DIAS-STP-83-16

ON THE SEQUENCE OF PEDAL TRIANGLES

by

J. G. Kingston, Department of Mathematics, University of Nottingham, Nottingham NG7 2RD

and

J. L. Synge, F.R.S., Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

CONTENTS

ON THE SEQUENCE OF PEDAL TRIANGLES

by J. G. Kingston* and J. L. Synge, F.R.S.**

* Department of Mathematics, University of Nottingham, Nottingham NG7 2RD

** School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Although geometers have studied the properties of triangles for over two thousand years, there still remain problems of interest involving operations performed infinitely often. A given triangle T_0 generates a sequence of triangles T_n where T_{n+1} is the pedal triangle of T_n. This sequence was discussed by Hobson (1897, 1925) but, while his formulae for the transition from T_n to T_{n+1} are correct, those for T_n in terms of T_0 are not. Lacking correct formulae, we experimented numerically, taking the angles of T_0 to be integers in degrees. To our surprise the angles in the pedal sequence became periodic with periods of twelve steps. The explanation of this curious fact led to a general investigation of pedal sequences, revealing that (a) the sequence may stop by degeneration of the triangle to a straight segment, (b) the angles may develop any periodicity, or (c) the sequence may proceed to infinity without periodicity. We give necessary and sufficient conditions on the angles of T_0 corresponding to these options, and discuss the periodic case in some detail.

1. Introduction

The pedal (or first pedal) triangle T' of a given triangle T is formed by joining the feet of the perpendiculars dropped from the vertices of T on the opposite sides, produced if necessary. Coxeter (1969) prefers the word orthic to pedal, but we shall use the latter word, following a well-established tradition.

The pedal of T' is the second pedal of T, and so on. The sequence stops iff we encounter a right-angled triangle; its pedal is a straight line segment. The condition for this is given in §3.

Hobson (1897, 1925) discussed pedal sequences, stating correctly that the sides and angles of T' are given in terms of the elements of T by the following formulae: if A,B,C, are all acute,

$$\begin{aligned} a' &= a\cos A, \quad b' = b\cos B, \quad c' = c\cos C; \\ A' &= \pi - 2A, \quad B' = \pi - 2B, \quad C' = \pi - 2C; \end{aligned} \tag{1.1}$$

if A is obtuse,

$$\begin{aligned} a' &= -a\cos A, \quad b' = b\cos B, \quad c' = c\cos C, \\ A' &= 2A - \pi, \quad B' = 2B, \quad C' = 2C, \end{aligned} \tag{1.2}$$

with similar formulae if B or C is obtuse. He gave formulae for the elements of the nth pedal, different according as n is odd or even. For n odd, his formula for the first angle of the nth pedal is

$$(1/3)(2^n + 1)\pi - 2^n A ; \tag{1.3}$$

if n = 1 (first pedal) this gives $\pi - 2A$, correct for A acute by (1.1) but false for A obtuse by (1.2). Consequently Hobson's formulae for the angles of the nth pedal must be rejected. In fact the occurrence of obtuse angles makes it impossible to write down any reasonably simple explicit formulae for the angles of the nth pedal. But Hobson's formulae for the sides of the nth pedal are correct to within a sign, and we may write for the first side of the nth pedal

$$a_n = \pm a \cos A \cos 2A \cos 4A \ldots \cos 2^{n-1}A, \qquad (1.4)$$

the sign being chosen to give a positive value.

2. The sides and angles of the nth pedal triangle

The notation A^i, a^i, $i = 1,2,3$ is now introduced to represent the angles A,B,C and sides a,b,c of the triangle T. The angles and sides of the nth pedal triangle will thus be represented as A_n^i and a_n^i. Equations (1.1) and (1.2) show that the angles progress according to the rule

$$A_1^i = 2EA^i (\text{mod } \pi), \quad i = 1,2,3, \qquad (2.1)$$

where E may be +1 or -1 according to the nature of the original triangle T, but is independent of i. For the angles of T_n,

$$A_n^i = 2^n E_n A^i (\text{mod } \pi) \qquad (2.2)$$

where $E_n = \pm 1$ so that

$$\sin A_n^i = E_n^i \sin 2^n A^i \qquad (2.3)$$

where $E_n^i = \pm 1$ and may differ for different i. It is known that the circumradius is halved when we pass from a triangle to its pedal, and so

$$R_n = 2^{-n} R. \qquad (2.4)$$

But $a_n^i = 2 R_n \sin A_n^i$, $a^i = 2 R \sin A^i$ and so (2.3) and (2.4) may be combined to give

$$a_n^i = 2^{-n} E_n^i a^i \sin 2^n A^i / \sin A^i . \qquad (2.5)$$

Since $\sin 2^n A^i = 2 \sin 2^{n-1} A^i \cos 2^{n-1} A^i$, (2.5) may be continually expanded until $\sin A^i$ cancels, becoming

$$a_n^i = E_n^i a^i \cos A^i \cos 2A^i \cos 4A^i \ldots \cos 2^{n-1} A^i, \qquad (2.6)$$

agreeing with (1.4).

Equations (2.2) and (2.5) enable A_n^i and a_n^i to be computed precisely; (2.5) determine a_n^i and <u>also</u> the sign of E_n^i (since $a_n^i > 0$); (2.2) then determine A_n^i.

3. Pedal degeneracy

The pedal of a right-angled triangle is a straight segment and this has no pedal; the pedal sequence stops. We call a triangle T pedally degenerate (PD) if a right angle occurs in T or in the pedal sequence which starts with T. It is evident from (2.3) that T is PD iff one of its angles is of the form

$$\pi(2m-1)/2^n, \tag{3.1}$$

where m and n are positive integers. Thus the angles responsible for PD are $\pi/2$, $\pi/4$, $\pi/8$, ... $3\pi/4$, $3\pi/8$, ... $5\pi/8$, $5\pi/16$, ...

A pedal sequence is infinite if it starts from a triangle which is not PD.

4. Pedal cycles

Let T_s and T_{s+n} be triangles in the pedal sequence which starts with T, assumed not PD. It is clear that if T_{s+n} is similar to T_s, then T_s is similar to T_{s+2n}, T_{s+3n}, ... We say then that we have an n-cycle starting with T_s.

We need some formulae in preparation for a theorem about n-cycles, these formulae being independent of the existence of an n-cycle. From (2.2)

$$A^i_{s+n} - A^i_s = E_{s+n}\, 2^s(2^n+E)A^i \pmod{\pi} \tag{4.1}$$

where $E = -E_s/E_{s+n} = \pm 1$ and is independent of i. Also, from (2.5)

$$a^i_{s+n}/a^i_s = -2^{-n} E^i \sin 2^{s+n} A^i / \sin 2^s A^i \tag{4.2}$$

where $E^i = \pm 1$ and depends on i.

THEOREM I: T_{s+n} is similar to T_s (and so T_s starts an n-cycle) iff T is not PD and its angles are of the form

$$A^i = \pi p^i / F(s,n,E),\ i = 1,2,3, \tag{4.3}$$

where $$F(s,n,E) = 2^s (2^n+E),\ E = \pm 1, \tag{4.4}$$

and p^i are positive integers satisfying

$$p^1 + p^2 + p^3 = F(s,n,E), \tag{4.5}$$

except that E=1 for n=1 and s=0 or 1.

Proof of sufficiency

We have to show that, if the angles of T are of the form (4.3), then T_{s+n} is similar to T_s. From (4.3)

$$A^i = \pi p^i / (2^s(2^n + E)) ,$$

so that

$$2^{s+n} A^i = \pi p^i - E\, 2^s A^i$$

and hence

$$\sin 2^{s+n} A^i = F^i \sin 2^s A^i ,$$

where $F^i = \pm 1$ and depends on i. Since a^i_s and a^i_{s+n} are positive, (4.2) now shows that

$$a^i_{s+n} / a^i_s = 2^{-n} ,$$

proving that T_{s+n} is similar to T_s .

Proof of necessity

We have to show that if T_{s+n} is similar to T_s, then the angles of T are as in (4.3). Since $A^i_{s+n} = A^i_s$, (4.1) gives

$$2^s\, (2^n+E)\, A^i = 0 \pmod{\pi}$$

where $E = \pm 1$ and is independent of i, so that A^i are indeed of the form (4.3). The p^i obviously satisfy (4.5). Necessity is now proved.

The exceptional cases noted after (4.5) are due to the fact that for n=1 and s=0 or 1 there are no partitions of F(s,n,E) if E = -1.

It might appear from the preceding theorem that, if the angles of T are as in (4.3), then the n-cycle does not start until we reach T_s. But that is not necessarily the case. Suppose for example that s=1, so that the partition in (4.5) is

$$p^1 + p^2 + p^3 = 2(2^n+E) = 2F(0,n,E) , \qquad (4.6)$$

This partition can be realised by even numbers, $p^i = 2q^i$, and when these are substituted in (4.3) we get

$$A^i = \pi q^i / F(0,n,E), \quad \Sigma q^i = F(0,n,E), \qquad (4.7)$$

and the theorem tells us that T_n is similar to T_o (=T). The general conclusion is that for any positive integer s we can push back the beginning of the n-cycle from T_s to T by choosing $p^i = 2^s q^i$ in the partition (4.5).

5. Examples

The only triangle which maintains its form under the pedal process is the equilateral triangle: here we have a monocycle (n=1), and ~~there~~ it seems that there is no more to be said. But Theorem I shows that we may start with a scalene triangle which, after a delay represented by s, settles down into a monocycle. Choose, for example, n=1, s=4, E=1, so that F = 16 × 3 = 48. We are now to take a partition of 48, but taking care to avoid PD. Thus 48 = 3 + 5 + 40, 3/48 = 1/16 and this is PD by (3.1). But 48 = 4 + 7 + 37 is not PD. To see how the sequence develops, we go back to (1.1) and (1.2), starting with the angles $4\pi/48$, $7\pi/48$, $37\pi/48$. The factor $\pi/48$ being understood, so that a right angle is represented by 24, and with obtuse angles marked by an asterisk, the calculation proceeds as follows:

4	7	37*
8	14	26*
16	28*	4
32*	8	8
16	16	16
16	16	16

(5.1)

The monocycle has been established in the fourth step.

We have verified Theorem I by numerical calculation for various values of s, n and E. In §7 we shall discuss the dodekacycle from which this work originated. As one further example here we shall take the heptacycle (n=7) without delay (s=0), but for both values of E.

For E=1, we have $F = 2^7 + 1 = 129 = 3 \times 43$, and we see from (3.1) that no partition can be PD. Let us see what happens to a triangle which is nearly equilateral, using the same notation as earlier, a right angle being represented by $64\frac{1}{2}$:

42	43	44	
45	43	41	
39	43	47	
51	43	35	(5.2)
27	43	59	
75*	43	11	
21	86*	22	
42	43	44	

The cycle is completed in seven steps.

For E = -1, we have $F = 2^7 - 1 = 127$, which is prime and no partition can give PD. With a right angle at $63\frac{1}{2}$, and a triangle nearly equilateral, we have

40	43	44	
47	41	39	
33	45	49	
61	37	29	(5.3)
5	53	69*	
10	106*	11	
20	85*	22	
40	43	44	

The cycle is completed in seven steps.

6. Pedal ancestry

Given a triangle T_o which is not PD, it generates a unique infinite pedal sequence, which may be cyclic or, in general, not so. A triangle T_{-1} of which T_o is the pedal may be called the antipedal of T_o, and so we are led to consider all triangles T_{-n} (with n positive) which include T_o in the pedal sequence generated by them. To use a biological term, these constitute the ancestry of T_o .

Consider the following relations between the angles A^i_n, A^i_{n+1} of the triangles T_n and T_{n+1}:

$$A^i_{n+1} = 2e_n A^i_n - \pi e_n d^i_n, \qquad e_n = \pm 1, \qquad d^i_n = 0 \text{ or } \pm 1. \qquad (6.1)$$

Although these formulae as they stand are highly ambiguous, they are in fact equivalent to the basic formulae (1.1) and (1.2). Let Σ denote summation with respect to i. Then

$$\Sigma A^i_n = \Sigma A^i_{n+1} = \pi, \qquad (6.2)$$

and so (6.1) implies

$$\Sigma d^i_n = 2 - e_n. \qquad (6.3)$$

If $e_n = -1$, then

$$d^i_n = (1,1,1), \qquad (6.4)$$

and (6.1) gives precisely (1.1); since $A^i_n = \frac{1}{2}\pi - A^i_{n+1}$, all angles of T_n are acute. But if $e_n = 1$, we can satisfy (6.3) only by choosing

$$d^i_n = (1,0,0) \qquad (6.5)$$

or some permutation of that; this gives by (6.1)

$$A^i_{n+1} = (2\,A^1_n - \pi,\ 2\,A^2_n.\ 2\,A^3_n)\ , \qquad (6.6)$$

which is the same as (1.2), and it is obvious that A^1_n is obtuse.

Thus, when allied to the angle-sum equations (6.2), (6.1) is equivalent to our basic equations, and has the advantage that we may read it the other way round to give antipedals:

$$A_n^i = \tfrac{1}{2}\, e_n\, A_{n+1}^i + \tfrac{1}{2}\, \pi\, d_n^i, \tag{6.7}$$

where

$$e_n = \pm 1, \qquad d_n^i = 0 \text{ or } \pm 1, \qquad \Sigma d_n^i = 2 - e_n. \tag{6.8}$$

But there is a big difference now: the ambiguities now refer to the angles we are seeking (those of T_n) and not to the given triangle T_{n+1}. There is in fact a fourfold ambiguity in the antipedal: any triangle has four parents, each step backward in the pedal sequence being determined only by our decision to make T_n acute-angled, or to have a specified angle obtuse.

Every person now living (male or female) had a unique ancestor ten thousand years ago if one goes back along a male line or a female line. Similarly every triangle has a unique line of ancestors if it proceeds through acute-angled triangles or through triangles in which some specified angle is obtuse.

To follow the acute ancestry, we are to choose $e_n = -1$, and $d_n^i = (1,1,1)$ so that (6.7) gives

$$A_n^i = \tfrac{1}{2}\,(\pi - A_{n+1}^i) \tag{6.9}$$

or

$$A_n^i - \pi/3 = -\tfrac{1}{2}\,(A_{n+1}^i - \pi/3)\,. \tag{6.10}$$

Thus the ultimate acute ancestor of any triangle (itself acute or obtuse) is an equilateral triangle, infinitely large since the circumradius is doubled in each step backward.

As for the obtuse ancestry, with A_n^1 obtuse, we are to use (6.7) with $e_n = 1$ and d_n^i as in (6.5). Thus at each step we halve the angles A_n^2 and A_n^3; the ultimate ancestor has angles $(\pi,0,0)$.

The fourfold ambiguity in the antipedal renders the backward extension of a pedal cycle complicated, since, to reproduce it, we would need to choose the proper parent at each step. Our knowledge of the distribution of obtuse angles in a cycle is so far purely experimental. But note that no cycle, except the 1-cycle, can consist of purely acute angles, for (6.10) shows that, as a sequence of acute triangles progresses, each acute angle differs from $\pi/3$ by twice as much as its predecessor. Obviously one angle will sooner or later become obtuse.

7. The likelihood of dodekacycles

The preceding results are the outcome of observing the existence of 12-cycles in numerical experiments with triangles whose angles were each an integral number of degrees. This was initially proved by noting that $2^2\ (2^{12}-1)$ is a multiple of 180, the index in 2^2 indicating a possible delay of 2 before the dodekacycle commences. A more complete statement follows.

THEOREM II: Consider a triangle T in which the angles are integral in degrees, but does not have an angle of 45^o, 90^o or 135^o. Then T generates a 12-cycle. If T has any odd angles, this cycle starts with the second pedal of T; if all the angles of T are even, but not divisible by 4, the cycle starts with the first pedal T_1; and if all the angles of T are divisible by 4, the cycle starts with T itself.

Proof: The exclusion of the specified angles implies, by (3.1), that the sequence is infinite. Since the angles are integral in degrees, in radians they are of the form

$$A^i = \pi\, q^i/180, \qquad (7.1)$$

where q^i are positive integers satisfying

$$\Sigma q^i = 180. \qquad (7.2)$$

Define positive integers by

$$p^i = 91\, q^i, \qquad (7.3)$$

so that

$$\Sigma p^i = 91 \times 180 = 4 \times 4095 = 4\ (2^{12} - 1) = F(2,12,-1) \qquad (7.4)$$

by (4.4). Thus we have

$$A^i = \pi p^i \ /\ (180 \times 91) = \pi\ p^i \ /\ F\ . \qquad (7.5)$$

These are of the form (4.3) with $s = 2$, $n = 12$, $E = -1$, and so by Theorem I they generate a 12-cycle. As for the possible delay in starting the 12-cycle, these are covered by the remarks at the end of §4. That completes the proof.

To illustrate the delays, here are examples, the angles being given in degrees:

61	63	56	62	64	54	64	68	48
58	54	68	56	52	72	52	44	84
64	72	44	68	76	36	76	92*	12
52	36	92*	44	28	108*	152*	4	24
104*	72	4	88	56	36	124*	8	48
28	144*	8	4	68	108*	68	16	96*
56	108*	16	8	136*	36	136*	32	12
112*	36	32	16	92*	72	92*	64	24
44	72	64	32	4	144*	4	128*	48
92*	36	52	64	8	108*	8	76	96*
4	72	104*	128*	16	36	16	152*	12
8	144*	28	76	32	72	32	124*	24
16	108*	56	28	116*	36	64	68	48
32	36	112*	56	52	72			
64	72	44						

In the first example the 12-cycle starts with T_2, in the second with T_1, and the third with T.

We have to thank the Babylonians for dividing the semicircle into 180 degrees, since otherwise pedal cycles might have remained undiscovered. But a closer examination shows that cycles would have been found had we taken angles integral in seconds of arc or integral in grades (100 grades = 90^o) or, more generally,

(in radians)

$$A^i = \pi q^i / M, \tag{7.6}$$

where q^i and M are positive integers. By Theorem I we know that, if T has such angles, and is not PD, it generates an n-cycle iff there exist positive integers p^i, s, n (with s = 0 permitted) to satisfy

$$q^i \, 2^s \, (2^n + E) = p^i \, M, \tag{7.7}$$

with E = 1 or -1.

A solution (not unique) can always be found by making use of the Euler function ϕ whose relevant properties (Dickson, 1939) are as follows: $\phi(m)$ is the number of positive integers, not exceeding the positive integer m, which are relatively prime to m: if p_1, p_2, ... p_k are the distinct prime factors of m then

$$\phi(m) = m\,(1 - p_1^{-1})\,(1 - p_2^{-1}) \ldots (1 - p_k^{-1}): \tag{7.8}$$

if a is prime to m,

$$a^{\phi(m)} \equiv 1 \pmod{m}. \tag{7.9}$$

THEOREM III: If a triangle T has angles

$$A^i = \pi q^i / M,$$

where q^i, M are positive integers, then T generates an n-cycle with $n = \frac{1}{2}\,\phi(M_1)$ where M_1 is the odd integer obtained from M by dividing out all powers of 2 contained in M.

Proof: Write

$$M = 2^t \, M_1 \tag{7.10}$$

where t is zero or a positive integer and M_1 is an odd integer. The form of $\phi(M_1)$ given in (7.8) shows that $\phi(M_1)$ is even and it may be shown that

$$2^{\phi(M_1)/2} = E_1 \pmod{M_1} \tag{7.11}$$

where E_1 = 1 or -1 and where M_1 is odd, $M_1 \geq 3$ (See Appendix A.) A solution of (7.7) may be obtained by taking

$$s = t, \quad E = -E_1, \quad n = \phi(M_1)/2 \tag{7.12}$$

$$p^i = \gamma q^i, \tag{7.13}$$

where γ is the positive integer determined by

$$2^{\phi(M_1)/2} - E_1 = \gamma M_1 , \tag{7.14}$$

and so the theorem is proved.

Let us apply the theorem to the case of a triangle with angles integral in degrees. If $M = 180 = 2^2 \times 45$, so that $s = 2$, $M_1 = 45$, $\phi(M_1) = 24$, Theorem III shows that we must get 12-cycles as already proved by Theorem II. For M = 180 Theorem III gives the "best" result in that triangles exist whose shortest pedal cycle is a 12-cycle.

What cycles are generated by T if its angles are integral in grades? With $M = 200 = 2^3 \times 25$, $s = 3$, $M_1 = 25$ and $\phi(M_1) = 20$, so that Theorem III indicates 10-cycles. Here are two experiments, first with a highly obtuse triangle and second with one which is nearly equilateral. The angles are measured in grades, so that the sum is 200 and the right angle 100.

1	2	197*
2	4	194*
4	8	188*
8	16	176*
16	32	152*
32	64	104*
64	128*	8
128*	56	16
56	112*	32
112*	24	64
24	48	128*
48	96	56
104*	8	88
8	16	176*

65	67	68
70	66	64
60	68	72
80	64	56
40	72	88
120*	56	24
40	112*	48
80	24	96
40	152*	8
80	104*	16
160*	8	32
120*	16	64
40	32	128*
80	64	56

In each case we get a delay of 3 steps (corresponding to s=3) and, as the shortest cycle in each sequence is a 10-cycle, Theorem III again gives the best possible result.

However this is not always the case and the estimate $\phi(M_1)/2$ for cycle length can be wildly out as the following example (Ribenboim, 1972) shows: when $M = M_1 = 1093^2$ (1093 is prime), $\phi(M_1)/2 = 1093.546 = 598,778$. But $2^{182} \equiv -1 \pmod{M_1}$, so that a solution of (7.7) may be found with n=182, and we may conclude that there exists a cycle of length 182, a considerable improvement on $\phi(M_1)/2$.

8. Spin and rotation in a pedal sequence

Having labelled the vertices of a triangle T with the letters ABC, the labelling of its pedal T' is defined by putting A'B'C' on the vertices of T' which lie on the sides opposite ABC. Thus a pedal sequence consists of labelled triangles in each of which the alphabetic passage round the triangle is either clockwise or the reverse; following the usual convention for rotations, we shall say that a triangle has spin +1 if the passage is counter-clockwise and -1 if it is clockwise, and we shall denote the spin by σ.

If T has spin σ, its pedal T' has spin σ if T is acute-angled and spin $-\sigma$ if T has an obtuse angle. This is seen at once from inspection.

THEOREM IV: Let T start an n-cycle so that the angles of T_n are the same as those of T. Let n_o be the number of obtuse angles in the set T to T_{n-1}. Let σ be the spin of T and σ_n that of T_n. Then $\sigma_n = \pm\sigma$ according as n_o is even or odd.

This follows immediately from the preceding statement. In the examples (5.2) and (5.3) and those of §7 we have n_o = 2,3,10,8,10,9,7; we have no theory to predict such values and they remain merely observed facts.

To bring out certain points, consider another example. Put $s=0$, $n=3$, $E=-1$ in (4.3), so that $F=7$, and p^i form a partition of 7, say 1,2,4. The angles of T are then $\pi/7$, $2\pi/7$, and $4\pi/7$, and, with the factor $\pi/7$ understood, the sequence runs

1	2	4*
2	4*	1
4*	1	2
1	2	4*

(8.1)

A tricycle is completed. The first point is that all four triangles are similar, but different, not only in size, but because the labelling of the several angles changes. The second point is that, although we see four asterisks, $n_o=3$ for one complete cycle because we do not include the last line of entries.

To deal with the third point, we need to clarify our terms. Two triangles are *congruent* if the three sides of one are equal to those of the other (Coxeter 1969, p.5). But two congruent triangles may have different orientations, so that, to superimpose them, we must either take one out of the plane and turn it over, or, equivalently, reflect it in a line. As a simple notation we suggest C_s to denote congruence with the *same* orientation and C_d to denote congruence with *different* orientations. The following theorem can be verified by a simple sketch:

THEOREM V: Let T and T' be two congruent triangles. Let the vertices of T be labelled A,B,C and those of T' labelled A',B',C' so that for the angles we have A=A', B=B', C=C'. Then T and T' are C_s if they have the same spin and C_d if their spins differ.

Since $n_o=3$ in (8.1), it follows from Theorem IV that T and T_3 (magnified by a factor 8) are C

(a) Rotation in pedal cycles

Starting from a triangle T which is not PD, we may derive from it a pedal sequence, infinite and in general not cyclic in respect of angles. There is no sense in speaking of the "rotation" of the triangles in such a sequence. But it is different if T starts an n-cycle. Then T_n is congruent to T if it is magnified by a factor 2^n with respect to any point. This magnification does not change the directions of the sides of T_n, and so, if T_n has the same spin as T, we can transform T into the magnified T_n by a translation and rotation, the rotation being independent of centre of magnification. Indeed the magnification is unnecessary if we are to consider only rotation.

To study this rotation, we start with the following theorem:

THEOREM VI: Let T be a triangle with vertices A,B,C and T' its pedal with vertices A',B',C'. Let R(BC), R(CA), R(AB) be the angles through which the directed sides BC, CA, AB must be turned to make them coincide with the directed sides B'C', C'A', A'B'. Then

$$R(BC) + R(CA) + R(AB) = \begin{cases} \pi & \text{(mod } 2\pi\text{) if T is acute-angled,} \\ 0 & \text{(mod } 2\pi\text{) if T has an obtuse angle.} \end{cases} \qquad (8.2)$$

Proof: Let T be acute-angled. If the pedal T' is removed, we are left with three triangles (Fig.1) each congruent to T but with different orientation. We

Fig. 1 here

change the direction of BC to that of A'B' by rotation through the angle A, and A'B' to B'C' by rotation through an angle 2B, so that R(BC) = A +2B. The other rotations are given by cyclic permutation, and so we get the first part of (8.2). This result is easy to remember if we take T equilateral, in which case each of the three rotations is obviously π.

The second part of (8.2) may also be proved from a figure, but it is complicated. An algebraic proof is given in §9, and the result will be assumed here. Again it is easy to remember the result from a particular case, namely that in which T has angles $2\pi/3$, $\pi/6$, $\pi/6$; the pedal T' is then equilateral.

Consider now an n-cycle which starts with T and is completed with T_n, a triangle similar to T. When magnified by a factor 2^n, T_n is congruent to T, but they may be either C_s or C_d. Let n_o as before be the number of obtuse angles in the set T to T_{n-1} (or equivalently T_1 to T_n) and n_a the number of acute-angled triangles in that set, so that

$$n_o + n_a = n. \qquad (8.3)$$

We have to consider four cases:

	n_o	n_a	n
(i)	even	even	even
(ii)	even	odd	odd
(iii)	odd	even	odd
(iv)	odd	odd	even

(8.4)

Case (i)

Theorem IV tells us that spin is conserved ($\sigma_n = \sigma$); by Theorem V then T and T_n (magnified) are C_s; the rotations of the three sides are therefore the same, say θ_n, and by (8.2), since n_a is even, we have

$$\theta_n = 2\pi r/3, \qquad r = 0, +1, \ldots \qquad (8.5)$$

On completion of the cycle, the triangle has rotated through one of the angles

$$\theta_n = 0,\ 2\pi/3,\ -2\pi/3 \qquad (\text{all mod } 2\pi)\ . \qquad (8.6)$$

Case (ii)

Again T and T_n (magnified) are C_s, so that all three sides have the same rotation θ_n, which by (8.2) is

$$\theta_n = \frac{1}{3}(n_a\pi + 2\pi r), \quad r = 0, \pm 1, \ldots$$
$$= (2s+1)\pi/3, \quad s = 0, \pm 1, \ldots \qquad (8.7)$$

so that on completion of the cycle the triangle has rotated through one of the angles

$$\theta_n = \pi/3, \quad \pi, \quad -\pi/3 \quad (\text{all mod } 2\pi). \qquad (8.8)$$

Cases (iii) and (iv)

With n_o odd, T and T_n (magnified) are C_d, and the sum of the rotations of the sides as in (8.2) has no simple meaning. We shall not pursue these cases further.

(b) Repeated cycles

If we repeat an n-cycle, we get a 2n-cycle, starting with T and ending with T_{2n}. Since this 2n-cycle contains an even number of triangles of each kind, Case (i) alone applies, and by (8.6) T_{2n} is rotated relative to T through one of the angles

$$\theta_{2n} = 0, \ 2\pi/3, \ -2\pi/3 \quad (\text{all mod } 2\pi). \qquad (8.9)$$

If we repeat again, we get a 3n-cycle, starting with T and ending with T_{3n}. Since multiplication by 3 does not alter parity, all cases in (8.4) apply, and we get an interesting result. In Case (i) the rotation is

$$\theta_{3n} = 0 \pmod{2\pi}, \qquad (8.10)$$

so that the sides of T_{3n} are parallel to those of T, while in Case (ii) the rotation is

$$\theta_{3n} = \pi \pmod{2\pi}, \qquad (8.11)$$

so that the sides are parallel but reversed in direction.

If we repeat the 3n-cycle, obtaining a 6n-cycle from T to T_{6n}, again only Case (i) applies, and we get the following theorem:

THEOREM VII: If T starts a sequence of six n-cycles completed by T_{6n}, those two triangles are similarly oriented (C_s) and, if T_{6n} is magnified by a factor 2^{6n} with respect to any point, the resulting triangle may be obtained from T by giving T a translation.

This translation will of course depend on our choice of centre of magnification, such as perhaps the centroid of T. But we have not been able to find the translation for this or any other choice.

9. Algebraic methods

When we pin logical reasoning to a diagram, we take the risk that the diagram fails to cover all cases to which logic applies. This is particularly true for pedal sequences in which we have to deal with triangles of both types - acute-angled and obtuse-angled. These require different diagrams. Further, when diagrams become complicated, reasoning based on them is sometimes hard to follow. Algebraic methods have the advantage that they cover all cases; they have of course the disadvantage that they tend to bring us out of contact with rapid visual intuitions.

Define a triangle T by a triad (X,Y,Z), these being either the position vectors of the vertices of T relative to some arbitrarily chosen origin or complex numbers in an Argand diagram with arbitrarily chosen axes. Define the directed sides by

$$x = Y - Z, \quad y = Z - X, \quad z = X - Y, \tag{9.1}$$

so that

$$x + y + z = 0. \tag{9.2}$$

If T' is the pedal of T, it is easy to show that its vertices are

$$\begin{aligned} X' &= \tfrac{1}{2}(Y + Z + ux), \\ Y' &= \tfrac{1}{2}(Z + X + vy), \\ Z' &= \tfrac{1}{2}(X + Y + wz), \end{aligned} \tag{9.3}$$

where

$$u = (b^2 - c^2) / a^2 = \sin(B-C) / \sin A, \qquad (9.4)$$

v and w being obtained by cyclic permutation; (a,b,c) and (A,B,C) are the sides and angles of T; (u,v,w) satisfy two equations:

$$ua^2 + vb^2 + wc^2 = 0, \qquad (9.5)$$

and

$$u + v + w + uvw = 0, \qquad (9.6)$$

or equivalently

$$\Pi(1+u) = \Pi(1-u), \qquad (9.7)$$

these being cyclic products; (9.5) follows from the concurrence of three prependiculars to the sides of T, and (9.6) from the concurrence of three lines drawn from the vertices to the opposite sides of T (Ceva's theorem).

We might write (9.3) in matrix form, the elements of the matrix involving only (u,v,w), and proceed to follow a pedal sequence by multiplying matrices. Since the iteration of (9.3) tells the whole story of a pedal sequence, this would be the obvious plan if the goal were to find the pedal sequence arithmetically, starting from some given triangle. But the numbers (u,v,w) would have to be revised at each step and their structure (9.4) is such that such a method is not likely to reveal general properties of mathematical interest.

By subtraction (9.3) gives formulae for the transformation of the directed sides

$$x' = \tfrac{1}{2}(-x + vy - wz), \text{ etc.} \qquad (9.8)$$

By (9.2) these may be written

$$\begin{aligned} x' &= \tfrac{1}{2} \quad (1+v)y + (1-w)z \;, \\ y' &= \tfrac{1}{2} \quad (1+w)z + (1-u)x \;, \qquad (9.9) \\ z' &= \tfrac{1}{2} \quad (1+u)x + (1-v)y \;. \end{aligned}$$

Iteration of these equations would give the whole history of a pedal sequence to within a translation.

We shall now use these formulae to fill in the proof of Theorem VI, treating x,y,z as complex numbers. Writing P for the products in (9.7), we get

$$8x'y'z' = 2xyzP + \Sigma(1-u^2)(1+v)x^2y + \Sigma(1-u^2)(1-w)x^2z, \qquad (9.10)$$

Σ denoting a cyclic sum. By (9.4)

$$1 + u = 2 \sin B \cos C / \sin A, \qquad 1 - u = 2 \cos B \sin C / \sin A,$$

$$P = 8 \cos A \cos B \cos C, \qquad (9.11)$$

$$(1 - u^2)(1 + v) = P \sin^2 C / \sin^2 A,$$

$$(1 - u^2)(1 - w) = P \sin^2 B / \sin^2 A.$$

Thus (9.10) gives

$$8x'y'z'/(xyz) = P(2 + \Sigma(x/z)\sin^2 C/\sin^2 A + \Sigma(x/y)\sin^2 B/\sin^2 A). \qquad (9.12)$$

If we now substitute for x from (9.2), the two summations together give

$$- \Sigma(b^2 + c^2)/a^2 - \Sigma(y/z)c^2/a^2 - \Sigma(z/y)b^2/a^2. \qquad (9.13)$$

Now it is easy to see that

$$y/z = - (b/c)\, e^{i\sigma A}, \qquad (9.14)$$

where σ is the spin of T, and (9.13) reduces to -3. Hence

$$x'y'z'/(xyz) = - P/8 = - \cos A \cos B \cos C. \qquad (9.15)$$

The essential point here is that the quotient of two complex numbers comes out real, the argument being independent of acuteness or obtuseness. Since $\arg(x'/x)$ is the angle of rotation from x to x', we see from (9.15) that the sum of the rotations from x to x', from y to y' and from z to z' is π (mod 2π) when T is acute-angled and 0 mod (2π) when T is obtuse-angled.

We shall now use the algebraic method, with vectors rather than complex numbers, to establish a result needed later. This result is indeed well known, but the usual proof uses a diagram, or rather two, to take care of both acute- and obtuse-angled triangles.

Let H, K and G be respectively the orthocentre, circumcentre and centroid of T, so that Euler's equation reads

$$H + 2K = 3G. \qquad (9.16)$$

Magnify T' by a factor 2 with H as centre; this generates points X", Y", Z" such that

$$X'' = 2X' - H, \text{ etc.} \tag{9.17}$$

Take the origin at the circumcentre of T so that K = 0; then (9.16) gives

$$H = 3G = X + Y + Z. \tag{9.18}$$

By (9.3) we have from (9.17)

$$\begin{aligned} X'' &= -X + ux, \\ X''^2 &= X^2 - 2uXx + u^2x^2, \text{ etc.}, \end{aligned} \tag{9.19}$$

scalar products being written without dots. Now H-X' is perpendicular to x, and so

$$(H - X')x = 0, \tag{9.20}$$

or by (9.18) and (9.3)

$$(Y + Z)x + (2X - ux)x = 0. \tag{9.21}$$

But $\quad (Y + Z)x = (Y + Z)(Y - Z) = Y^2 - Z^2 = 0, \qquad (9.22)$

since X,Y,Z lie on the circumcircle of T. Hence by (9.19)

$$X''^2 = Y''^2 = Z''^2 = R^2, \tag{9.23}$$

where R is the circumradius of T. This is the desired result: the points X",Y",Z" lie on the circumcircle of T.

10. Representation on a circle

A triangle T is completely specified if we know (i) its circumcentre K, (ii) its circumradius R and (iii) the positions of its vertices A,B,C on the circumcircle. Let T' be the pedal of T. Then, from the result at the end of the preceding section, magnification by a factor 2 with centre at H, the orthocentre of T, carries the vertices of T' onto the circumcircle of T, giving us three points which describe T' to within a translation. Proceeding in this way, we may represent all members of a pedal sequence by triads of points on a single circle, the circumcircle of the triangle from which the sequence starts.

To illustrate this, let us put s=0, n=4, E=-1 in (4.4), so that F=15, and start with the partition (1,3,11). Then, with a factor $\pi/15$ understood, the tetracycle is as follows:

1	3	11*	
2	6	7	
11*	3	1	(10.1)
7	6	2	
1	3	11*	

Since n_o and n_a are both even, Case (i) of (8.4) applies, so that spin is conserved, and if we repeat the cycle twice to obtain T_{12}, we get by (8.5) a total rotation 0 (mod 2π), so that T_{12} is similar and parallel to T, its sides reduced by a factor 2^{-12}, so that it is extremely small.

But if we adopt the magnification process described above, using as centre of magnification for each triangle the orthocentre of its predecessor, and denote by $\bar{T}_r$ the magnified version of T_r, then all vertices lie on the circumcircle of T, as shown in Fig. 2, which goes as far as $\bar{T}_6$. This triangle is identical with T except for the exchange in position of two vertices: A_6=C, C_6=A, these letters referring to the vertices of the magnified triangles. Evidently another 6 steps will produce $\bar{T}_{12}$ in exactly the same position as T, with A_{12}=A, B_{12}=B, C_{12}=C. This is entirely consistent with Theorem VII which predicts that six 4-cycles produce a triangle which when magnified by 2^{24} will be a pure translation of the original triangle. In the present representation this means that $\bar{T}_{24}$ is in exactly the same position as T, which is verified for this example.

Fig. 2 here

Representation on a circle permits us to write down, in a single form irrespective of acuteness or obtuseness, formulae for an angular step in the pedal process, including rotation of sides. From a triangle T with vertices (X,Y,Z), we generate, as in (9.19), the magnified pedal T" with vertices (X",Y",Z") where

$$X'' = -X + u(Y-Z), \qquad u = (b^2-c^2)/a^2, \text{ etc.} \qquad (10.2)$$

Although derived in a vector context, we can now use complex variables, since the equations are linear. Write

$$X = Re^{i\xi}, \qquad Y = Re^{i\eta}, \qquad Z = Re^{i\zeta}, \qquad (10.3)$$

with double primes inserted for X", Y", Z", so that (10.2) gives

$$e^{i\xi''} = -e^{i\xi} + u(e^{i\eta} - e^{i\zeta}). \qquad (10.4)$$

Now

$$\begin{aligned} a^2 &= R^2(e^{i\eta} - e^{i\zeta})(e^{-i\eta} - e^{i\zeta}) \\ &= 2R^2(1 - \cos(\eta-\zeta)), \end{aligned} \qquad (10.5)$$

and so

$$u = \frac{\cos(\xi-\eta) - \cos(\xi-\zeta)}{1 - \cos(\eta - \zeta)}, \qquad (10.6)$$

with v and w given by cyclic permutation. When these are substituted into (10.4) and its cyclic companions, we have formulae for ξ'',η'',ζ'', and hence for X",Y",Z", the vertices of the magnified pedal. From these the angles of that triangle are easily obtained. The advantage of this approach lies in the fact that we avoid reference to the fact that the sum of the angles of a triangle is π; ξ,η,ζ are completely arbitrary, except for the avoidance of pedal degeneracy.

11. Conclusion

We have discovered two facts: the negative fact that Hobson was in error and the positive fact that pedal cycles exist. But we have been unable to locate the pedal point, by which we mean that point which is the limit of an infinite pedal sequence, cyclic or not. Another unsolved problem is to locate the in-point, which is the limit of a sequence in which the vertices of T_{n+1} are the points of contact of the sides of T_n with its inscribed circle. There are no cycles in this sequence since the angles tend steadily to $\pi/3$.

One of us (J.L.S.) thanks a colleague, Professor J. T. Lewis, for useful discussions.

Appendix A

Theorem: Let n and a be positive integers greater than unity, relatively prime and with n odd. Then

$$a^{\frac{1}{2}\phi(n)} \equiv E(a,n) \pmod n \qquad (A.1)$$

where E(a,n) is 1 or -1 according to the following rules:

(i) If n is a prime p, then E(a,p) = 1 iff the quadratic congruence

$$x^2 \equiv a \pmod p \qquad (A.2)$$

has a solution.

(ii) If n is a power of a prime p, then E(a,n) = E(a,p).

(iii) If n is divisible by at least two different primes, then E(a,n) = 1.

(iv) If a = 2 and n is a prime p, then

$$\begin{aligned} E(2,p) &= 1 \quad \text{if } p \equiv \pm 1 \pmod 8 \\ &= -1 \quad \text{if } p \equiv \pm 3 \pmod 8). \end{aligned} \qquad (A.3)$$

Proof:

(i) Here $\phi(n) = p-1$ and

$$a^{\frac{1}{2}\phi(n)} = a^{\frac{1}{2}(p-1)} = 1 \text{ or } -1 \qquad (A.4)$$

according as the congruence (A.2) has, or has not, a solution (Vinogradov 1954). This determines $E(a,p)$.

(ii) Here $n = p^m$, and we know that the result is true for $m = 1$. We proceed by induction using

$$\phi(p^{m+1}) = p\,\phi(p^m), \qquad (A.5)$$

and assuming that

$$a^{\frac{1}{2}\phi(p^m)} = E_m + k_m p^m, \qquad (A.6)$$

where $E_m = \pm 1$ and k_m is an integer. Raising to the power p, we get

$$a^{\frac{1}{2}\phi(p^{m+1})} \equiv E_{m+1} \pmod{p^{m+1}}, \qquad (A.7)$$

where

$$E_{m+1} = E_m^{\,p}. \qquad (A.8)$$

Since p is odd, all the E's have the same value, viz. $E(a,p)$, and so (ii) is proved.

(iii) Now $n = n_1 n_2 \ldots n_r$ where these factors are powers of different primes and r exceeds 1, this being in fact the factorisation of any odd composite number. Since these factors are relatively prime we have

$$\phi(n) = \phi(n_1)\,\phi(n_2)\,\ldots\,\phi(n_r) \qquad (A.9)$$

and so

$$a^{\frac{1}{2}\phi(n)} = [a^{\frac{1}{2}\phi(n_1)}]^{N_1}, \quad N_1 = \phi(n_2)\,\phi(n_3)\,\ldots\,\phi(n_r). \qquad (A.10)$$

Thus by (ii)

$$a^{\frac{1}{2}\phi(n_1)} = E + kn_1\,, \quad E = \pm 1. \quad k = \text{integer}. \qquad (A.11)$$

Since the n's are odd, the ϕ's are even; thus N_1 is even, and the ambiguity in E disappears when we raise to the power N_1, obtaining

$$a^{\frac{1}{2}\phi(n)} = 1 + c_1 n_1, \quad c_1 = \text{integer}. \qquad (A.12)$$

Similarly

$$a^{\frac{1}{2}\phi(n)} = 1 + c_2 n_2, \quad \text{etc.}$$

and so there exist integers c_1, c_2, ..., c_r such that

$$c_1 n_1 = c_2 n_2 = \ldots = c_r n_r. \qquad (A.13)$$

Hence c_1 contains n_2, n_3, ... n_r as factors, and so

$$a^{\frac{1}{2}\phi(n)} \equiv 1 \pmod{n}; \qquad (A.14)$$

This establishes (iii).

(iv) Now a = 2, and n is a prime p, so that this is a special case of (i). Thus it is a question of seeing whether the congruence

$$x^2 \equiv 2 \pmod{p} \qquad (A.15)$$

has solutions. But we can do it otherwise by equating two different expressions for the Legendre symbol (Vinogradov 1954, pp. 81, 85);

$$\left(\frac{2}{p}\right) \equiv 2^{\frac{1}{2}(p-1)} \pmod{p} \equiv (-1)^{(p^2-1)/8}. \qquad (A.16)$$

Now any odd prime p may be written in one of the forms $8m\pm1$, $8m\pm3$, with m an integer; $(p^2-1)/8$ is even in the first case and odd in the second. Thus (iv) is established.

As for E(2,n), the results (ii) and (iii) are available:

$$E(2,p^m) = E(2,p), \quad E(2,n) = 1, \qquad (A.17)$$

when n is divisible by at least two different primes.

References

Coxeter, H. S. M. 1969 Introduction to Geometry (second edition), pp. 5, 18. New York: John Wiley and Sons.

Dickson, L. E. 1939 Modern Elementary Theory of Numbers pp. 9-12, Chicago: The University Press.

Hobson, E. W. 1897 A Treatise on Plane Trigonometry (second edition), pp. 194-200. Cambridge: University Press.

Hobson, E. W. 1925 A Treatise on Plane Trigonometry (sixth edition), pp. 197-203. Cambridge: University Press.

Ribenboim, P. 1972 Algebraic Numbers, p. 64. New York: Wiley-Interscience.

Vinogradov, I. M. 1954 Elements of Number Theory, pp. 80, 81, 85. U.S.A. Dover Publications Inc.

Fig 2 - Duplicate copy - labelled

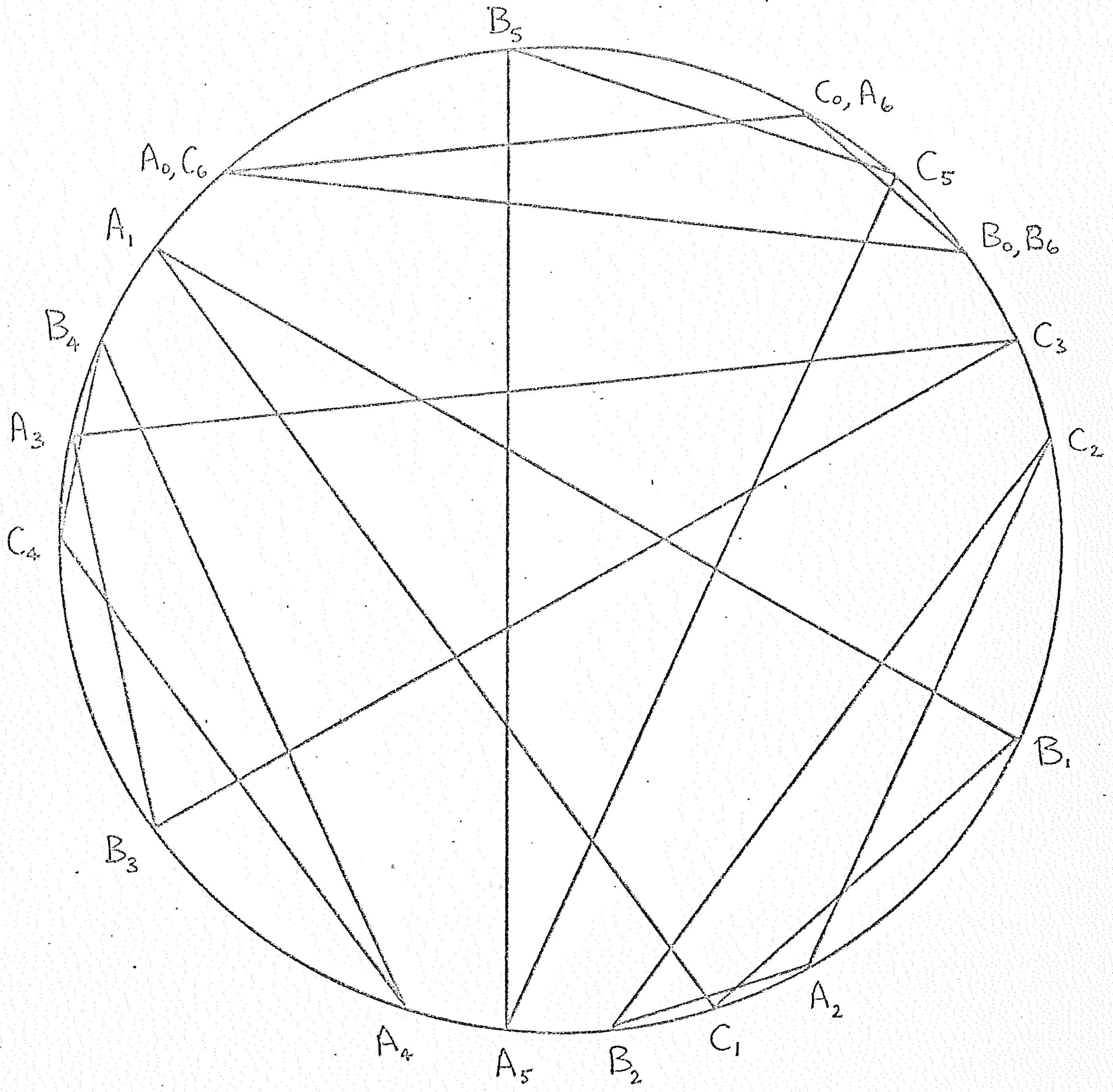

Fig. 2

Fig 1 : Duplicate copy – labelled.

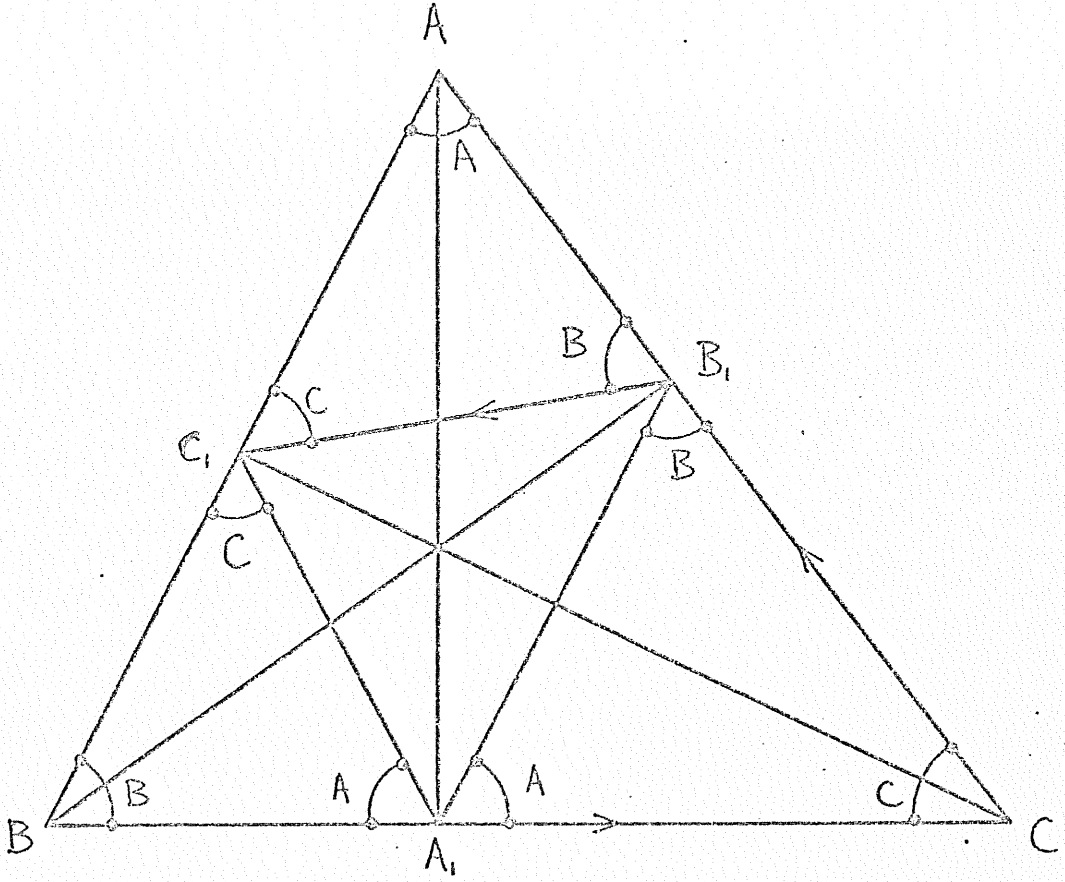

Fig. 1