, 4_f 4_{\bar{f}}) \phi^{\bar{1}\bar{1}} \phi^{\bar{2}\bar{2}} \phi^{3_f 3_{\bar{f}}} \phi^{4_f 4_{\bar{f}}} \\ &+ 2 \sum_{1\bar{1}} \sum_{2_f 2_{\bar{f}}} \sum_{3\bar{3}} \sum_{4_f 4_{\bar{f}}} V(\bar{1}\bar{1}, 2_f 2_{\bar{f}}, \bar{3}\bar{3}, 4_f 4_{\bar{f}}) \phi^{\bar{1}\bar{1}} \phi^{2_f 2_{\bar{f}}} \phi^{\bar{3}\bar{3}} \phi^{4_f 4_{\bar{f}}} + \dots \end{aligned} \quad (2.11)$$

The ellipsis indicate omitted terms that are unimportant for the 2-point function calculation. The notation is that of equations (2.6) and (2.9), and  $\sum_{1\bar{1}} = \sum_{11} - \sum_{1,1_f}$ . The relative factor in the above is 4 to 2 since there are 4 ways to contract two neighboring fields (i.e. planar diagrams) and only two different ways to contract non-neighboring fields (non-planar diagrams).

Instead of integrating out only one shell, one could integrate out an arbitrary number of them. For example, integrating out  $q^2$  shells gives

$$\begin{aligned} \Delta S_2^{(q^2)} &= \dots + 4 \sum_{1\bar{1}} \sum_{2\bar{2}} \sum_{3_f 3_{\bar{f}}} \sum_{4_f 4_{\bar{f}}} V(\bar{1}\bar{1}, \bar{2}\bar{2}, 3_f 3_{\bar{f}}, 4_f 4_{\bar{f}}) \phi^{\bar{1}\bar{1}} \phi^{\bar{2}\bar{2}} \phi^{3_f 3_{\bar{f}}} \phi^{4_f 4_{\bar{f}}} \\ &+ 2 \sum_{1\bar{1}} \sum_{2_f 2_{\bar{f}}} \sum_{3\bar{3}} \sum_{4_f 4_{\bar{f}}} V(\bar{1}\bar{1}, 2_f 2_{\bar{f}}, \bar{3}\bar{3}, 4_f 4_{\bar{f}}) \phi^{\bar{1}\bar{1}} \phi^{2_f 2_{\bar{f}}} \phi^{\bar{3}\bar{3}} \phi^{4_f 4_{\bar{f}}} + \dots \end{aligned} \quad (2.12)$$

with now

$$\sum_{1_f 1_{\bar{f}}} = \sum_{k=2l_1-(q-1)}^{2l_1} \sum_{m=-k}^k \sum_{p=2l_2-(q-1)}^{2l_2} \sum_{n=-p}^p, \quad \sum_{1\bar{1}} = \sum_{11} - \sum_{1_f 1_{\bar{f}}}$$

while the partition function (2.7) takes the form

$$\mathcal{Z} = \hat{N} \int \mathcal{D}\hat{\phi} e^{-\hat{S}_l - \hat{S}_l^{int} - \langle \Delta S_2^{q^2} \rangle_f + \frac{(\Delta S_2^{q^2})_f - \langle \Delta S_2^{q^2} \rangle_f^2}{2} \hat{I} + \dots}, \quad \text{where } \mathcal{D}\hat{\phi} = \prod_{1\bar{1}} \frac{d\bar{\phi}^{1\bar{1}} d\phi^{1\bar{1}}}{2\pi} \quad (2.13)$$

For  $l_1 = l_2 = l$ , the full one-loop corresponds to integrating over all shells i.e.  $q = 2l+1$ . The corresponding effective action is

$$\langle \Delta S_2^{(q^2)} \rangle_{|q=2l+1} = \frac{1}{R^2} \frac{\lambda_{4,l}}{4!} \sum_{k_1 m_1 p_1 n_1} \left[ \delta \mu_l^P + \delta \mu_l^{NP}(k_1, p_1) \right] |\phi^{k_1 m_1 p_1 n_1}|^2. \quad (2.14)$$

The 2-point function computation readily gives us the renormalized mass:

$$\mu_l^2(k_1, p_1) = \mu_l^2 + \frac{1}{R^2} \frac{\lambda_{4,l}}{4!} \left[ \delta \mu_l^P + \delta \mu_l^{NP}(k_1, p_1) \right] \quad (2.15)$$

where the planar contribution given by

$$\delta \mu_l^P = 4 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b), \quad A(a, b) = \frac{(2a+1)(2b+1)}{a(a+1) + b(b+1) + R^2 \mu_l^2}. \quad (2.16)$$

On the other hand, the non-planar contribution is

$$\begin{aligned}\delta\mu_l^{NP}(k_1, p_1) &= 2 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b) (-1)^{k_1+p_1+a+b} B_{k_1 p_1}(a, b), \quad \text{where} \\ B_{ab}(c, d) &= (2l+1)^2 \left\{ \begin{matrix} a & l & l \\ c & l & l \end{matrix} \right\} \left\{ \begin{matrix} b & l & l \\ d & l & l \end{matrix} \right\}.\end{aligned}\quad (2.17)$$

The symbol  $\left\{ \right\}$  in  $B_{ab}(c, d)$  is the standard  $6j$  symbol (see for example [17]). As is immediately obvious from these expressions, both planar and non-planar graphs are finite and well-defined for all finite values of  $l$ . However, a measure for the fuzzy UV-IR mixing is the difference  $\Delta$  between planar and non-planar contributions, which we define below:

$$\begin{aligned}\delta\mu_l^P + \delta\mu_l^{NP}(k_1, p_1) &= \Delta\mu_l^P + \frac{1}{2} \Delta(k_1, p_1), \quad \Delta\mu_l^P = 6 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b) \\ \Delta(k_1, p_1) &= 4 \sum_{a=0}^{2l} \sum_{b=0}^{2l} A(a, b) \left[ (-1)^{k_1+p_1+a+b} B_{k_1 p_1}(a, b) - 1 \right].\end{aligned}\quad (2.18)$$

Were this difference  $\Delta(k_1, p_1)$  to vanish, we would recover the usual contribution to the mass renormalization as expected in a commutative field theory. The fact that this difference is not zero in the limit of large IRR's  $l$ , i.e.  $l \rightarrow \infty$ , is what is meant by UV-IR mixing on fuzzy  $S^2 \times S^2$ . Indeed this may be taken as the definition of the UV-IR problem on general fuzzy spaces. In fact (2.18) can also be taken as the regularized form of the UV-IR mixing on  $\mathbf{R}^4$ . Removing the UV cut-off  $l \rightarrow \infty$  while keeping the infrared cut-off  $R$  fixed = 1 one can show that  $\Delta$  diverges as  $l^2$ , i.e

$$\Delta(k_1, p_1) \rightarrow (8l^2) \int_{-1}^1 \int_{-1}^1 \frac{dt_x dt_y}{2 - t_x - t_y} \left[ P_{k_1}(t_x) P_{p_1}(t_y) - 1 \right], \quad (2.19)$$

where, for simplicity, we have assumed  $\mu_l \ll l$  [12]. (2.19) is worse than the case of two dimensions [see equation (3.20) of [12]], in here not only the difference survives the limit but also it diverges. This means in particular that the UV-IR mixing can be largely controlled or perhaps understood if one understands the role of the UV cut-off  $l$  in the scaling limit and its relation to the underlying star product on  $S_P^2$ .

## 2.2 The 4-Point Function

The computation of higher order correlation functions become very complicated, but this exercise is necessary if we want to compute for example the beta-function. It is also useful to put forward key features which will be needed (in the future) to study noncommutative matrix gauge theories and their continuum limits. We will only look at the four-point function here.

Our starting point is (2.13), which tells us that integrating out  $q^2$  shells produces the following correction to the 4-point function:

$$\begin{aligned}\langle (\Delta S_2^{(q^2)})^2 \rangle_f &= W(\hat{1}, \hat{2}, \hat{3}, \hat{5}; \hat{4}, \hat{6}, \hat{7}, \hat{8}) \phi^{1235} \left[ \langle \phi^{4_f 4_f} \phi^{6_f 6_f} \rangle_f \langle \phi^{7_f 7_f} \phi^{8_f 8_f} \rangle_f \right. \\ &\quad \left. + \langle \phi^{4_f 4_f} \phi^{7_f 7_f} \rangle_f \langle \phi^{6_f 6_f} \phi^{8_f 8_f} \rangle_f + \langle \phi^{4_f 4_f} \phi^{8_f 8_f} \rangle_f \langle \phi^{7_f 7_f} \phi^{6_f 6_f} \rangle_f \right],\end{aligned}$$

Here,  $W(\hat{1}, \hat{2}, \hat{3}, \hat{5}; \hat{4}, \hat{6}, \hat{7}, \hat{8}) = W_2(\hat{1}\hat{1}, \hat{2}\hat{2}, 4_f 4_f, 6_f 6_f) W_2(\hat{3}\hat{3}, \hat{5}\hat{5}, 7_f 7_f, 8_f 8_f)$  such that  $W_2(\hat{1}\hat{1}, \hat{2}\hat{2}, 3_f 3_f, 4_f 4_f) = 4V(\hat{1}\hat{1}, \hat{2}\hat{2}, 3_f 3_f, 4_f 4_f) + 2V(\hat{1}\hat{1}, 3_f 3_f, \hat{2}\hat{2}, 4_f 4_f)$ ,  $\phi^{1235} = \phi^{\hat{1}\hat{1}} \phi^{\hat{2}\hat{2}} \phi^{\hat{3}\hat{3}} \phi^{\hat{5}\hat{5}}$ , and the notation is that of equations (2.6), (2.9) and (2.12). Inserting the free propagator (2.8) above yields the 4-point function

$$\frac{\langle (\Delta S_2^{(q^2)})^2 \rangle_f - \langle \Delta S_2^{(q^2)} \rangle_f^2}{2} = \sum_{i1} \sum_{22} \sum_{33} \sum_{55} R^4 \frac{\lambda_4}{4!} \phi^{\hat{1}\hat{1}} \phi^{\hat{2}\hat{2}} \phi^{\hat{3}\hat{3}} \phi^{\hat{5}\hat{5}} \delta\lambda_4(1235) \quad (2.20)$$

where

$$\begin{aligned}\delta\lambda_4(1235) &= \frac{\lambda_4}{4!} \sum_{k_4, k_6=f} \sum_{p_4, p_6=f} \frac{A(k_4, p_4) A(k_6, p_6)}{(2k_4+1)(2k_6+1)(2p_4+1)(2p_6+1)} \left[ 8\eta_1^{(1)} \eta_2^{(1)} \right. \\ &\quad \left. + 16\eta_1^{(2)} \eta_2^{(2)} + 4\eta_1^{(3)} \eta_2^{(3)} + 8\eta_1^{(4)} \eta_2^{(4)} \right].\end{aligned}\quad (2.21)$$

The first graph in (2.21) is the usual one-loop contribution to the 4-point function, i.e the two vertices are planar. The fourth graph contains also two planar vertices but with the exception that one of these vertices is twisted, i.e with an extra phase. The second graph contains on the other hand one planar vertex and one non-planar vertex, whereas the two vertices in the third graph are both non-planar. The analytic expressions for  $\eta_i^{(a)} \equiv \eta_i^{(a)}(k_4 k_6; 1235) = \sum_{m_4=-k_4}^{k_4} \sum_{m_6=-k_6}^{k_6} \rho_i^{(a)}(k_4 k_6; 1235)$  are given by

$$\begin{aligned}\rho_i^{(1)} &= (-1)^{m_4+m_6} V_i(\hat{1}\hat{2} 4_f 6_f) V_i(\hat{3}\hat{5} - 4_f - 6_f), & \rho_i^{(2)} &= (-1)^{m_4+m_6} V_i(\hat{1}\hat{2} 4_f 6_f) V_i(\hat{3} - 4_f \hat{5} - 6_f) \\ \rho_i^{(3)} &= (-1)^{m_4+m_6} V_i(\hat{1} 4_f \hat{2} 6_f) V_i(\hat{3} - 4_f \hat{5} - 6_f), & \rho_i^{(4)} &= (-1)^{m_4+m_6} V_i(\hat{1}\hat{2} 4_f 6_f) V_i(\hat{3}\hat{5} - 6_f - 4_f),\end{aligned}$$

where the lower index in  $\eta$ 's and  $\rho$ 's labels the sphere whereas the upper index denotes the graph, and the notation  $-4_f 4_f$  stands for  $(k_4, -m_4, p_4, -n_4)$  in contrast with  $4_f 4_f = (k_4, m_4, p_4, n_4)$ .

By using extensively the different identities in [17] we can find after a long calculation that the above 4-point function has the form

$$\begin{aligned}\delta\lambda_4(1235) &= \frac{\lambda_4}{4!} \left[ 8\delta\lambda_4^{(1)}(1235) + 16\delta\lambda_4^{(2)}(1235) + 4\delta\lambda_4^{(3)}(1235) + 8\delta\lambda_4^{(4)}(1235) \right], \quad \text{where} \\ \delta\lambda_4^{(a)}(1235) &= \sum_{k_4, k_6=f} \sum_{p_4, p_6=f} A(k_4, p_4) A(k_6, p_6) \nu_1^{(a)}(k_4 k_6; 1235) \nu_2^{(a)}(p_4 p_6; 1235), \quad a = 1 \dots 4,\end{aligned}\quad (2.22)$$

The label  $f$  stands for the shells we integrated over and hence it corresponds to  $q^2 = (2l+1)^2$  for the full one-loop contribution. The planar amplitudes, in the first  $\mathbb{R}^2$  factor for example, are given by

$$\nu_1^{(1)} = \sum_k (-1)^{k+k_4+k_6} \delta_k(1235) E_{k_1 k_2}^{k_4 k_6}(k) E_{k_3 k_5}^{k_4 k_6}(k), \quad \nu_1^{(4)} = \sum_k \delta_k(1235) E_{k_1 k_3}^{k_4 k_6}(k) E_{k_2 k_5}^{k_4 k_6}(k) \quad (2.23)$$

whereas the non-planar amplitudes are given by

$$\nu_1^{(2)} = \sum_k (-1)^{k_3+k_4} \delta_k(1235) E_{k_1 k_2}^{k_4 k_6}(k) F_{k_3 k_5}^{k_4 k_6}(k), \quad \nu_1^{(3)} = \sum_k (-1)^{k_2+k_3} \delta_k(1235) F_{k_1 k_2}^{k_4 k_6}(k) E_{k_3 k_5}^{k_4 k_6}(k) \quad (2.24)$$

with

$$\begin{aligned}F_{k_1 k_2}^{k_4 k_6}(k) &= (2l+1) \sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_4 & l & l \\ k_6 & l & l \\ k & k_1 & k_2 \end{matrix} \right\} \\ E_{k_1 k_2}^{k_4 k_6}(k) &= (2l+1) \sqrt{(2k_1+1)(2k_2+1)} \left\{ \begin{matrix} k_1 & k_2 & k \\ l & l & l \end{matrix} \right\} \left\{ \begin{matrix} k_4 & k_6 & k \\ l & l & l \end{matrix} \right\}.\end{aligned}\quad (2.25)$$

The “fuzzy delta” function  $\delta_k(1235)$  is defined by

$$\delta_k(1235) = (-1)^m C_{k_1 m_1 k_2 m_2}^{km} C_{k_3 m_3 k_5 m_5}^{k-m}. \quad (2.26)$$

The justification for this name will follow shortly.

The full effective action at one-loop of the above scalar field theory on  $S_P^2 \times S_P^2$  is obtained by adding the two quantum actions (2.14) and (2.20) to the classical action (2.4).

### 3 Continuum Planar Limits

We can now state with some detail the continuum limits in which the fuzzy spheres approach (in a precise sense) the noncommutative planes. There are primarily two limits of interest to us: one is the canonical large stereographic projection of the spheres onto planes, while the second is a new flattening limit which we will argue corresponds to a conventional cut-off.

For simplicity, consider a single fuzzy sphere with cut-off  $l$  and radius  $R$ , and define the fuzzy coordinates  $x_i^F = \theta L_i$  (i.e.  $x_\pm^F = x_\mp^F \pm i x_3^F$ ) where  $\theta = R/\sqrt{l(l+1)}$ . The stereographic projection [9, 10] to the noncommutative plane is realized as

$$y_+^F = 2R x_+^F \frac{1}{R - x_3^F}, \quad y_-^F = 2R \frac{1}{R - x_3^F} x_-^F. \quad (3.1)$$

In the large  $l$  limit it is obvious that these fuzzy coordinates indeed approach the canonical stereographic coordinates. A planar limit can be defined from above as follows:

$$\theta'^2 = \frac{R^2}{\sqrt{l(l+1)}} = \text{fixed as } l, R \rightarrow \infty. \quad (3.2)$$

In this limit, the commutation relation becomes

$$[y_+^{NC}, y_-^{NC}] = -2\theta'^2, \quad y_\pm^{NC} \equiv y_\pm^F = x_\pm^F, \quad (3.3)$$

where we have substituted  $L_3 = -l$  corresponding to the north pole. The above commutation relation may also be put in the form

$$[x_1^{NC}, x_2^{NC}] = -i\theta'^2, \quad x_a^{NC} \equiv x_a^F, \quad a = 1, 2 \quad (3.4)$$

The minus sign is simply due to our convention for the coherent states on co-adjoint orbits. The extension to the case of two fuzzy spheres is trivial.

A second way to obtain the noncommutative plane is by taking the limit

$$\theta = \frac{R}{\sqrt{l(l+1)}} = l, R \rightarrow \infty. \quad (3.5)$$

A UV cut-off is automatically built into this limit: the maximum energy a scalar mode can have on the fuzzy sphere is  $2l(2l+1)/R^2$ , which in this scaling limit is  $4/\theta^2$ . There are no modes with energy larger than this value. To understand this limit a little better, let us restrict ourselves to the north pole  $\vec{n} = \vec{n}_0 = (0, 0, 1)$  where we have  $\langle \vec{n}_0, l | L_3 | \vec{n}_0, l \rangle = -l$  and  $\langle \vec{n}_0, l | L_a | \vec{n}_0, l \rangle = 0$ ,  $a = 1, 2$ . The commutator  $[L_1, L_2] = iL_3 = -il$ , so the noncommutative coordinates on this noncommutative plane “tangential to the north pole” can be given either simply by  $x_a^F$  as above. This now defines a strongly noncommuting plane, viz

$$[x_a^F, x_b^F] = -il\theta^2 \epsilon_{ab}. \quad (3.6)$$

Or alternatively one can define the noncommutative coordinate by  $X_a^{NC} \equiv \sqrt{\frac{\theta}{R}} x_a^F$ , satisfying

$$[X_a^{NC}, X_b^{NC}] = -i\theta^2 \epsilon_{ab}. \quad (3.7)$$

In the convention used here,  $\epsilon_{12} = 1$  and  $\epsilon_{ac}\epsilon_{cb} = -\delta_{ab}$ .

Intuitively, the second scaling limit may be understood as follows. Noncommutativity introduces a short distance cut-off of the order  $\delta X = \sqrt{\frac{\theta^2}{2}}$  because of the uncertainty relation  $\Delta X_1^{NC} \Delta X_2^{NC} \geq \frac{\theta^2}{2}$ . However, the Laplacian operators on generic noncommutative planes do not reflect this short distance cut-off, as they are generally taken to be the same as the commutative Laplacians. On the above noncommutative plane (3.7) the cut-off  $\delta X$  effectively translates into the momentum space as some cut-off  $\delta P = \frac{1}{\sqrt{2}\theta^2}$ . This is because of (and in accordance with) the commutation relations  $[X_a^{NC}, P_b^{NC}] = i\delta_{ab}$ ,  $P_a^{NC} = -\frac{1}{\theta^2} \epsilon_{ab} X_b^{NC}$ , giving us the uncertainty relations  $\Delta X_a^{NC} \Delta P_b^{NC} \geq \frac{\delta_{ab}}{2}$ . Since one can not probe distances less than  $\delta X$ , energies above  $\delta P$  should not be

accessible either, i.e.  $[P_a^{NC}, P_b^{NC}] = -\frac{i}{\theta^2} \epsilon_{ab}$ . The fact that the maximum energy of a mode is of order  $1/\theta$  in the second scaling limit ties in nicely with this expectation.

The limit (3.5) may thus be thought of as a regularization prescription of the noncommutative plane which takes into account our expectation of “UV-finiteness” of noncommutative quantum field theories.

#### 3.1 Field Theory in the Canonical Planar Limit

We are now in a position to study what happens to the scalar field theory in the limit (3.4). First we match the spectrum of the Laplacian operator on each sphere with the spectrum of the Laplacian operator on the limiting noncommutative plane as follows

$$a(a+1) = R^2 p_a^2, \quad (3.8)$$

where  $p_a$  is of course the modulus of the two dimensional momentum on the noncommutative plane which corresponds to the integer  $a$ , and has the correct mass dimension. However since the range of  $a$ 's is from 0 to  $2l$ , the range of  $p_a^2$  will be from 0 to  $\frac{2l(2l+1)}{R^2} = l\Lambda'^2 \rightarrow \infty$ ,  $\Lambda' = 2/\theta'$ . In other words, all information about the UV cut-off is lost in this limit.

Let us see how the other operators in the theory scales in the above planar limit. It is not difficult to show that the free action scales as

$$\sum_{a,b} \sum_{m_a, m_b} \left[ R^2 a(a+1) + R^2 b(b+1) + R^4 \mu_i^2 \right] |\phi^{ab m_a m_b}|^2 \simeq \int_{\sqrt{l}\Lambda'} \frac{d^2 \vec{p}_a d^2 \vec{p}_b}{\pi^2} \left[ p_a^2 + p_b^2 + M^2 \right] |\phi_{NC}^{p_a p_b \phi_a \phi_b}|^2. \quad (3.9)$$

The scalar field is assumed to have the scaling property  $\phi_{NC}^{p_a p_b \phi_a \phi_b} \simeq R^4 \phi^{ab m_a m_b}$ , which gives the momentum-space scalar field the correct mass dimension of  $-3$  [recall that  $[\phi^{ab m_a m_b}] = M$ ]. The  $\phi_a$  and  $\phi_b$  above (not to be confused with the scalar field!) are the angles of the two momenta  $\vec{p}_a$  and  $\vec{p}_b$  respectively, i.e.  $\phi_a = \frac{\pi m_a}{a+\frac{1}{2}}$  and  $\phi_b = \frac{\pi m_b}{b+\frac{1}{2}}$ . This formula is exact, and can be simplified further when quantum numbers  $a$ 's and  $b$ 's are large: the  $\phi_a$  and  $\phi_b$  will be in the range  $[-\pi, \pi]$ . It is also worth pointing out that the mass parameter  $M$  of the planar theory is exactly equal to that on the fuzzy spheres, i.e.  $M = \mu_\infty$ , and no scaling is required. This is in contrast with [12] but only due to our definition of the fuzzy action (2.4).

With these ingredients, it is not then difficult to see that the flattening limit of the planar 2-point function (2.16) is given by

$$\delta M^P \equiv \frac{\delta \mu_i^P}{R^2} = 16 \int \int \frac{p_a p_b d p_a d p_b}{p_a^2 + p_b^2 + M^2} \quad (3.10)$$

which is the 2-point function on noncommutative  $\mathbb{R}^4$  with a Euclidean metric  $\mathbb{R}^2 \times \mathbb{R}^2$ . By rotational invariance it may be rewritten as

$$\delta M^P = \frac{4}{\pi^2} \int_{\sqrt{l}\Lambda'} \frac{d^4 k}{k^2 + M^2}. \quad (3.11)$$

We do now the same exercise for the non-planar 2-point function (2.17). Since the external momenta  $k_1$  and  $p_1$  are generally very small compared to  $l$ , one can use the following approximation for the  $6j$ -symbols [17]

$$\left\{ \begin{matrix} a & l & l \\ b & l & l \end{matrix} \right\} \approx \frac{(-1)^{a+b}}{2l} P_a(1 - \frac{b^2}{2l^2}), \quad l \rightarrow \infty, \quad a \ll l, \quad 0 \leq b \leq 2l, \quad (3.12)$$

By putting in all the ingredients of the planar limit we obtain the result

$$\delta M^{NP}(k_1, p_1) \equiv \frac{\delta \mu_i^{NP}}{R^2} = 8 \int_0^\infty \int_0^\infty \frac{p_a p_b d p_a d p_b}{p_a^2 + p_b^2 + M^2} P_{k_1}(1 - \frac{\theta'^4 p_a^2}{2R^2}) P_{p_1}(1 - \frac{\theta'^4 p_b^2}{2R^2}).$$

Although the quantum numbers  $k_1$  and  $p_1$  in this limit are very small compared to  $l$ , they are large themselves i.e.  $1 \ll k_1, p_1 \ll l$ . On the other hand, the angles  $\nu_a$  defined by  $\cos \nu_a = 1 - \frac{\theta'^4 p_a^2}{2R^2}$  can be considered for all

practical purposes small, i.e.  $\nu_a = \frac{\theta'^2 p_a}{R}$  because of the large  $R$  factor, and hence we can use the formula (see for eg [21], page 72)

$$P_n(\cos \nu_a) = J_0(\eta) + \sin^2 \frac{\nu_a}{2} \left[ \frac{J_1(\eta)}{2\eta} - J_2(\eta) + \frac{\eta}{6} J_3(\eta) \right] + O(\sin^4 \frac{\nu_a}{2}), \quad (3.13)$$

for  $n \gg 1$  and small angles  $\nu_a$ , with  $\eta = (2n+1) \sin \frac{\nu_a}{2}$ . To leading order we then have

$$P_{k_1}(1 - \frac{\theta'^4 p_a^2}{2R^2}) = J_0(\theta'^2 p_{k_1} p_a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_a e^{i\theta'^2 \cos \phi_a p_{k_1} p_a}.$$

This result becomes exact in the strict limit of  $l, R \rightarrow \infty$  where all fuzzy quantum numbers diverge with  $R$ . We get then

$$\delta M^{NP}(p_{k_1}, p_{p_1}) = \frac{2}{\pi^2} \int \int \int \frac{(p_a d p_a d \phi_a)(p_b d p_b d \phi_b)}{p_a^2 + p_b^2 + M^2} e^{i\theta'^2 p_{k_1}(p_a \cos \phi_a)} e^{i\theta'^2 p_{p_1}(p_b \cos \phi_b)}.$$

By rotational invariance we can set  $\theta'^2 B^{\mu\nu} p_{k_1\mu} p_{p_1\nu} = \theta'^2 p_{k_1}(p_a \cos \phi_a)$ , where  $B^{12} = -1$ . In other words, we can always choose the two-dimensional momentum  $p_{k_1}$  to lie in the  $y$ -direction, thus making  $\phi_a$  the angle between  $\vec{p}_a$  and the  $x$ -axis. The same is also true for the other exponential. We thus obtain the canonical non-planar 2-point function on the noncommutative  $\mathbb{R}^4$  (with Euclidean metric  $\mathbb{R}^2 \times \mathbb{R}^2$ ). Again by rotational invariance, this non-planar contribution to the 2-point function may be put in the compact form

$$\delta M^{NP}(p) = \frac{2}{\pi^2} \int_{\sqrt{l}\Lambda'} \frac{d^4 k}{k^2 + M^2} e^{i\theta'^2 p B k}. \quad (3.14)$$

The structure of the effective action in momentum space allows us to deduce the star products on the underlying noncommutative space. For example, by using the tree level action (3.9) together with the one-loop contributions (3.11) and (3.14) one can find that the effective action obtained in the large stereographic limit (3.2) is given by

$$\int_{\sqrt{l}\Lambda'} \frac{d^4 \vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + M^2 + \frac{g_4^2}{6} \left[ 2 \int_{\sqrt{l}\Lambda'} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + M^2} + \int_{\sqrt{l}\Lambda'} \frac{d^4 k}{(2\pi)^4} \frac{e^{i\theta'^2 \vec{p} B \vec{k}}}{k^2 + M^2} \right] \right] |\phi_1(\vec{p})|^2 \quad (3.15)$$

where  $g_4^2 = 8\pi^2 \lambda_4$  and  $\phi_1(\vec{p}) = 4\pi\sqrt{2} \phi_{NC}^{p_a p_b \phi_a \phi_b}$  and  $\sqrt{l}\Lambda' \rightarrow \infty$ . This effective action can be obtained from the quantization of the action

$$\int d^4 x \left[ \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} M^2 \phi_1^2 + \frac{g_4^2}{4!} \phi_1 *' \phi_1 *' \phi_1 *' \phi_1 \right],$$

where  $\phi_1 \equiv \phi_1(x^{NC}) = \int \frac{d^4 p}{(2\pi)^4} \phi_1(\vec{p}) e^{-ipx^{NC}} = \phi_1^\dagger$  and  $*'$  is the canonical (or Moyal-Weyl) star product

$$f *' g(x^{NC}) = e^{\frac{i}{2} \theta'^2 B^{\mu\nu} \partial_\mu^\star \partial_\nu^\star} f(y) g(z) |_{y=z=x^{NC}} \quad (3.16)$$

This is consistent with the commutation relation (3.4) and provides a nice check that the canonical star product on the sphere derived in [22] (also given here by equation (2.2)) reduces in the limit (3.2) to the above Moyal-Weyl product (3.16). In the above,  $B$  is the antisymmetric tensor which can always be rotated such that the non vanishing components are given by  $B^{12} = -B_{21} = -1$  and  $B^{34} = -B_{43} = -1$ .

In fact one can read immediately from the above effective action that the planar contribution is quadratically divergent as it should be, i.e.

$$\Delta M^P = \frac{1}{64\pi^2} \delta M^P = \int_{\sqrt{l}\Lambda' \rightarrow \infty} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + M^2} = \frac{1}{16\pi^2} l \Lambda'^2 \rightarrow \infty, \quad (3.17)$$

whereas the non-planar contribution is clearly finite

$$\begin{aligned} \Delta M^{NP}(p) &= \frac{1}{32\pi^2} \delta M^{NP}(p) = \int_{\sqrt{l}\Lambda' \rightarrow \infty} \frac{d^4 k}{(2\pi)^4} \frac{e^{i\theta'^2 \vec{p} B \vec{k}}}{k^2 + M^2} \\ &= \frac{1}{8\pi^2} \left[ \frac{2}{E^2 \theta'^4} + M^2 \ln(\theta'^2 E M) \right], \quad \text{where } E^\nu = B^{\mu\nu} P_\mu. \end{aligned} \quad (3.18)$$

This is the answer of [5]: it is singular at  $P = 0$  as well as at  $\theta' = 0$ .

### 3.2 A New Planar Limit With Strong Noncommutativity

As explained earlier, the limit (3.5) possesses the attractive feature that a momentum space cut-off is naturally built into it. In addition to obtaining a noncommutative plane in the strict limit, UV-IR mixing is completely absent. But while the new scaling is simply stated, obtaining the corresponding field theory is somewhat subtle. We will need to modify the Laplacian on the fuzzy sphere to project our modes with momentum greater than  $2\sqrt{l}$ . In other words, the noncommutative theory on a plane with UV cut-off  $\theta$  is obtained not by flattening the full theory on the fuzzy sphere, but only a “low energy” sector, corresponding to momenta upto  $2\sqrt{l}$ .

In order to clarify the chain of arguments, we will first implement naively the limit (3.5) and show that it corresponds to a strongly noncommuting plane. Finite noncommuting plane is only obtainable if we pick a specific low energy sector of the fuzzy sphere before taking the limit as we will explain in the next section.

Our rule for matching the spectrum on the fuzzy sphere with that on the noncommutative plane is the same as before, namely  $a(a+1) = R^2 p_a^2$ . However because of (3.5), the range of  $p_a^2$  is now from 0 to  $\frac{2l(2l+1)}{R^2} = \frac{4}{\theta^2}$ . The kinetic part of the action will scale in the same way as in (3.9), only now the momenta  $\vec{p}$ 's in (3.9) are restricted such that  $p \leq \Lambda$ . With this scaling information, we can see that the planar contribution to the 2-point function is given by

$$\delta m^P \equiv \frac{\delta \mu_l^P}{R^2} = \frac{4}{\pi^2} \int_{k \leq \Lambda} \frac{d^4 k}{k^2 + \mu_l^2}, \quad \Lambda = \frac{2}{\theta}. \quad (3.19)$$

We can similarly compute the non-planar contribution to the 2-point function using (3.12). The motivation for using this approximation is more involved and can be explained as follows. In the planar limit  $l, R \rightarrow \infty$ , it is obvious that the relevant quantum numbers  $k_1$  and  $p_1$  are in fact much larger compared to 1, i.e.  $k_1 \sim R p_{k_1} \gg 1$  and  $p_1 \sim R p_{p_1} \gg 1$ , since  $R \simeq \theta l$ . However (3.12) can be used only if  $k_1, p_1 \ll l$ , or equivalently  $\frac{k_1}{l} = \frac{2 p_{k_1}}{\Lambda} \ll 1$  and  $\frac{p_1}{l} = \frac{2 p_{p_1}}{\Lambda} \ll 1$ . This is clearly true for small external momenta  $p_{k_1}$  and  $p_{p_1}$ , which is exactly the regime of interest in order to see if there is UV-IR mixing. The condition for the reliability of the approximation (3.12) is then  $\theta p_{external} \ll 1$ . We will sometimes refer to this condition as “ $\theta$  small”, the precise meaning of this phrase being “momentum scale of interest is much smaller than  $1/\theta$ ”. We thus obtain

$$\delta m^{NP}(k_1, p_1) \equiv \frac{\delta \mu_l^{NP}}{R^2} = 8 \int_0^\Lambda \int_0^\Lambda \frac{p_a p_b d p_a d p_b}{p_a^2 + p_b^2 + \mu_l^2} P_{k_1}(1 - \frac{\theta^2 p_a^2}{2}) P_{p_1}(1 - \frac{\theta^2 p_b^2}{2}). \quad (3.20)$$

Now the angles  $\nu_a$ 's of (3.13) are defined by  $\cos \nu_a = 1 - \frac{\theta^2 p_a^2}{2}$ , and since  $\theta p \ll 1$ , these angles are still small. They are therefore given to the leading order in  $\theta p$  by  $\nu_a = \theta p_a + \dots$  where the ellipsis indicate terms third order and higher in  $\theta p$ . By using (3.13) we again have

$$P_{R p_{k_1}}(1 - \frac{\theta^2 p_a^2}{2}) = J_0(R \theta p_{k_1} p_a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_a e^{i R \theta \cos \phi_a p_{k_1} p_a}. \quad (3.21)$$

Using rotational invariance we can rewrite this as

$$\delta m^{NP}(p) = \frac{2}{\pi^2} \int_{k \leq \Lambda} \frac{d^4 k}{k^2 + \mu_l^2} e^{i R \theta p B k}. \quad (3.22)$$

One immediate central remark is in order: the noncommutative phase contains now a factor  $R\theta$  instead of the naively expected factor of  $\theta^2$ . This is in contrast with the previous case of canonical planar limit, where the strength of the noncommutativity  $\theta'^2$  defined by the commutation relation (3.4) is exactly what appears in the noncommutative phase of (3.14). In other words this naive implementation of (3.5) yields in fact the strongly noncommuting plane (3.6) instead of (3.7). Also we can similarly to the previous case put together the tree level action (3.9) with the one-loop contributions (3.19) and (3.22) to obtain the effective action

$$\int_\Lambda \frac{d^4 \vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + \mu_l^2 + \frac{g_4^2}{6} \left[ 2 \int_\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \mu_l^2} + \int_\Lambda \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \mu_l^2} e^{i R \theta \vec{p} B \vec{k}} \right] \right] |\phi_3(\vec{p})|^2. \quad (3.23)$$

As before  $g_4^2 = 8\pi^2\lambda_4$ , whereas  $\phi_3(\vec{p}) = l^{3/2}\phi_2(\sqrt{l}\vec{p})$ ,  $\phi_2(\vec{p}) = 4\pi\sqrt{\frac{2}{l^3}}\phi_{NC}(\frac{\vec{p}}{\sqrt{l}})$  with  $\phi_{NC}(\vec{p}) \equiv \phi_{NC}^{p_a p_b \phi_a \phi_b} = R^4 \phi^{abm_a m_b}$  (in the metric  $\mathbb{R}^2 \times \mathbb{R}^2$ ). It is not difficult to see that the one-loop contributions  $\delta m^P$  and  $\delta m^{NP}(p)$  given in (3.19) and (3.22) can also be given by the equations

$$\begin{aligned}\bar{\Delta}m^P &= \frac{l}{64\pi^2}\delta m^P = \int_{\sqrt{l}\Lambda \rightarrow \infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + l\mu_l^2} \\ \bar{\Delta}m^{NP}(p) &= \frac{l}{32\pi^2}\delta m^{NP}(\frac{p}{\sqrt{l}}) = \int_{\sqrt{l}\Lambda \rightarrow \infty} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + l\mu_l^2} e^{i\theta^2 \vec{p}B\vec{k}}.\end{aligned}\quad (3.24)$$

We have already computed that the leading terms in  $\bar{\Delta}m^P$  and  $\bar{\Delta}m^{NP}(p)$  are given by

$$\bar{\Delta}m^P = \frac{l}{16\pi^2} \left[ \Lambda^2 - \mu_l^2 \ln(1 + \frac{\Lambda^2}{\mu_l^2}) \right], \quad \bar{\Delta}m^{NP}(p) = \frac{1}{8\pi^2} \left[ \frac{2}{E^2\theta^4} + l\mu_l^2 \ln(\theta^2 \sqrt{l}E\mu_l) \right], \quad \text{where } E^\nu = B^{\mu\nu}p_\mu.$$

Obviously then we obtain

$$\delta m^P = 4 \left[ \Lambda^2 - \mu_l^2 \ln(1 + \frac{\Lambda^2}{\mu_l^2}) \right], \quad \delta m^{NP}(p) = 4\mu_l^2 \ln(\theta^2 E\mu_l). \quad (3.25)$$

If we now require the mass  $\mu_l$  in (3.9) to scale as  $\mu_l^2 = \frac{m^2}{l}$  (the reason will be clear shortly), then one can deduce immediately that the planar contribution  $\delta m^P$  is exactly finite equal to  $4\Lambda^2$ , whereas the non-planar contribution  $\delta m^{NP}(p)$  vanishes in the limit  $l \rightarrow \infty$ .

Remark finally that despite the presence of the cut-off  $\Lambda$  in the effective action (3.23), this effective action can still be obtained from quantizing

$$\int d^4x \left[ \frac{1}{2}(\partial_\mu \phi_3)^2 + \frac{1}{2}\mu_l^2 \phi_3^2 + \frac{g_4^2}{4!} \phi_3 * \phi_3 * \phi_3 * \phi_3 \right], \quad (3.26)$$

only we have to regularize all integrals in the quantum theory with a cut-off  $\Lambda = 2/\theta$ . [ $\phi_3 \equiv \phi_3(x^F) = \int \frac{d^4p}{(2\pi)^4} \phi_3(\vec{p}) e^{-ipx^F} = \phi_3^\dagger$ , and the star product  $*$  is the Moyal-Weyl product given in (3.16) with the obvious substitution  $\theta' \rightarrow R\theta$ ].

### 3.3 A New Planar Limit With Finite Noncommutativity

Nevertheless, the action (3.23) can also be understood in some way as the effective action on the noncommutative plane (3.7) with finite noncommutativity equal to  $\theta^2$ . Indeed by performing the rescaling  $\vec{p} \rightarrow \frac{\vec{p}}{\sqrt{l}}$  we get

$$\int_{\sqrt{l}\Lambda} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + m^2 + \frac{g_4^2}{6} \left[ 2 \int_{\sqrt{l}\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} + \int_{\sqrt{l}\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{i\theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_2(\vec{p})|^2. \quad (3.27)$$

We have already the correct noncommutativity  $\theta^2$  in the phase and the only thing which needs a new interpretation is the fact that the cut-off is actually given by  $\sqrt{l}\Lambda \rightarrow \infty$  and not by the finite cut-off  $\Lambda$ . [Remark that if we do not reduce the cut-off  $\sqrt{l}\Lambda$  again to the finite value  $\Lambda$ , the physics of (3.27) is then essentially that of canonical noncommutativity, i.e the limit (3.5) together with the above rescaling of momenta is equivalent to the limit (3.2)].

Now having isolated the  $l$ -dependence in the range of momentum space integrals in the effective action (3.27), we can argue that it is not possible to get rid of this  $l$ -dependence merely by changing variables. Actually, to correctly reproduce the theory on the noncommutative  $\mathbb{R}^4$  given by (3.5) and (3.7), we will now show that one must start with a modified Laplacian (or alternately propagator) on the fuzzy space [24]. For this, we replace the Laplacian  $\Delta = [L_i^{(a)}, [L_i^{(a)}, \dots]]$  (see equation (2.3),  $a = 1, 2$ ) on each fuzzy sphere which has the canonical obvious spectrum  $k(k+1)$ ,  $k = 0, \dots, 2l$ , with the modified Laplacian

$$\Delta_j = \Delta + \frac{1}{\epsilon}(1 - P_j). \quad (3.28)$$

Here  $P_j$  is the projector on all the modes associated with the eigenvalues  $k = 0, \dots, j$ , i.e.

$$P_j = \sum_{k=0}^j \sum_{m=-k}^k |k, m\rangle \langle k, m|,$$

The integer  $j$  thus acts as an intermediate scale, and using the modified propagator gives us a low energy sector of the full theory. We will fix the integer  $j$  shortly.

With this modified Laplacian, modes with momenta larger than  $j$  do not propagate: as a result, they make no contribution in momentum sums that appear in internal loops. In other words, summations like  $\sum_0^{2l}$  (which go over to integrals with range  $\int_0^\Lambda$ ) now collapse to  $\sum_0^j$  (where the integrals now are of the range  $\int_0^{\Lambda_j}$ , with  $\Lambda_j = \frac{j}{\sqrt{l}}\Lambda$ ).

The new flattening limit is now defined as follows: start with the theory on  $S_F^2 \times S_F^2$ , but with the modified propagator (3.28). First take  $\epsilon \rightarrow 0$ , then  $R, l \rightarrow \infty$  with  $\theta = R/l$  fixed. This gives us the effective action (3.27) but with with momentum space cut-off  $\sqrt{l}\Lambda_j = \frac{j}{\sqrt{l}}\Lambda$ , i.e.

$$\int_{\sqrt{l}\Lambda_j} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + m^2 + \frac{g_4^2}{6} \left[ 2 \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} + \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{i\theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_2(\vec{p})|^2 \quad (3.29)$$

This also tells us that the correct choice of the intermediate scale is  $j = [2\sqrt{l}]$  for which  $\sqrt{l}\Lambda_j = \Lambda$ . For this value of the intermediate cut-off, we obtain the noncommutative  $\mathbb{R}^4$  given by (3.5) and (3.7).

By looking at the product of two functions of the fuzzy sphere, we can understand better the role of the intermediate scale  $j (= [2\sqrt{l}])$ . The fuzzy spherical harmonics  $T_{l_a m_a}$  go over to the usual spherical harmonics  $Y_{l_a m_a}$  in the limit of large  $l$ , and so does their product, provided their momenta are fixed. Alternately, the product of two fuzzy spherical harmonics  $T$ 's is "almost commutative" (i.e. almost the same as that of the corresponding  $Y$ 's) if their angular momentum is small compared to the maximum angular momentum  $l$ , whereas it is "strongly noncommutative" (i.e. far from the commutative regime) if their angular momenta are sufficiently large and comparable to  $l$ . The intermediate cut-off tells us precisely where the product goes from one situation to the other: Working with fields having momenta much less than  $[2\sqrt{l}]$  leaves us in the approximately commutative regime, while fields with momenta much larger than  $[2\sqrt{l}]$  take us in the strongly noncommutative regime. In other words, the intermediate cut-off tells us where commutativity and noncommutativity are in delicate balance. Indeed by writing (3.29) in the form

$$\begin{aligned}& \int_{\sqrt{l}\Lambda_j} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + m^2 + \frac{g_4^2}{6} \left[ 2 \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} + \int_{\sqrt{l}\Lambda_j} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{i\theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_2(\vec{p})|^2 = \\ & \int_{\Lambda} \frac{d^4\vec{p}}{(2\pi)^4} \frac{1}{2} \left[ \vec{p}^2 + \mu_{l,j}^2 + \frac{g_4^2}{6} \left[ 2 \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \mu_{l,j}^2} + \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \mu_{l,j}^2} e^{i(\frac{j}{\sqrt{l}})^2 \theta^2 \vec{p}B\vec{k}} \right] \right] |\phi_3^{(j)}(\vec{p})|^2.\end{aligned}$$

$[\mu_{l,j}^2 = l\mu_l^2(\frac{2\sqrt{l}}{j})^2]$ ,  $\phi_3^{(j)}(\vec{p}) = (\frac{j}{\sqrt{l}})^3 \phi_2(\frac{j}{\sqrt{l}}\vec{p})$ ,  $\phi_3^{(2l)} \equiv \phi_3$ . For  $j \ll [2\sqrt{l}]$ ,  $(\frac{j}{\sqrt{l}})^2 \theta^2 \rightarrow 0$  and this is the effective action on a commutative  $\mathbb{R}^4$  with cut-off  $\Lambda = 2/\theta$ . For  $j \gg [2\sqrt{l}]$  this effective action corresponds to canonical noncommutativity if we insist on the first line above as our effective action or to strongly noncommuting  $\mathbb{R}^4$  if we consider instead the effective action to be given by the second line. For the value  $j = [2\sqrt{l}]$ , where we obtain the noncommutative  $\mathbb{R}^4$  given by (3.5) and (3.7), there seems to be a balance between the above two situations and one can also expect the UV-IR mixing to be smoothen out.

To show this we write first the one-loop planar and non-planar contributions for  $j = [2\sqrt{l}]$ , viz

$$\Delta m^P = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}, \quad \Delta m^{NP}(p) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{i\theta^2 \vec{p}B\vec{k}}.$$

We can evaluate these integrals by introducing a Schwinger parameter  $(k^2 + m^2)^{-1} = \int d\alpha \exp(-\alpha(k^2 + m^2))$ .

Explicitly, we obtain for the planar contribution

$$\begin{aligned}\Delta m^P &= \frac{1}{16\pi^2} \left[ -\Lambda^2 \int \frac{d\alpha}{\alpha} e^{-\alpha(m^2+\Lambda^2)} + \int \frac{d\alpha}{\alpha^2} e^{-\alpha m^2} \left( 1 - e^{-\alpha\Lambda^2} \right) \right] \\ &= \frac{1}{16\pi^2} \left[ \Lambda^2 + m^2 \ln \frac{m^2}{m^2 + \Lambda^2} \right].\end{aligned}\quad (3.30)$$

Obviously the above planar function diverges quadratically as  $\Lambda^2$  when  $\theta \rightarrow 0$ , i.e. the noncommutativity acts effectively as a cut-off.

Next we compute the non-planar integral. To this end we introduce as above a Schwinger parameter and rewrite the integral as follows

$$\begin{aligned}\Delta m^{NP}(p) &= \frac{1}{16\pi^4} \int_0^\infty d\alpha e^{-\alpha m^2 - \frac{\epsilon^4 E^2}{4\alpha}} \int_\Lambda d^4 k e^{-\alpha \left[ \vec{k} - \frac{i\vec{E}}{2\alpha} \right]^2} \\ &= \frac{1}{16\pi^4} \sum_{r=0}^\infty (\theta^2)^r \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \frac{i^{r-s}}{s!(r-2s)!} \left[ \int_0^\infty d\alpha \left( \frac{E^2}{4i\alpha} \right)^s e^{-\alpha m^2 - \frac{\epsilon^4 E^2}{4\alpha}} \left[ \int_\Lambda d^4 k e^{-\alpha k^2} (\vec{k} \vec{E})^{r-2s} \right] \right], \quad E^\nu = B^{\mu\nu} p^\mu.\end{aligned}$$

In above we have also used the fact that  $\theta$  is small in the sense we explained earlier (i.e.  $E\theta \ll 1$ ) and in accordance with [16] to expand the second exponential around  $\theta = 0$ . This is also because the cut-off  $\Lambda$  is inversely proportional to  $\theta$ . [In the last line we used the identity  $\sum_{p=0}^\infty \sum_{q=0}^p A_{q,p-q} = \sum_{r=0}^\infty \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} A_{s,r-2s}$ ,  $\lfloor \frac{r}{2} \rfloor = \frac{r}{2}$  for  $r$  even and  $\lfloor \frac{r}{2} \rfloor = \frac{r-1}{2}$  for  $r$  odd]. It is not difficult to argue that the inner integral above vanishes unless  $r$  is even. Using also the fact that the cut-off  $\Lambda$  is rotationally invariant one can evaluate the inner integral as follows. We have

$$\int_\Lambda d^4 k e^{-\alpha k^2} (\vec{k} \vec{E})^n = 4\pi^2 E^n (n-1)!! \left[ \frac{1}{(2\alpha)^{\frac{n}{2}+2}} - \Lambda^n e^{-\alpha\Lambda^2} \sum_{q=-1}^{\frac{n}{2}} \frac{1}{(n-2q)!!} \frac{1}{(2\alpha)^{q+2}} \frac{1}{\Lambda^{2q}} \right],$$

where  $n$  is an even number given by  $n = r - 2s$ .

We can now put the above non-planar function in the form

$$\Delta m^{NP}(p) = \frac{1}{16\pi^2} \sum_{N=0}^\infty \frac{1}{N!} \left( \frac{\theta^2 E}{2} \right)^{2N} \int \frac{d\alpha}{\alpha^{N+2}} e^{-\alpha m^2 - \frac{\epsilon^4 E^2}{4\alpha}} \sum_{M=0}^N C_N^M (-1)^M \left[ 1 - \sum_{P=0}^{M+1} \frac{(\alpha\Lambda^2)^P}{P!} e^{-\alpha\Lambda^2} \right]. \quad (3.31)$$

$[C_N^M = \frac{N!}{M!(N-M)!}]$ . The first term in this expansion corresponds exactly to the case of canonical noncommutativity where instead of  $\Lambda$  we have no cut-off, i.e.

$$\begin{aligned}\Delta m^{NP}(p) &= \frac{1}{16\pi^2} \sum_{N=0}^\infty \frac{1}{N!} \left( \frac{\theta^2 E}{2} \right)^{2N} \int \frac{d\alpha}{\alpha^{N+2}} e^{-\alpha m^2 - \frac{\epsilon^4 E^2}{4\alpha}} \sum_{M=0}^N C_N^M (-1)^M + \dots \\ &= \frac{1}{8\pi^2} \left[ \frac{2}{\theta^4 E^2} + m^2 \ln(m\theta^2 E) \right] + \dots \equiv \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) + \dots\end{aligned}$$

As expected this term provides essentially the canonical UV-IR mixing. As it turns out this singular behaviour is completely regularized by the remaining  $N = 0$  term in (3.31), i.e.

$$\begin{aligned}\Delta m^{NP}(p) &= \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) + \frac{1}{16\pi^2} \int \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{\epsilon^4 E^2}{4\alpha}} \left[ - \sum_{P=0}^1 \frac{(\alpha\Lambda^2)^P}{P!} e^{-\alpha\Lambda^2} \right] + \dots \\ &= \frac{1}{16\pi^2} I^{(2)}(m^2, \frac{\theta^4 E^2}{4}) - \frac{1}{16\pi^2} \left[ I^{(2)}(m^2 + \Lambda^2, \frac{\theta^4 E^2}{4}) + \Lambda^2 I^{(1)}(m^2 + \Lambda^2, \frac{\theta^4 E^2}{4}) \right] + \dots\end{aligned}\quad (3.32)$$

The integrals  $I^{(L)}(x, y)$  are given essentially by Hankel functions, viz

$$\begin{aligned}I^{(1)}(x, y) &= \int_0^\infty \frac{d\alpha}{\alpha} e^{-x\alpha - \frac{y}{\alpha}} = \frac{1}{2} \left[ i\pi H_0^{(1)}(2i\sqrt{xy}) + h.c. \right] \\ I^{(L)}(x, y) &= \int_0^\infty \frac{d\alpha}{\alpha^L} e^{-x\alpha - \frac{y}{\alpha}} = \frac{1}{2} \left[ \frac{i\pi}{L-1} \left( \frac{x}{y} \right)^{\frac{L-1}{2}} \sqrt{xy} e^{\frac{i\pi}{2}} [H_{L-2}^{(1)}(2i\sqrt{xy}) + H_L^{(1)}(2i\sqrt{xy})] + h.c. \right], \quad L > 1.\end{aligned}$$

Hankel functions admit the series expansion  $H_0^{(1)}(z) = \frac{2i}{\pi} \ln z + \dots$  and  $H_\nu^{(1)}(z) = -\frac{i(\nu-1)!}{\pi} \left( \frac{z}{2} \right)^\nu + \dots$  for  $\nu > 0$  when  $z \rightarrow 0$ . In this case the mass  $m$  and the external momentum  $E$  are both small compared to the cut-off  $\Lambda = 2/\theta$  and thus the dimensionless parameters  $z \equiv \sqrt{xy} = 2\frac{m}{\Lambda} \frac{E}{\Lambda}$  or  $z \equiv \sqrt{xy} = 2\sqrt{1 + \frac{m^2}{\Lambda^2}} \frac{E}{\Lambda}$  are also small, in other words we can calculate for example  $I^{(1)}(x, y) = -2 \ln(2\sqrt{xy})$ ,  $I^{(2)}(x, y) = 2x \ln(2\sqrt{xy}) + \frac{1}{y}$  and  $I^{(L)}(x, y) = \frac{(L-2)!}{y^{L-1}} [1 - \frac{xy}{(L-2)(L-1)}]$  for  $L \geq 3$ . Thus the first term  $N = 0$  in the above sum (i.e. equation (3.32)) is simply given by

$$\Delta m^{NP}(p) = -\frac{m^2}{16\pi^2} \ln(1 + \frac{\Lambda^2}{m^2}) + \dots \quad (3.33)$$

As one can see it does not depend on the external momentum  $p$  at all. In the commutative limit  $\theta \rightarrow 0$ , this diverges logarithmically as  $\ln \Lambda$  which is subleading compared to the quadratic divergence of the planar function. Higher corrections can also be computed and one finds essentially an expansion in  $\frac{\Lambda^2 E}{2} = E\theta = 2\frac{E}{\Lambda}$  given by

$$\begin{aligned}\Delta m^{NP}(p) &= -\frac{m^2}{16\pi^2} \ln(1 + \frac{\Lambda^2}{m^2}) \\ &+ \frac{\Lambda^2}{16\pi^2} I^{(1)}(x, y) \sum_{p=2}^\infty \frac{1}{p!} \left( \frac{\Lambda\theta^2 E}{2} \right)^{2(p-1)} \eta_{p-1,p-2} + \frac{1}{16\pi^2} I^{(2)}(x, y) \sum_{p=2}^\infty \frac{1}{p!} \left( \frac{\Lambda\theta^2 E}{2} \right)^{2p} \eta_{p,p-2} \\ &+ \frac{1}{16\pi^2} \sum_{N=1}^\infty \left( \frac{\theta^4 E^2}{4} \right)^N I^{(N+2)}(x, y) \sum_{p=2}^\infty \frac{1}{p!} \left( \frac{\Lambda\theta^2 E}{2} \right)^{2p} \eta_{p+N,p-2}.\end{aligned}$$

$[x = m^2 + \Lambda^2, y = \frac{\theta^4 E^2}{4}, \eta_{p+N,p-2} = \sum_{M=0}^{p-2} \frac{(-1)^M}{M!(p+N-M)!}]$ . It is not difficult to find that the leading terms in the limit of small external momenta (i.e.  $E/\Lambda \ll 1$ ) are effectively given by

$$\Delta m^{NP}(p) = -\frac{m^2}{16\pi^2} \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) - \frac{E^2}{4\pi^2} \ln \left( 4 \frac{E}{\Lambda} \sqrt{1 + \frac{m^2}{\Lambda^2}} \right) \left[ 1 + O \left( \frac{E^2}{\Lambda^2} \right) \right] + \frac{E^2}{8\pi^2} \left[ 1 + O \left( \frac{E^2}{\Lambda^2} \right) \right]. \quad (3.34)$$

Clearly in the strict limit of small external momenta when  $E \rightarrow 0$ , we have  $E^2 \ln E \rightarrow 0$  and the non-planar contribution does not diverge (only the first term in (3.34) survives this limit as it is independent of  $E$ ) and hence there is no UV-IR mixing. The limit of zero noncommutativity is singular but now this divergence has the nice interpretation of being the divergence recovered in the non-planar 2-point function when the cut-off  $\Lambda = \frac{2}{\theta}$  is removed. This divergence is however logarithmic and therefore is sub-leading compared to the quadratic divergence in the planar part.

The effective action (3.29) with  $j = [2\sqrt{I}]$  can be obviously obtained from quantizing the action (3.26) with the replacements  $\mu_i^2 \rightarrow m^2$ ,  $\phi_3 \rightarrow \phi_2 \equiv \phi_2(X^{NC}) = \int \frac{d^4 p}{(2\pi)^4} \phi_2(\vec{p}) e^{-ip \cdot X^{NC}} = \phi_2^\dagger$  and where as before we have to regularize all integrals in the quantum theory with a cut-off  $\Lambda = 2/\theta$ . The star product  $*$  is the Moyal-Weyl product given in (3.16) with the substitutions  $\theta' \rightarrow \theta$ ,  $x^{NC} \rightarrow X^{NC}$ . This effective action can also be rewritten in the form

$$\int d^4 x \left[ \frac{1}{2} \partial_\mu \phi_2 *_\Lambda \partial_\mu \phi_2 + \frac{1}{2} m^2 \phi_2 *_\Lambda \phi_2 + \frac{g_4^2}{4!} \phi_2 * \phi_2 * \phi_2 * \phi_2 \right], \quad (3.35)$$

which is motivated by the fact that the *effective* star product defined by

$$\begin{aligned}f *_\Lambda g(X^{NC}) &= \int_\Lambda \frac{d^4 p}{(2\pi)^4} f(\vec{p}) \int_\Lambda \frac{d^4 k}{(2\pi)^4} g(\vec{k}) e^{-ip \cdot X^{NC}} * e^{-ik \cdot X^{NC}} \\ &= \int d^4 y' d^4 z' \delta_\Lambda^4(y') \delta_\Lambda^4(z') f(y - y') * g(z - z')|_{y=z=X^{NC}},\end{aligned}\quad (3.36)$$



is such that  $\int d^4x f *_{\Lambda} g(x) = \int_{\Lambda} \frac{d^4p}{(2\pi)^4} f(\vec{p})g(-\vec{p})$ . The distribution  $\delta_{\Lambda}^4(y')$  is not the Dirac delta function  $\delta^4(y')$  but rather  $\delta_{\Lambda}^4(y') = \int_{\Lambda} \frac{d^4p}{(2\pi)^4} e^{-ip'y'}$ , i.e.  $\delta_{\Lambda}^4(y')$  tends to the ordinary delta function in the limit  $\Lambda \rightarrow \infty$  of the commutative plane where the above product (3.36) also reduces to the ordinary point-wise multiplication of functions. If the cut-off  $\Lambda$  was not correlated with the non-commutativity parameter  $\theta$ , then the limit  $\Lambda \rightarrow \infty$  would had corresponded to the limit where the product (3.36) reduces to the Moyal-Weyl product given in equation (3.16). This way of writing the effective action (i.e. (3.35)) is to insist on the fact that all integrals are regularized with a cut-off  $\Lambda = 2/\theta$ . In other words the above new star product which appears only in the kinetic part of the action is completely equivalent to a sharp cut-off  $\Lambda$  and yields therefore exactly the propagator (3.28) with which only modes  $\leq \Lambda$  can propagate.

We should also remark here regarding non-locality of the star product (3.36). At first sight it seems that this non-locality is more severe in (3.36) than in (3.16), but as it turns out this is not entirely true: in fact the absence of the UV-IR mixing in this product also suggests this. In order to see this more explicitly we first rewrite (3.36) in the form

$$f *_{\Lambda} g(X^{NC}) = \int d^4y' d^4z' f(y')g(z')K_{\Lambda}(y', z'; X^{NC})$$

$$K_{\Lambda}(y', z'; X^{NC}) = \delta_{\Lambda}^4(y - y') * \delta_{\Lambda}^4(z - z')|_{y=z=X^{NC}}.$$

The kernel  $K_{\Lambda}$  can be computed explicitly and is given by

$$K_{\Lambda}(y', z'; X^{NC}) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \delta_{\Lambda}^4(X^{NC} - y' + \frac{\theta^2}{2} Bk) e^{ik(z' - X^{NC})}.$$

For the moment, let us say that  $\Lambda$  and  $\theta$  are unrelated. Then, taking  $\Lambda$  to infinity gives [1, 5]

$$K(y', z'; X^{NC}) = \frac{16}{\theta^8 \det B} \frac{1}{(2\pi)^4} e^{\frac{2i}{\theta^2}(z' - X^{NC})B^{-1}(y' - X^{NC})}.$$

If we have for example two functions  $f$  and  $g$  given by  $f(x) = \delta^4(x - p)$  and  $g(x) = \delta^4(x - p)$ , i.e. they are non-zero only at one point  $p$  in space-time, their star product which is clearly given by the kernel  $K(p, p; X^{NC})$  is non-zero everywhere in space-time. The fact that  $K$  is essentially a phase is the source of the non-locality of (3.16) which leads to the UV-IR mixing.

On the other hand the kernel  $K_{\Lambda}(p, p; X^{NC})$  with finite  $\Lambda$  can be found in two dimensions (say) to be given by

$$K_{\Lambda}(p, p; X^{NC}) = \frac{1}{\pi^2 \theta^4} \int_{-\theta}^{\theta} da \delta_{\Lambda}(a + L_1) e^{\frac{2i}{\theta^2} L_2 a} \int_{-\theta}^{\theta} db \delta_{\Lambda}(b + L_2) e^{-\frac{2i}{\theta^2} L_1 b},$$

with  $L_a = X_a^{NC} - p_a$ ,  $a = 1, 2$ . If we now make the approximation to drop the remaining  $\Lambda$  (since the effects of this cut-off were already taken anyway) one can see that the above integral is non-zero only for  $-\theta + p_1 \leq X_1^{NC} \leq \theta + p_1$  and  $-\theta + p_2 \leq X_2^{NC} \leq \theta + p_2$  simultaneously. In other words the star product  $K_{\Lambda}(p, p; X^{NC})$  of  $f(x)$  and  $g(x)$  is also localized around  $p$  within an error  $\theta$  and is equal to  $\frac{1}{\pi^2 \theta^4}$  there. The star product (3.36) is therefore effectively local.

Final remarks are in order. First we note that the effective star product (3.36) leads to an effective commutation relations (3.7) in which the parameter  $\theta^2$  is multiplied by an overall constant equal to  $\int d^4y' d^4z' \delta_{\Lambda}^4(y') \delta_{\Lambda}^4(z')$ , we simply skip the elementary proof. Remark also that this effective star product is non-associative as one should expect since it is for all practical purposes equivalent to a non-trivial sharp momentum cut-off  $\Lambda$  [23].

The last remark is to note that the prescription (3.28) can also be applied to the canonical limit of large stereographic projection of the spheres onto planes, and in this case one can also obtain a cut-off  $\Lambda' = \frac{2}{\theta'}$  with  $j$  fixed as above such that  $j = \lfloor 2\sqrt{l} \rfloor$ . The noncommutative plane (3.4) defined in this way is therefore completely equivalent to the above noncommutative plane (3.7).

### 3.4 The Continuum Planar Limit of the 4-Point Function

We now undertake the task of finding the continuum limit of the above 4-point function (equations (2.20) and (2.22)) which we expect to correspond to the 4-point function on the noncommutative  $\mathbb{R}^4$ . This expectation is motivated of course by the result of the last sections on the 2-point function. As it turns out this is also the case here and as an explicit example we work out the continuum flattening limit of the planar amplitudes.

The planar diagrams are  $\delta\lambda_4^{(1)}$  and  $\delta\lambda_4^{(4)}$ . First, let us recall that in above the indices 4 and 6 refer to internal momenta whereas 1, 2, 3 and 5 refer to external momenta. Next, since we are interested in the planar limits (in which  $R, l \rightarrow \infty$ ) of the 4-point function, we can use the asymptotic formula

$$\left\{ \begin{matrix} a & b & c \\ d+l & e+l & f+l \end{matrix} \right\} = \frac{(-1)^{a+b+d+e}}{\sqrt{(2l+1)(2c+1)}} C_{af-ed-f}^{cd-e}, \quad l \rightarrow \infty, \quad (3.40)$$

which allows us to approximate in the limit the “fuzzy delta” function (2.26) as follows:

$$\delta_k(1235) = (2l+1)(2k+1)(-1)^{k_1+k_2+k_3+k_5+m} \delta_{m_1+m_2+m_3+m_5,0}$$

$$\times \left\{ \begin{matrix} k_1 & k_2 & k \\ m_2+l & -m_1+l & l \end{matrix} \right\} \left\{ \begin{matrix} k_3 & k_5 & k \\ m_5+l & -m_3+l & l \end{matrix} \right\}. \quad (3.41)$$

We have also used the properties of the Clebsch-Gordan coefficients to obtain the selection rule  $m = m_1 + m_2 = -m_3 - m_5$ , thus justifying the name. The next selection rule comes from the fact that the function  $E_{k_1 k_2}^{k_4 k_6}(k) E_{k_3 k_5}^{k_4 k_6}(k)$  in the planar diagrams (2.23) is proportional in the large  $l$  limit (by virtue of equation (3.40)) to  $C_{k_1 0 k_2 0}^{k_4 0} C_{k_3 0 k_5 0}^{k_4 0} (C_{k_4 0 k_6 0}^{k_4 0})^2$ , whereas on the other hand these Clebsch-Gordan coefficients are such that  $C_{a 0 b 0}^{c 0} \neq 0$  only if  $a+b+c$  is even. This means in particular that  $k+k_4+k_6$  is even,  $k+k_1+k_2$  is even and  $k+k_3+k_5$  is even, and hence one can argue in different ways that one can have for example

$$k_1 + k_2 = k_3 + k_5, \quad k_1 + k_2 = k_4 + k_6, \quad k_3 + k_5 = k_4 + k_6. \quad (3.42)$$

For obvious reasons we will only focus on this sector. As a consequence of these rules, the planar graphs  $\nu_i^{(1)}$  and  $\nu_i^{(4)}$  are equal. Indeed for large  $l$ , one can easily show that these diagrams take the form

$$\nu_i^{(1)} = \nu_i^{(4)} \simeq (2l+1)^3 (-1)^{m_1+m_2} \delta_{m_1+m_2+m_3+m_5,0} \delta_{m_1+m_2+m_4+m_6,0} \delta_{k_1+k_2,k_3+k_5} \delta_{k_1+k_2,k_4+k_6}$$

$$\times a_k(1235) S(46; 1235) \quad \text{where}$$

$$S(46; 1235) = \sum_k (2k+1) \left\{ \begin{matrix} k_4 & k_6 & k \\ l & l & l \end{matrix} \right\}^2 \left\{ \begin{matrix} k_1 & k_2 & k \\ l & l & l \end{matrix} \right\}^2 \left\{ \begin{matrix} k_3 & k_5 & k \\ l & l & l \end{matrix} \right\}^2, \quad (3.43)$$

where  $a_k(1235) = \sqrt{\prod_{i=1}^{2,3,5} (2k_i+1)}$ . As in the case of the 2-point function we have assumed that the external momenta  $k_1, k_2, k_3$  and  $k_5$  are such that  $k_i < l$ ,  $i = 1, 2, 3, 5$ . It is also expected that the approximation sign becomes an exact equality only in the strict limit. Furthermore from the properties of the 6j-symbols, only the values  $0 \leq k \leq k_4 + k_6$  will contribute to the sum  $\sum_k$ . Lastly we have also invoked in (3.43) the fact that for each fixed pair  $(k_4, k_6)$  which is integrated over in (2.22) the azimuth numbers  $(m_4, m_6)$ , although they are already summed over, conspire such that their sum is  $m_4 + m_6 = -m_1 - m_2$ . From [17] we can now use the identity

$$\left\{ \begin{matrix} k_4 & k_6 & k \\ l & l & l \end{matrix} \right\}^2 = \sum_{X_1} (-1)^{X_1} (2X_1+1) \left\{ \begin{matrix} k_4 & l & l \\ X_1 & l & l \end{matrix} \right\} \left\{ \begin{matrix} k_6 & l & l \\ X_1 & l & l \end{matrix} \right\} \left\{ \begin{matrix} k & l & l \\ X_1 & l & l \end{matrix} \right\}, \quad (3.44)$$

etc. The delta function  $\delta_{k_1+k_2,k_3+k_5}$  makes it safe to treat the internal momenta  $k_4$  and  $k_6$  as if they were small (recall that  $k_4$  and  $k_6$  are non-negative integers), and  $0 \leq k \leq k_4 + k_6$  means that  $k$  can be treated as small as well. One can therefore use the result (3.12) to rewrite the above equation as

$$\left\{ \begin{matrix} k_4 & k_6 & k \\ l & l & l \end{matrix} \right\}^2 = \frac{1}{(2l+1)^3} \sum_{X_1=0}^{2l} (2X_1+1) P_{k_4} \left( 1 - \frac{X_1^2}{2l^2} \right) P_{k_6} \left( 1 - \frac{X_1^2}{2l^2} \right) P_k \left( 1 - \frac{X_1^2}{2l^2} \right),$$

etc. As we have already established, in the large  $l$  limit we can approximate this sum by the integral

$$\left\{ \begin{matrix} A & B & k \\ l & l & l \end{matrix} \right\}^2 = \frac{2R^2}{(2l+1)^3} \int_0^\Lambda p_{x_1} dp_{x_1} P_A \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right) P_B \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right) P_k \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right). \quad (3.45)$$

with  $(A, B) = (k_1, k_2), (k_3, k_5)$  and  $(k_4, k_6)$ . We are obviously using the flattening limit (3.5), i.e.  $\theta = \frac{R}{\sqrt{l(l+1)}}$ ,  $\Lambda = \frac{2}{\theta}$  for reasons which will become self-evident shortly. Using the result (3.21) we have

$$P_A \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right) P_B \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right) = \frac{1}{(2\pi)^2} \int d\phi_A d\phi_B e^{iR\theta \vec{p}_{x_1} \wedge (\vec{p}_A + \vec{p}_B)}. \quad (3.46)$$

where  $\vec{p}_A \wedge \vec{p}_B = B^{\mu\nu} p_A^\mu p_B^\nu$ , with  $B^{12} = -1$ , and  $\phi_A, \phi_B$  have the interpretation of angles between  $\vec{p}_A$  and  $\vec{p}_B$  respectively and the  $x$ -axis. Similarly we have

$$P_k \left(1 - \frac{\theta^2 p_{x_1}^2}{2}\right) P_k \left(1 - \frac{\theta^2 p_{x_2}^2}{2}\right) P_k \left(1 - \frac{\theta^2 p_{x_3}^2}{2}\right) = \frac{1}{(2\pi)^3} \int d\alpha_1 d\alpha_2 d\alpha_3 e^{iR\theta \vec{p}_k \wedge (\vec{p}_{x_1} + \vec{p}_{x_2} + \vec{p}_{x_3})}, \quad (3.47)$$

where now the angles  $\alpha_i$ 's are the angles between the vectors  $\vec{p}_{x_i}$ 's and the  $x$ -axis. Since  $R$  is large, the integrals (3.46) with  $(A, B) = (k_1, k_2), (k_3, k_5)$  and  $(k_4, k_6)$  are dominated by those values of  $\vec{p}_{x_1}, \vec{p}_{x_2}$  and  $\vec{p}_{x_3}$  such that  $\vec{p}_{x_1} = \vec{p}_{k_4} + \vec{p}_{k_6}$ ,  $\vec{p}_{x_2} = \vec{p}_{k_1} + \vec{p}_{k_2}$  and  $\vec{p}_{x_3} = \vec{p}_{k_3} + \vec{p}_{k_5}$  respectively, and correspondingly the integral (3.47) is dominated by  $\vec{p}_{x_1} + \vec{p}_{x_2} + \vec{p}_{x_3} = \vec{p}_{k_4} + \vec{p}_{k_6}$ . This is clearly a valid approximation because the conservation law  $\vec{p}_{k_1} + \vec{p}_{k_2} + \vec{p}_{k_3} + \vec{p}_{k_5} = 0$  is expected to hold (as we explain below) and because of the large factor of  $R$  appearing in the different phases in (3.46) and (3.47). After we apply the conservation law we may reinterpret the angles (say)  $\alpha_1$  and  $\alpha_2$  as the angles made by  $\vec{p}_{k_4}$  and  $\vec{p}_{k_6}$  and the  $x$ -axis respectively. Using all these ingredients one can convince ourselves that the sum over  $k$  in (3.43) behaves in the limit as

$$S(46; 1235) \simeq \frac{1}{2l+1} \left\{ \begin{matrix} k_4 & l & l \\ k_6 & l & l \end{matrix} \right\}^2 \left\{ \begin{matrix} k_1 & l & l \\ k_2 & l & l \end{matrix} \right\} \left\{ \begin{matrix} k_3 & l & l \\ k_5 & l & l \end{matrix} \right\}.$$

We now proceed to the task of rewriting this sum in terms of the noncommutative plane variables. To this end we use the representation (3.21) in the form

$$\begin{aligned} P_{k_4}^2 \left(1 - \frac{\theta^2 p_{k_6}^2}{2}\right) &= \int \frac{d\phi_{k_4}}{2\pi} \cos(R\theta \sin \phi_{k_4} p_{k_4} p_{k_6}) \int \frac{d\phi_{k_6}}{2\pi} \cos(R\theta \sin \phi_{k_6} p_{k_4} p_{k_6}) \\ &\simeq \int \frac{d\phi_{k_4}}{2\pi} \frac{d\phi_{k_6}}{2\pi} \cos^2(R\theta \vec{p}_{k_4} \wedge \vec{p}_{k_6}), \end{aligned}$$

where we have used the large  $R$  limit to go to the last line, i.e. since the angles  $\phi_4 = \phi_6 \simeq 0$  dominate the integrals in the limit, the two cosines become essentially equal. We have also reinforced explicitly the symmetry of (3.43) under the exchange  $k_4 \leftrightarrow k_6$  on each 6j-symbol in  $\mathcal{S}$  above (as is also the case in (3.43)). The  $\phi_{k_4}$  and  $\phi_{k_6}$  have the natural interpretation of angles between the vectors  $\vec{p}_{k_4}$  and  $\vec{p}_{k_6}$  respectively and the  $x$ -axis of the plane. For the case  $(A, B) = (k_1, k_2)$ , we can use

$$P_{k_1} \left(1 - \frac{\theta^2 p_{k_2}^2}{2}\right) = \int \frac{d\phi}{2\pi} e^{iR\theta \cos \phi p_{k_1} p_{k_2}}.$$

However, here  $\phi$  cannot be interpreted as the angle between  $\vec{p}_{k_1}$  (or  $\vec{p}_{k_2}$ ) with any specific axis, but if  $\phi_{12}$  is the angle between the two vectors  $\vec{p}_{k_1}$  and  $\vec{p}_{k_2}$  then we can define  $x = \phi + \phi_{12}$ , and write

$$P_{k_1} \left(1 - \frac{\theta^2 p_{k_2}^2}{2}\right) = \int_{\phi_{12}}^{2\pi + \phi_{12}} \frac{dx}{2\pi} e^{-iR\theta \sin x \vec{p}_{k_1} \wedge \vec{p}_{k_2}} e^{iR\theta \cos x \vec{p}_{k_1} \vec{p}_{k_2}} \simeq e^{-iR\theta \vec{p}_{k_1} \wedge \vec{p}_{k_2}}.$$

As before, since  $R$  is large, the integral is dominated by the value  $\cos x = 0$  or  $x = \frac{\pi}{2}$ . We can then evaluate the above sum  $\mathcal{S}$  explicitly and find

$$S(46; 1235) = \frac{1}{(2l+1)^5} \int \frac{d\phi_{k_4}}{2\pi} \frac{d\phi_{k_6}}{2\pi} \cos(R\theta \vec{p}_{k_1} \wedge \vec{p}_{k_2}) \cos(R\theta \vec{p}_{k_3} \wedge \vec{p}_{k_5}) \cos^2(R\theta \vec{p}_{k_4} \wedge \vec{p}_{k_6}),$$

where the symmetry of (3.43) under the exchanges  $k_1 \leftrightarrow k_2$  and  $k_3 \leftrightarrow k_5$  is now explicit. This is essentially the phase of the 4-point function found in [18]. In order to see this fact more clearly, we first show that (3.43) takes now the form

$$\begin{aligned} \nu_1^{(1)} = \nu_1^{(4)} &\simeq \frac{a_k(1235)}{R^4} \int \frac{d\phi_{k_4}}{2\pi} \frac{d\phi_{k_6}}{2\pi} \cos(R\theta \vec{p}_{k_1} \wedge \vec{p}_{k_2}) \cos(R\theta \vec{p}_{k_3} \wedge \vec{p}_{k_5}) \cos^2(R\theta \vec{p}_{k_4} \wedge \vec{p}_{k_6}) \\ &\times \delta^2(\vec{p}_{k_1} + \vec{p}_{k_2} + \vec{p}_{k_3} + \vec{p}_{k_5}) \delta^2(\vec{p}_{k_1} + \vec{p}_{k_2} + \vec{p}_{k_4} + \vec{p}_{k_6}), \end{aligned}$$

where we have also made the following interpretation of the limiting form of the 2-dimensional fuzzy delta function

$$(-1)^{\frac{m}{2}} \frac{R^2}{2l+1} \delta_{k, k_0} \delta_{m, -m_0} \rightarrow \delta(\vec{p}_k + \vec{p}_{k_0}). \quad (3.48)$$

The factor  $(-1)^{\frac{m}{2}}$  is motivated by (2.8), the factor  $2l+1$  is needed in order for (3.48) to diverge correctly (in the limit) when  $k = k_0$  and  $m = -m_0$ , while the  $R^2$  factor is to restore the correct mass dimension for the delta function. An identical formula will of course hold for the other  $\mathbb{R}^2$  factor, i.e.

$$\begin{aligned} \nu_2^{(1)} = \nu_2^{(4)} &\simeq \frac{a_p(1235)}{R^4} \int \frac{d\phi_{p_4}}{2\pi} \frac{d\phi_{p_6}}{2\pi} \cos(R\theta \vec{p}_{p_1} \wedge \vec{p}_{p_2}) \cos(R\theta \vec{p}_{p_3} \wedge \vec{p}_{p_5}) \cos^2(R\theta \vec{p}_{p_4} \wedge \vec{p}_{p_6}) \\ &\times \delta^2(\vec{p}_{p_1} + \vec{p}_{p_2} + \vec{p}_{p_3} + \vec{p}_{p_5}) \delta^2(\vec{p}_{p_1} + \vec{p}_{p_2} + \vec{p}_{p_4} + \vec{p}_{p_6}). \end{aligned}$$

By putting the above functions  $\nu_1^{(1,4)}$  and  $\nu_2^{(1,4)}$  in equation (2.22), we easily obtain the 4-dimensional one-loop planar contributions  $\delta\lambda_4^{(1)}$  and  $\delta\lambda_4^{(4)}$  and consequently the planar contribution to the 4-point function  $\delta\lambda_4^P$ . Indeed we have

$$\begin{aligned} \delta\lambda_4^{(1)}(1235) = \delta\lambda_4^{(4)}(1235) &= \frac{a(1235)}{R^4 \pi^4} \delta^4(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_5) \\ &\times \int_\Lambda d^4 \vec{p}_4 \frac{\cos(R\theta \vec{p}_1 \wedge \vec{p}_2) \cos(R\theta \vec{p}_3 \wedge \vec{p}_5) \cos^2(R\theta \vec{p}_4 \wedge \vec{p}_6)}{(\vec{p}_4^2 + \frac{m^2}{4})((\vec{p}_1 + \vec{p}_2 + \vec{p}_4)^2 + \frac{m^2}{4})}, \end{aligned}$$

where the notation (in the metric  $\mathbb{R}^2 \times \mathbb{R}^2$ ) is  $p_4^2 = p_{k_4}^2 + p_{p_4}^2$ ,  $d^4 p_4 = \frac{1}{4} dp_{k_4}^2 dp_{p_4}^2 d\phi_{k_4} d\phi_{p_4}$ ,  $\delta^4(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_5) = \delta^2(\vec{p}_{k_1} + \vec{p}_{k_2} + \vec{p}_{k_3} + \vec{p}_{k_5}) \delta^2(\vec{p}_{p_1} + \vec{p}_{p_2} + \vec{p}_{p_3} + \vec{p}_{p_5})$  and  $\vec{p}_1 \wedge \vec{p}_2 = \vec{p}_{k_1} \wedge \vec{p}_{k_2} + \vec{p}_{p_1} \wedge \vec{p}_{p_2}$  and  $a(1235) = a_k(1235) a_p(1235)$ . The associated effective action in this case can now easily be computed and we find

$$\frac{g_4^2}{4!} \int_{\Lambda} \frac{d^4 \vec{p}_1}{(2\pi)^4} \frac{d^4 \vec{p}_2}{(2\pi)^4} \frac{d^4 \vec{p}_3}{(2\pi)^4} \frac{d^4 \vec{p}_5}{(2\pi)^4} \delta\lambda_4^P(1235) \delta^4(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_5) \hat{\phi}_2(\vec{p}_1) \hat{\phi}_2(\vec{p}_2) \hat{\phi}_2(\vec{p}_3) \hat{\phi}_2(\vec{p}_5), \quad (3.49)$$

where

$$\delta\lambda_4^P(1235) = \frac{32}{3} g_4^2 \int_{\Lambda} \frac{d^4 \vec{p}_4}{(2\pi)^4} \frac{\cos(\theta^2 \vec{p}_1 \wedge \vec{p}_2) \cos(\theta^2 \vec{p}_3 \wedge \vec{p}_5) \cos^2(\theta^2 \vec{p}_4 \wedge \vec{p}_6)}{(\vec{p}_4^2 + m^2)((\vec{p}_1 + \vec{p}_2 + \vec{p}_4)^2 + m^2)}. \quad (3.50)$$

We have employed in above the same definitions as those of the 2-point function used in (3.26), namely  $g_4^2 = 8\pi^2 \lambda_4$  and  $\hat{\phi}_2(\vec{p}_1) = 4\pi \sqrt{\frac{2}{3}} \hat{\phi}_{NC}(\frac{\vec{p}_1}{\Lambda})$ . However, the noncommutative field  $\hat{\phi}_{NC}(\vec{p}_1)$  is now reinterpreted such that we have  $\hat{\phi}_{NC}(\vec{p}_1) \equiv \hat{\phi}_{NC}^{p_{k_1} p_{p_1} \phi_{k_1} \phi_{p_1}} = R^4 \phi^{k_1 p_1 m_1 n_1} \sqrt{(2k_1+1)(2p_1+1)}$  or, in other words,  $\hat{\phi}_{NC}(\vec{p}_1) = \phi_{NC}(\vec{p}_1) \sqrt{(2k_1+1)(2p_1+1)}$ .

## 4 Conclusion

We have investigated in some detail the problem of obtaining theories on noncommutative  $\mathbb{R}^4$  starting from finite matrix models defined on  $S_F^2 \times S_F^2$ . Particular attention was paid to a new limit that gives a theory on noncommutative  $\mathbb{R}^4$  with a UV cut-off proportional to the inverse of the noncommutativity parameter  $\theta$ , and without any mixing between UV and IR degrees of freedom.

The new scaling is implemented via the introduction of an intermediate scale  $[2\sqrt{l}]$ . Intuitively, this intermediate scale carries information about the transition between commutative and noncommutative regimes of the theory: if we only use modes with momenta much smaller than this intermediate scale, the theory becomes commutative, whereas modes with momenta much larger take us to the noncommutative regime.

It would be interesting to extend this analysis to theories on  $S_F^2$  and  $S_F^2 \times S_F^2$  that have fermionic and gauge [25] degrees of freedom, as well as supersymmetric theories [26]. We also see no obstacle to using this method to study theories that are obtained from Kaluza-Klein reduction on fuzzy  $S^4$  [27], as well as gauge theory on fuzzy  $CP^2$  [28].

**Acknowledgments:** It is a pleasure to thank A. P. Balachandran, Denjoe O'Connor and Peter Prešnajder for numerous discussions. SV would like to thank DIAS for warm hospitality during the final stage of this project. The work of SV is supported in part by DOE grant DE-FG03-91ER40674.

## References

- [1] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73** 977-1029 (2001); [hep-th/0106048](#).
- [2] A.P. Balachandran, B.P. Dolan, J-H. Lee, X. Martin and Denjoe O'Connor, J. Geom. Phys. **43**, 184-204 (2002); [hep-th/0107099](#).
- [3] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, J. Geom. Phys. **42**, 28-53 (2002); [hep-th/0103023](#).
- [4] S. P. Trivedi and S. Vaidya, JHEP **0009**, 041 (2000); [hep-th/0007011](#).
- [5] S. Minwalla, M. V. Raamsdonk and N. Seiberg, JHEP **0002**, 020 (2000); [hep-th/9912072](#).
- [6] J. Ambjorn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, JHEP **0005**, 023 (2000); [hep-th/0004147](#).
- [7] S. Vaidya and B. Ydri, [hep-th/0209131](#).
- [8] S. Vaidya, Phys.Lett. **B512**, 403-411 (2001); [hep-th/0102212](#).
- [9] G. Alexanian, A. Pinzul and A. Stern, Nucl.Phys. **B600** 531-547 (2001); [hep-th/0010187](#).
- [10] I. Kishimoto, JHEP **0103**, 025 (2001); [hep-th/0103018](#).
- [11] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, Berlin, 1986.
- [12] C-S. Chu, J. Madore and H. Steinacker, JHEP **0108**, 038 (2001); [hep-th/0106205](#).
- [13] B. P. Dolan, D. O'Connor and P. Prešnajder, JHEP **0203**, 013 (2002); [hep-th/0109084](#).
- [14] H. Grosse, C. Klimčík and P. Prešnajder, Int.J.Theor.Phys. **35** 231-244 (1996); [hep-th/9505175](#). Commun.Math.Phys. **185** 155-175 (1997); [hep-th/9507074](#).
- [15] R. Estrada, J. M. Gracia-Bondia and J. C. Varilly, J. Math. Phys. **30** 2789 (1989).
- [16] R. Wulkenhaar, [hep-th/0206018](#).

- [17] D. A. Varshlovich, A. N. Moskalev and V. R. Khersonsky, *Quantum Theory of Angular Momentum*, World Scientific, New Jersey, 1988.
- [18] I. Aref'eva, D. Belov and A. Koshelev, Phys.Lett. **B476**, 431 (2000); [hep-th/9912075](#).
- [19] B.Ydri, "Fuzzy Physics", [hep-th/0110006](#), Ph.D Thesis.
- [20] K. G. Wilson and J. Kogut, Phys. Repts.**12C**, 75 (1974).
- [21] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Chelsea Publishing Company, New York, 1949.
- [22] A. P. Balachandran, B. P. Dolan, J. Lee, X. Martin and D. O'Connor, [hep-th/0107099](#).
- [23] B. Ydri, Phys. Rev. **D63**, 025004 (2001); [hep-th/0003232](#).
- [24] D. O'Connor, S. Vaidya and B. Ydri, unpublished.
- [25] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, Nucl. Phys. **B604**, 121-147 (2001); [hep-th/0101102](#).
- [26] A. P. Balachandran, S. Kurkcuglu and E. Rojas, JHEP **0207**, 056 (2002); [hep-th/0204170](#).
- [27] D. O'Connor and J. Medina, in preparation.
- [28] Y. Kitazawa, Nucl. Phys. **B642**, 210-226 (2002); [hep-th/0207115](#).