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STOCHASTIC PROCESSES IN ASTROPHYSICS: STELLAR FORMATION AND GALACTIC EVOLUTION

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CONTENTS

1 - INTRODUCTION

The application of stochastic methods to astrophysical problems has a long and colorful history. Perhaps the first area of investigation, in analogy with the successes of nineteenth century statistical mechanics, was that of stellar dynamics. The sidereal universe was treated as a gas of massive bodies in a phase space whose natural coordinates referred to the galactic plane and center. This treatment, initiated by Schwarzschild (1) and Eddington (2) was capped with the review by Chandrasekhar(3). It included the elucidation of the velocity distribution function, the variation of the velocity dispersion with galactocentric position and age (later in part explained by Spitzer and Schwarzschild (4)) and the discovery of the rich field of statistical stellar dynamics (see e.g. reviews by Kurth (5), Mihalas and Routly (6), Oort (7), Mihalas and Binney (8) and references therein). At about the same time, the methods were applied to stochastic line broadening in atomic systems and eventually to stellar atmospheres (see Mihalas (9), Griem (10), Chandrasekhar (3)). The application of such methods to stable fluctuations in the brightness of the Milky Way and the statistical mechanics of gravitational encounters between stars has also been one of great productivity (Barucha -Reid (11), Chandrasekhar and Munch (12)). Turbulence theory has been applied to dynamo models (see review by Parker (13)), and to propagation of cosmic rays through the galaxy by diffusive motion in both energy and spacetime (see Ginzburg and Ptuskin (14) for review).

Since these areas have been covered extensively in the literature (15) we shall not add to the already groaning mass of tomes on the subject with this survey. Instead, we shall concentrate on those processes which have in the past decade been brought to bear on the problem of galactic evolution, star formation, and global problems of the large scale distribution of the galactic population of the universe. Most of these methods fall into categories of broad generality and can in effect be labeled by the methods employed in their investigation rather than the area being studied. We shall therefore proceed with this review by separating the problems according to the method rather than the topic being investigated. This is done for several reasons. The field of stochastic phenomena in astrophysics in particular and physical science in general has taken on the appearance of a growth industry since the introduction of computer methods about 15 years ago. The time for diffusion of techniques between fields has been decreasing, and still there is some lack of communication between theorists in different disciplines concerning the similarity of approaches. Since, for example, population ecology has spawned many of the nonlinear methods used in the modelling of the chemical history of the galaxy, and since percolation techniques can be used for any form of lattice-dominated phase transition from QCD to galactic structure, we feel that this separation by method will assist the reader by allowing for easy comparison between techniques and setting in the different areas of astrophysical investigation. Having presented our philosophical justification, then, we should now present our basic categories of methods:

Coagulation phenomena: agglomeration and fragmentation calculations which bear direct kinship to the Fokker-Planck treatments but also include discussion of expectation of N-body systems for which the initial distribution function and dynamics cannot or are not treated in the continuum limit.

Percolation: application of local interactions to the problem of generating long range order, including phase transitions and morphogenesis, to large scale discrete models.

Langevin systems: nonlinear deterministic and stochastic representations of interacting states or populations with and without consideration of spatial variance.

Fokker-Planck equations: fully stochastic realizations of the Langevin systems and the Monte-Carlo simulations to which they give rise .

2 - AGGLOMERATION PHENOMENA (coagulation equation applications)

One recent development in astrophysical stochastic processes has been the widespread use of coagulation calculations for both nucleation phenomena and dust formation and processes that relate to the distribution function for masses and mass ratios in forming stellar systems. The use of the coagulation equation for the study of star and stellar system formation in particular has been quite recent and warrents a review.

(a) SOLAR SYSTEM FORMATION.

Perhaps the first studies to employ agglomeration were those related to the formation of the solar system. The use of the coagulation equation, essentially a macroscopic version of the master equation for systems which can be treated as being controlled by one independent variable and time, has been of some importance in recent simulations of the process of star formation. Employed for some time in the study of nucleation (16), the results were first used extensively by Safronov and his collaborators (17) in the modelling of planetary formation in the solar nebula.

Various stages can be identified by the main physical process acting in the evolution of low-mass protosolar nebula models.

In the initial phase a disk was formed by the settling to the central plane of the dust grains. The original disk structure can be specified by the application of some rather basic hydrodynamical constraints. Generally it is assumed that the disk is dominated by turbulent heating and is not self-gravitating (Shakura and Sunyaev (18), Pringle(19)). The grains can grow up to 1 cm, due to the condensation they undergo in the cooling nebula. The turbulence can prevent the disk from becoming dense enough to suffer the gravitational instability which could fragment it in higher mass pieces (Weidenschilling (20)). The mechanism that allows the growth of the grains up to dimensions of the order of one meter or more, is the coagulation.

The grains [illegible] will proceed to collide with eachother and then both fragment and stick (Safronov (17), Pechernikova et al (21), Nakagawa et al (22), Morfill (23)). The growth of the individual grains, or fragments, will then Smoluchowski equation of the form :

$$\frac{\partial N(\mu,t)}{\partial t}=\frac{1}{2}\int_0^{\mu}\alpha(\mu,\mu')N(\mu',t)N(\mu-\mu',t)d\mu'-\int_0^{\mu}N(\mu)N(\mu')\alpha(\mu,\mu')d\mu' \qquad (2\text{-}1)$$

where $\alpha(\mu,\mu')$ is the sticking probability, μ is the mass of the particle and $N(\mu,t)$ is the number of particles of mass μ at time t. In the case of particles in the solar nebula, there are several arguments which have been aduced to supply some phenomenological form for α . These are outlined by Nakagawa et al (22), Safronov and Ruzmaikina (24) and Wetherill (25). Simple simulations can be carried out with such systems, assuming that the collision rules are well specified, and the results show that the formation of large scale agglomerated bodies can proceed quite easily. The approximation which is made in the treatment by Nakagawa et al (22) and Morfill(23) is that of a continuous medium, in which case the equation for the evolution of the size distribution is a diffusion equation. Such an equation, the limit of the full master equation for the system, can be solved numerically for the distribution of surface density (not particle sizes explicitly) as a function of heliocentric distance. A mass spectrum can be calculated as a function of distance by introducing the agglomeration conditions explicitly and then solving for the variation of an initial particle spectrum. The evolution of that distribution can then be followed as a function of time.

Within this new higher density medium, it is possible for gravitational clumping to nonetheless occur (Safronov (17), Goldreich and Ward (26), Coradini et al.(27))) and for bodies of sizes of the order of 1 Km to form.

The subsequent evolution of this swarm of planetesimals is governed by their relative gravitational interactions and collisions. With the exception of a thermodynamical model based on phase transitions (Farinella et al (28)), this problem has been treated numerically (29); the approach is to follow explicit ly the evolution of the system of planetesimals as an N-body problem. The individual particles are labelled with the appropriate orbital parameters and the equations of motion for a large number of planetesimals are then integrated The distribution of the particles initially simulates the structure of the protosolar disk, and the subsequent evolution of the protoplanets can be followed allowing for the same kind of collisional dynamics first assumed by Dole (30) and Isaacman and Sagan (31) in their models for synthetic solar systems.

When a fragment grows up to 1000 Km, it can capture the residual gas in the nebula and this is the main process of growth together with the rare collisions with residual planetesimals.

The asteroid belt, which long suffered for lack of interest, has provoked for its aspect of planetesimal system a series of theoretical works. Its dynamical structure is intriguing for the presence of chaotic regions in the distribution of asteroids in phase space (32), revealing the intrinsic non-deterministic nature of

N-body non-integrable gravitational problem, and the possibility of testing Kolmogorov-Arnold-Moser theory (33). The collisional evolution, as determined by several dynamical models (34), based on the coagulation equation and other statistical treatments, like Monte Carlo calculations, affects the physical properties of these objects. Detailed observations, like photometric lightcurves, can test the prediction of the theory.

(b) STAR FORMATION - INITIAL MASS FUNCTION.

The basic data for stochastic simulations of galaxies and their constituent populations and metallicity evolution is the initial mass function (IMF), which represents the mass distribution with which stars are presumed to form. Derived from the observed distribution of luminosity among field stars (van Rhijn (35), Philip and Upgren (36) and references therein), and from clusters stars (Salpeter (37), Miller and Scalo (38)), its derivation involves many detailed corrections for both stellar evolution and abundance variations among the observed population. The methods for achieving the IMF from the observed distribution are most thoroughly outlined by Miller and Scalo (38), but can be stated briefly, since they also relate to an accurate testing of various proposed stochastic methods.

It should be noted that the first problems encountered for stellar distributions are quite similar to those with which studies of galaxies and their intrinsic properties have to deal.

Given the observed distribution of stellar masses among both cluster and field stars, the problem is to correct for evolutionary and metallicity effects. The actual mass is rarely measured; instead, one observes the bolometric brightness and surface temperature or color. The first correction which must then be made is to determine the age of the star, and correct for the variation in these parameters with time. If the star is in a cluster, then it is possible to at least get a handle on the appropriate age for all the members and to use theoretical interiors models to trace all of the stars back in time and surface parameters to the stage of hydrogen core burning, the so-called zero-age main sequence (ZAMS). The population of stars along this part of the evolutionary history of the cluster is presumably the population with which the cluster formed. The same is true for field stars if the relative populations of the various main sequence groups (which are ordered by temperature or spectral type) can be determined from a complete survey (that is complete down to some limiting apparent brightness with appropriate statistical corrections for the fainter inaccessible part of the sample). The rate of star formation can then be assumed, and the IMF obtained from modelling the population statistics.

When this is done, some parametric form for the mass spectrum has to be assumed. The initial approximation, that of a power law, is referred to as Salpeter mass function, following Salpeter (37). This approximation, of course, cannot apply over the entire range of possible masses, since the lower masses produce divergence in the total population. It is usual to specify three parameters: the upper and lower mass cutoffs and the exponent. While not useful in a fundamental way for explicating the origin of the mass spectrum, it is a convenient parametrization for models of star formation and the populations of external galaxies.

If we imagine that the stellar population has been formed continuously over a period of time, but that the distribution function for the formation of new stars is stable (in a stochastic sense) then the IMF should be reflected in the more evolved members of the sample population. That is, the mean age of stars should be younger for the most massive, and the relative number of massive to low mass stars should be stable. The primary complication to this would be, and in fact is, that the stars of the upper main sequence do loose a considerable amount of mass during post-hydrogen core exhaustion evolution, and therefore can re-populate the lower mass tail of the IMF. However this phenomenon is not really under control in the theoretical work of tracing back the sample to the ZAMS, under the constraint of a history of star formation, since the algorithms for the computation of the evolution of these massive, stars high luminosity, dominated by mass loss, are not well understood at present (see Chiosi (39), Abbott (40), Shore and Sanduleak (41)). For the lower mass stars, specifically for those less than 10 $M_{\odot}$, the situation is a good deal better. These stars evolve essentially conservatively until the extreme red giant stage, which represent only a few stars in any sample and therefore can be ignored.

The average star encountered in the field will be evolved and therefore corrections for the expansion of the envelope and consequent increase in the luminosity must be applied, as well as changes in the metallicity (gradient over both space and time) must also be applied to the lower mass stars.

It is still necessary to make assumptions, many of them ad hoc, concerning the time dependence of the star formation rate (see Section 4 for the Langevin nonlinear treatment which allows to circumvent this problem as well), and then evolve a chosen population forward in time to obtain the final IMF.

While the ideal way of proceeding would appear to be a fully stochastic simulation of the population, including the effects of mass loss and rotation, all of which are chosen from distribution functions which are known at present, this has only been done for a few cluster models by Elmegreen and Mathieu (42). It would be most useful to apply this to the field population in general. In addition, models are currently being studied which allow for the formation of stars of different masses by using different reaction channels in the Langevin systems of the following Sections in order to see whether the IMF is a stable stochastic function of time. If it changes, the star formation rate, the metallicity evolution of the disk and the IMF variations become inexorably linked and impossible to separate (Shore and Duncan (43)).

Broadly speaking, the treatment of fragmentation of molecular clouds falls into two categories in its astrophysical guises: analytic approximations to the coagulation equation and numerical simulations of accumulations among fragments or of fragmentation during collisions or collapse of clouds. We shall review these together, since the analytic treatment often preceeds the introduction of numerical methods in a large variety of different contexts. However, since we are dealing in this review primarily with problems related to star formation and galactic evolution, we shall ignore the work that has been done on dust formation and nucleation of classical (chemical) systems. These have been extensively reviewed by Abraham (16), Burton

(16), Draine and Salpeter (16) for problems of astrophysical interest. We shall only refer to this literature for analogies which may be of some aid in establishing new directions for work on megascopic systems like interstellar clouds.

The idea that fragmentation is the source for stars in clouds was first expressed most comprehensively by Hoyle (44), who argued that the observed distribution of galactic masses, and the constituent stellar masses, can be represented as the result of a hierarchical fragmentation. The dominant mass in this case is the Jeans mass, the size of a self-gravitating blob which will be critically stable against collapse if it remains isothermal. This mass given by:

$$M_J = 1.86 \left[\frac{R}{G}\right]^{1/2} \frac{T^{1/2}}{\rho^{1/2}} \qquad (2\text{-}2)$$

where R is the perfect gas constant, G the gravitational constant, T is the temperature of the cloud and ρ is the mean density. Hoyle's consideration was elaborated on by Hunter (45) who followed the collapse of a sphere through the critical stage, and by Mestel (46) who argued that magnetic fields can play a critical role in supporting such a self-gravitating system against collapse. Indeed, if there is any charge present in the cloud at all, the magnetic field will actually dominate the stability of the system (Parker (13)). The scale of the critical mass is then the most probable mass to form at any stage in the collapse of a cloud (Larson (47), Palla et al (48)). Armed with this scale, it is possible to compute whether observed clouds in the intestellar medium are stable against fragmentation. The simple answer beingthat the average molecular cloud contains many thousands of Jeans masses and so should be unstable to spontaneous star formation. The support of such systems has alternatively been ascribed to magnetic fields (Mouschovias (49)) and turbulence (Fleck (50), Ferrini et al (50)).

It is such considerations that have given rise to the application of analytic treatments of the coagulation equation. The first such tratment for clouds is that of Nakano (51), who assumed that initially all the fragments have the Jeans mass. Taking several forms for the agglomeration coefficient which was assumed to be a simple mass dependent quantity (or constans), the coagulation equation was numerically integrated. For $\alpha_a(\mu,\mu')$ constant or $\alpha_b(\mu,\mu')=\mu^{\frac{1}{3}}+\mu'^{\frac{1}{3}}$the resultant mass distribution is peaked between log m of 1 and 2, while for $\alpha_d(\mu,\mu')=\mu+\mu'$ and $\alpha_c(\mu,\mu')=\alpha_b(\mu,\mu')^2$the distribution is a monotonically decreasing function of mass. In general coefficient α_a and α_b lead to something like a log-normal distribution, while α_d tends toward a power law. Case α_c is intermediate.

An elaboration on this procedure was provided by Norman and Silk (52) who treat the growth of T Tauri stars in turbulent clouds by the coagulation equation and provide an analytic solution for the evolution for a simple power law input. The turbulence of the cloud derives from the turning on of the T Tauri stage, which through stellar wind stirring maintains the turbulence required to support the clouds against collapse. The key feature of their treatment is a concentration on the low mass stars, which are near the Jeans mass for the cloud and consequently the most probable stars to form (Elmegreen (53)). This treatment, however, has several shortcomings. It ignores the infall of material, which may be of some importance in the evolution of the system (especially if there is any stimulated formation of fragments or agglomeration process) and it also ignores the decay of individual fragments. The latter may be due either to collapse and the subsequent formation of a star (if the mass is greater than the Jeans mass and the fragment is not supported either by turbulence or magnetic field) and the evaporation by stars which may already have formed. This is a relevant problem for the evolution of molecular clouds, since recent observations (Montmerle et al (54)) have shown that X-ray emission by T Tauri stars which have been formed previously in the cloud may alter the subsequent evolution of the fragments Therefore, one of the pressing problems in the theory of fragmentation and evolution of the cores of molecular clouds is the inclusion of the effects of previously present stars on the dynamics and thermal-ionization structure of the medium. While Norman and Silk (52) do treat feedback processes in a simple way, there is still lacking a general theory of the detailed effects of the appearance of stars on the internal structure and evolution of molecular clouds.

A simple treatment of fragmentation which includes the hierarchical model is due to Larson (55). Assuming successive, but random, bifurcations of a sample of collapsing fragments, he derives a binomial distribution for the mass spectrum:

$$f(n,m) = \left(\frac{1}{2}\right)^{n} \binom{n}{m} \qquad (2\text{-}3)$$

where there are n stages of fragmentation and each fragment has a mass fraction 2^{-m}. If the probability of fragmentation is p, then this generalizes to:

$$f(n,m;p) = p^{m}\,(1-p)^{n-m} \binom{n}{m} \qquad (2\text{-}4)$$

which for $n \longrightarrow N$, becomes a Gaussian distribution for the fragments:

$$f(N,m;p) = (2\pi p(1-p)N)^{-1/2} \exp\ -\left(\frac{(m-Np)^2}{2p(1-p)N}\right) \qquad (2\text{-}5)$$

Since each division is essentially a Poisson-type trial (that is there is a probability p at each step which is independent of the previous history of the fragment) there is a random, normal distribution of the fragments with the mean value being Np and the dispersion being 2p(1-p)N . The fluctuation is therefore of the order of $N^{1/2}$, as expected from such a Poisson trial. There are no terms related to the agglomeration of these fragments, which will tend to cause the distribution to evolve away from the Gaussian form. Larson points out that such a fragmentation scheme will ultimately lead, in its full realization, to a fractal distribution of masses (see Mandelbrot (57)) which is characteristic of turbulent distribution (relevant also for the properties of molecular clouds- Ferrini et al (50)). It should be noted that the IMF to which Larson (55) compares his data does not agree with that derived by Miller and Scalo (38), and his function is quite different from the results of the coagulation calculations. The treatment of statistical fragmentation has recently been discussed by Zinnecker (56) and Elmegreen (53) for clusters and binary systems formation. Returning to the picture advocated by Hoyle, they treat fragmentation as a multiplicative stochastic process in which the probability of fission of a fragment at any step is Markovian, that is, independent of the history of the fragment. In this fashion, they do not explicitly include the dynamics of the collapse phase. As in the case of the broken-stick distribution known from ecological models (see May (58) for a review), they obtain a log-normal mass spectrum which is in good agreement with the initial mass function obtained for galactic stars by Miller and Scalo (38) and which asymptotically provides the power law spectrum suggested by Salpeter (37). Elmegreen and Mathieu (42) have generalized this treatment to the formation of star clusters by arguing that the formation of massive stars shuts off the process of stable fragmentation-star formation within the cloud. Therefore, only in the most massive systems, where there is a likelihood of forming OB stars, will there be a short formation time. Elmegreen (53) argues that statistically long formation times (of the order 10^9 years for the typical system) may characterize the formation of clusters. The resultant mass spectrum should be dominated by the low mass stars. In addition, the long formation period may give rise to an intrinsic spread in the metallicity and time - dependent properties of the stars.

A novel approach to the fragmentation problem is due to Ferrini et al.(59) who describe the internal cloud dynamics by means of a nonlinear Lagrangian in a scalar field, whose modulus square represents the density distribution inside the cloud. By solving analytically the nonlinear resulting Klein-Gordon equation, they calculate the fragmentation spectrum. Although the model is not explicitly self-gravitating, the Jeans mass enters into the calculation for the growth time of a fragment and in the renormalized mass, and therefore the gravitational triggering and dynamics of the collapse are implicitly included in the model. The model is essentially a log-normal distribution, when the time scale for the propagation of perturbation in the nonlinear field is smaller than the local free fall time, and therefore agrees well with the Miller and Scalo (38) fits for the field stars. In addition, the IMF is of the form suggested by Elmegreen and Zinnecker for the mass spectrum resulting from multiplicative stochastic processes, as discussed above. Moreover, since the Jeans mass of the cloud depends on the possible presence of magnetic fields the expected cloud fragmentation spectrum can be derived under very general astrophysical assumptions.

The formation of multiple star systems has long plagued the theory of star formation. Usually termed the angular momentum problem, it is the product of the fact that molecular clouds rotate. Assume that the rotation rate for a 1 $M_{\odot}$ (solar mass) cloud is 10^{-15} s^{-1}, simply the galactic differential rotation rate for a 1 parsec cloud. If the angular momentum is conserved during the collapse, the resultant velocity of the finally formed single star would be enough to prevent stability (the rotational distortion of the final star will be such that an equatorial cusp is inevitable). Calculations using three dimensional fluid dynamical codes (60,61) show that the collapse of rotating clouds will certainly form multiple systems.

We refer the interested reader to the paper by Bossi et al.(62), where this problem is discussed in detail, in its parallel aspects of star formation and planetary system formation.

3 - PERCOLATION MODELS

There are two primary treatments of percolation in astrophysics, one connected with galactic structure and the other, far less well developed or understood, with magnetic dynamos. We shall concentrate on the first (see the reviews by Broadbeut and Hammersley (63) and Hammersley (64)). The morphological structure of disk galaxies has been a problem of long standing since the introduction of the Hubble classification scheme (Hubble (65)). This taxonomy uses both the nuclear spheroid (the so-called bulge to disk ratio) and the openness of the spiral arms in the disk to provide a classification. The deterministic basis of this scheme, that there is a sequential evolution between the taxa as originally described by Jeans (66) has been shown not to apply, but the system has survived because numerous other properties of galaxies appear to be well correlated with the Hubble type. A fully hydrodynamic (in effect deterministic) description is provided by the Density Wave theory due to Lin and Shu (67) and its nonlinear elaboration by Roberts (68). The theory is well reviewed by Toomre (69) and we shall only discuss it briefly, since it can be shown to fit into the stochastic scheme.

Disk systems composed by stars and presumably gas, and which are selfgravitating are intrinsically unstable to the growth of a spectrum of longitudinal density waves. The stellar population acts as a compressible fluid whose number density perturbation causes the gravitational potential to vary with azimuth and radius in the disk. This in turn produces accelerations which support the density wave, and in principle the system might be self-sustanding The resultant structure of the disk would be spiral in appearance, since the shear of the system will produce spiral longitudinal perturbations, and the pattern speed through the system should be fixed by the local condition of shearing, given by the epicyclic frequency. If the wave is sufficiently strong, nonlinear hydrodynamical models show that a standing shock will develop at the spiral arms, resulting in compression of the disk gaseous component and the possible driving of either star formation or cloud formation. Although this should in principle also feed into the distribution function for the stellar population, such an effect has not yet been included with sufficient generality in the (analytic) models to comment on this reaction with the basic density wave predictions. Toomre (69) and Zang (70) have both shown that the system is unstable to winding, and that the spiral pattern is indeed not stable on long (many rotation) timescales. It is in this context that the percolation models have been introduced.

The idea that star formation can have an induced component which depends on the available stellar population in the disk was first realized by Elmegreen and Lada (71) and Herbst and Assousa (72), and at about the same time by Mueller and Arnett (73) for stellar systems. We shall discuss the effect of such an assumption on models of star formation in local models in Section 4, now we can show that this picture carries over quite nicely into the global models. However, in these pictures, since there has been little analytic work on the large scale structure (with the exception of Shore (74) and Fujimoto and Ikeuchi(75)), we must confine the discussion to the more abstract aspects of percolation theory on differentially rotating planes and then discuss the re-interpretation of the results in light of the modelling that has been done to date.

The simulations of the stochastic models, for the study of the morphology of star forming disks as a function of the probability of stimulated star formation and the rotational velocity of the system, have been performed by several groups, notable Gerola and Seiden (76), Schulman, Seiden and Gerola (77), Comins (78), Madore and Freedman (79), Statler et al. (80), Feitzinger et al.(81).

The basic principle of all of these models is that star formation can be viewed as a percolation phenomenon, with the metastructure of the system resulting from the short range interaction between neighboring cells. The stimulus of Conway's Life game has proven of great importance in this field Governed by the rules that the activity of a cell is controlled by the neighboring population of active cells, and that the propagation of influence between two cells can occur with or without time delay, are the basic properties of the systems. This is augmented with the "dynamical" condition that the disk on which the percolation occurs can be treated as differentially rotating. Unlike the Conway game this means that there is a global structuring and replentishment of active sites. In addition, it implies that the percolation, if not taking place on a fixed grid, will have a variable critical probability from place to place in the disk. This latter problem is circumvented by assuming that the rotation curve for the disk is flat; that is the rotational frequency varies as the inverse of the radius. Such an expedient not only fixes the number of cells that are needed to treat the evolution of the disk, it also provides for a global percolation parameter. The models show that the peak of the rotation curve, that is the rate at which the cells are forced dynamically to communicate, is the basic percolation parameter. The primary success of the models has been the reproduction of stable spiral structure. The propagation of the spiral pattern occurs at the percolation velocity, determined only by the maximum of the rotational frequency. The rate of replentishment of a cell's gas is also determined by this velocity, since the differential rotation is scaled to this speed, and therefore the robustness of the global star formation is dominated by the maximum of the rotational velocity.

Predictions of the color, metallicity and star formation gradients across such disks can be compared directly with observations (Wray et al (83)) for a few galaxies, and also the rates of star formation with time can be derived as well. One of the most remarkable observations, which is in effect similar to that observed in chemical systems, is that modes of coherent oscillations are possible (Feitzinger et al (81)) which have the appearance of ring galaxies. The implicitly nonlinear stochastic models do not allow one to follow the details of the star formation as such, but rather present a morphology which can be classified in the same fashion as those of galaxies. This classification has been stressed by Shore and Comins (84) and also by the originators of the various systems (Hubble (65), Van den Bergh (85)). In the absence of detailed information about the rotation curve and distribution of stellar constituents, the only information available for a galaxy is global in nature --its morphology, total gas content, integrated luminosity and integrated spectral type. Ultimatly, the goal of the SSPSF or density wave models must, it seems fair to say, be to provide some causal connections between these gross features of these systems and the detailed "microphysics" by which they come into being.

4 - LANGEVIN SYSTEMS AND FOKKER-PLANCK EQUATION

Langevin systems of coupled nonlinear equations have been used recently in the modelling of galactic evolution, in the framework of the so-called one-zone-model (OZM). By OZM it is understood a model in which a certain limited region of a galaxy is considered, its content in gas, stars and eventually clouds is studied by neglecting the variations of the basic galactic properties across the region (Ostriker and Thuan (86), Tinsley (87)); it can be allowed for exchange of matter with the external medium.

The astrophysical problem of justifying on theoretical grounds the morphology of galaxies (spiral and elliptical, with their different content in stars and gas), their chemical evolution (initial rapid enreachment of metals i.e. any element heavier than Hydrogen and Helium) and finally the attempt to trace a classification based on different physical aspects of the evolution, has been tackled on employing the approach of cooperative systems. In these models a scenario is proposed where the large scale dynamics is related to the local microscopic interactions. At the same time a macroscopic description (e.g. the interplay of various phases, the metallicity) is derived by means of few (stochastic) variables.

The mathematical structure of the models is their unifying background: systems of nonlinear coupled differential equations with eventually nonlocal terms. Approximate analytical solutions have been calculated for linearized or reduced models and their asymptotic behaviours have been determined, while various numerical simulations have been performed for the complete models. The structure of the fixed points, their values and stability, have been analyzed and some preliminary correspondence between fixed points and morphological classes of galaxies is evident, for example the parallelism between low and high gas content with respectively elliptical and spiral galaxies.

Typical is the oscillating behaviour of the solutions, the astrophysical meaning of this phenomenon being straightforward: the bursts in star formation rate have been observed in young galaxies, and the color distribution of older ones is again an evidence of nonmonotonic star formation history. The burst time scale can be calculated from the parameters of the models. At the same time, the local models are particularly suited to describe irregular galaxies, while the nonlocal models reproduce the large scale pattern of spiral galaxies. Finally the chemical evolution of galaxies can be reproduced with great care. In conclusion, the common matrix of modelling is the synergetic behaviour of the system: a few variables would describe the evolution, while microphysics intervenes in the processes which determine the values of the parameters.

TABLE I

Shore (88) first introduced some Langevin models in which the effects of induced star formation on the evolution of a galaxy are investigated. In Table I we summarize all the models described in the literature; we presently analyze only the main features and we refer to the original papers for a more detailed discussion. Let us call s(t) and g(t) the mass fraction of stars and gas respectively. A simple model assumes the rate of star formation to be determined by the rate of depletion of interstellar gas as follows:

$$\begin{aligned} \dot{s}(t) &= -rs(t)+as(t)g(t) \\ \dot{g}(t) &= -dg(t)+f \end{aligned} \qquad (4\text{-}1)$$

where the rate of infall of halo material, f, is assumed constant, r is the rate of return of stellar material to the gas phase, a is the rate of induced star formation, and d is the rate of gas consumption. A more realistic model can be obtained from the generalized one zone three phase model consisting of diffuse gas, clouds and stars:

$$\begin{aligned} \dot{s}(t) &= -rs(t)+as(t)c(t)+bg(t)^{n} \\ \dot{c}(t) &= -dc(t)-a's(t)c(t)+eg(t)^{m} \\ \dot{g}(t) &= rs(t)+f-eg(t)^{m}-bg(t)^{n} \end{aligned} \qquad (4\text{-}2)$$

Now b is the modified Schmidt (89) rate (spontaneous star formation, assuming that clouds and not diffuse gas are responsible for spontaneous star formation) d is the rate of clouds destruction by background sources, a is the rate of formation of clouds out of diffuse gas.

The nonlinear one zone models can be generalized by the introduction of explicitly stochastic terms for any of the rates. The easiest, and physically most interesting, one to introduce is that of time-variable infall. In this case Ferrini et al (90) have shown that there are analytic solutions possible for the simple two component gas model, which agree well with both the equilibrium behaviour predicted by the linearized models and the deterministic systems.

Consider the system:

$$\dot{s}(t)=-rs(t)+as(t)g(t)$$
$$\dot{g}(t)=-dg(t)+r's(t)-hs(t)^n+f+F(t) \qquad (4-4)$$

where all of the variables have the same meaning as for the deterministic systems, but the additional term F(t) is assumed to be a random variable with the property:

$$\langle F(t)F(t')\rangle =2D\delta(t-t')$$

This is the approximation of a Wiener variable, which renders the system (4-4) a coupled Langevin system. The correlation function for F is therefore of the character of a diffusion coefficient, and the system can be solved by standard Lagrangian methods.

Assuming that the change in the gas fraction is negligible, Ferrini et al (90) solve the system by putting it into the form:

$$\dot{s}(t)=\frac{a}{d}(f-f_o)s(t)+\frac{ar'}{d}s(t)^2-\frac{ha}{d}s(t)^{n+1}+\frac{a}{d}s(t)F(t) \qquad (4\text{-}5)$$

where f_o is defined to be rd/a. This is then an equation associated with a potential function V(s), which is characteristic of systems with multiplicative noise and which is of the form:

$$\frac{d}{a}V(s) = -\frac{1}{2}(f-f_o)s^2-\frac{r'}{3}s^3 \quad +\frac{h}{n+2}s^{n+2} \qquad (4\text{-}6)$$

It should be noted that if n is of the orther 2, this is a cusp manifold (which is one which will show bifurcation behaviour of the form discussed by Shore and Comins (84)). The analysis of this system presented in Ferrini et al. (90) shows that this is indeed a system with multiple equilibrium states. Having this form for the potential, there exists a Fokker-Planck equation for the evolution of the probability distribution function for the stellar population of the system, which is one that yields the expectation for the number of stars as a function of time. The evolution equation for this system is:

$$\frac{\partial}{\partial t}P(s;t)=-\frac{a\partial}{d\partial s}[(f-f_o)s-hs^{n+1}+\frac{a}{d}Ds]P(s;t) \;+\frac{aD\partial^2}{d\,\partial s^2}[s^2 P(s;t)] \qquad (4\text{-}7)$$

which gives:

$$\langle s\rangle = \left(\frac{dh}{aDn}\right)^{-\frac{1}{n}} \frac{\Gamma\left(\frac{f-f_o}{aDn}d+\frac{1}{n}\right)}{\Gamma\left(\frac{(f-f_o)d}{aDn}\right)} \qquad \sigma^2(s)=\frac{aD}{dh} \qquad f>f_o \qquad (4\text{-}8)$$

As expected, the variance is linearly dependent on D, while the mean stellar fraction varies as D (noting that the case n=2 is then essentially that of a Poisson limit evolution).

A similar treatment in which the rates themselves can all be treated as random variables has been given by Shore (74). Here the treatment by Bartlett (91) is used, which allows for the evolution of the coupled system under the explicit assumption that $g=1-s$. This will not be true in the case of infall, but well approximates the evolution of the closed system. The Fokker-Planck equation in this case results from the interpretation of the coupled one zone evolution equations as both being Langevin equations. The infall characteristics are not specified, nor is it necessary to state at the outset what the variance of the rate coefficients is like. The equation is of the form:

$$\frac{\partial}{\partial t}P(s;t)=as(1-s)\frac{\partial^2}{\partial s^2}P(s;t)-d(1-s)\frac{\partial}{\partial s}P(s;t) \qquad (4\text{-}9)$$

which has as a solution:

$$P(s;t)=P_o e^{\lambda t}[{}_2F_1(\alpha,\beta_+,\gamma;s)+A_2F_1(\alpha,\beta_-,\gamma;s)] \qquad (4\text{-}10)$$

where

$$\gamma=-d/a \quad , \quad -(\alpha+\beta)=1+d/a \quad , \quad \alpha\beta = \lambda/a \qquad (4\text{-}11)$$

and $\beta_\pm$ are the roots of $\beta^2+(1+d/a)\beta+\lambda/a=0$

Again, this is a solution which shows multiple roots and the bifurcation character one is led to expect from the deterministic models.

A numerical treatment of these equations has been provided recently both by Ferrini and Marchesoni (92) and by Ferrini et al. (93). The analysis proceeds as follows.

The system (4-3) is coupled to the metallicity equation:

$$\frac{d}{dt}(Zg) = -(1-R)Z\dot{s} + P_Z\dot{s} + Z_f f \qquad (4-12)$$

where Z_f is the metallicity of the infalling material, P_Z is the rate of stellar production of Z and R is the rate of return of material to the gas phase. This is the same equation as explored in Tinsley (87) and Shore (88). The rate of star formation ψ , has been replaced by the specific rate from the nonlinear models. We can then couple this directly to the evolution equations for the system and solve for the stellar and gas contribution explicitly, without the usual assumptions of previous work about the rate of star formation with time (see e.g. Audouze and Tinsley (94), Pagel and Edmunds (95), Twarog (96), Miller and Scalo (38) for cases in which the rate of star formation must be assumed in the analysis of field star metallicity). The distribution function for the metallicity has been given by Ferrini et al (90), and the explicit deterministic evolution of the system has been solved by Shore (88). A version of the system with deterministic (that is constant) infall of halo material has been presented previously by Lynden-Bell (97) and agrees with the more detailed solution by Shore (88).

In the previous exploration of these systems the analytic solutions were sought which would describe the asymptotic stability of such a system. We now drop the assumptions required for the linearized treatment and examine the full system. We begin by assuming that the infall can be a stochastic variable in time. Such a behaviour is expected for a galaxy in a cluster, for example, for which collisional stripping or infall might be occurring, or for any region of the galaxy through which stars are randomly passing. In such a case as this last one, the rate coefficients should also be random functions. For the moment, we will examine only the metallicity evolution of the system. We have computed a set of models for reasonable values of the parameters with stochastic infall. The values chosen for the parameters are:

$a = a' = 0.10$ $h = 0.05$ $n = 1.84$ (Sanduleak (98))

$r = r' = 0.10$ $f = 0.04$

and $Z_f = 10^{-3}$. It was alo assumed that the infall had no metallicity dispersion, but that the infall had a mean value of f, with a dispersion ± 0.04 . We have, for comparison, also computed a fully deterministic model

with the same parameters. In all cases, the initial ratio g_o/s_o is 10^3 (essentially all gas) and $Z_o=10^{-3}$. The results are shown in Fig. 1. The stellar fraction rises quickly, with an asymptotically linear form, while the gas fraction decreases slowly and thereafter remains fixed. The metal abundance of the system saturates, even while the stars continue to increase. There is little difference in the evolution, in this particular system, of the stellar and gas phases with the deterministic rates. However, for a critical value of r=0.08, all other parameters kept fixed, the metallicity reaches an initial maximum and then slowly declines. This sort of behavior is reminiscent of Larson (99), which shows that when the star formation rate is very large in comparison with the death rate, the metallicity actually peaks during the early evolution of the system. FIGURE 1

We have also tried models in which the fluctuations increased in amplitude with time, and the metallicity also increased stochastically with time. Again the metallicity of the system saturates at approximately solar values, although the stellar fraction behaves like the stochastic system described previously. Finally, in Fig.2 we show the results for the evolution assuming that the infall decreases like e^{-pt} where p is a free parameter. This is the result of allowing f to FIGURE 2 fluctuate to zero during the course of the infall, although the rate can temporarily increase. Here, the stellar fraction saturates (decreasing ψ). Again, the metallicity changes with time in a fashion almost indistinguishable from the previous cases. In short, models of this sort seem to suggest that while the infall is necessary in order to explain the general evolution of the system, very different histories of this infall can produce essentially the same result in the final state. That our galaxy seems to have essentially constant metallicity, constant birthrate and increasing star to gas ratio at about the same rate as the birthrate suggest that the appropriate description is one with either constant or stochastic infall at a rate comparable with the rate of induced star formation.

In order to specify the nature of models further, we should add that a three-level system is the only one completely appropriate for the modelling of a galaxy. This has also been discussed recently by Seiden (100) who has labelled these phases "active" and "inactive". The active phase is the gas, since it is from this that the clouds and eventually the stars are formed. The inactive phase, or the clouds in our picture (Seiden and Gerola (76)) is a form of holding phase for the material.

It does not seem without interest to evaluate the linearized stochastic case, both from the standpoint of introducing the formalism and because there is some evidence that systems forming stars can be treated as self-regulated and therefore equilibrated systems (Franco and Cox (101), Norman and Silk (52), Franco and Shore (102)). Let us assume a simplified two level system:

$$\dot{s}(t) = as(t)g(t) - rs(t)$$
$$\dot{g}(t) = r's(t) - a's(t)g(t) + f \qquad (4\text{-}13)$$

so that we now have:

$$\frac{d}{dt}\underset{\sim}{X} = A\underset{\sim}{X} + \underset{\sim}{\Phi} \qquad (4\text{-}14)$$

where

$$\underset{\sim}{\Phi} = \begin{pmatrix} a_1 s_o g_o - r_1 s_o \\ s_o r'_1 - a'_o g_o + f \end{pmatrix} \qquad (4\text{-}15)$$

and

$$A = \begin{pmatrix} a_o g_o - r_o & a_o s_o \\ r'_o - a'_o g_o & -a'_o s_o \end{pmatrix} \qquad (4\text{-}16)$$

are the state operators for the system. Now consider the diagonalizing transformation T such that:

$$T^{-1}\underset{\sim}{\Phi} = \begin{pmatrix} (r_1/r_o - a'_1/a_o)r_o s_o + f \\ (a_1/a_o - r_1/r_o + a'_1/a_o - r_1/r_o)s_o r_o - f \end{pmatrix} \qquad (4\text{-}17)$$

The equation (4-14) can now be integrated to yield:

$$g(t) = c_o e^{-a_o s_o t} + \int_0^t (R-A)_x r_o s_o \exp(-a_o s_o [t-x])dx + $$
$$+ \int_0^t \exp[a_o s_o (x-t)] f(x)dx \qquad (4\text{-}18)$$

The important thing to note here, where we can take c_o to be given by the initial conditions, is that the entire process can be viewed as a stochastic Ito equation, nowehere assuming anything about the differentiability of the infall parameter f(t). The advantage of this approach for the generalization of the evolution equations to the stochastic regime in which we do not linearize the system is therefore clear; the infall (or outflow, we need not specify the sign of f) can be a stepwise continuous function of time, or even discontinuous. The rate of change of the gas fraction in the system will be determinable regardless.

The reason for dwelling on this point is that the evolution of any galactic system will be influenced by random processes occurring in the environment in which it finds itself. The chance encounter between two galaxies in a cluster will cause time-dependent but stochastic in character infall. The occurrance of supernovae in portions of the system will be random in time, and the input of energy to the interstellar medium and consequent alteration of the local conditions for star formation will also be inherently stochastic in nature. Thus, having in hand a qualitative formalism for the analysis of such effects may serve as a useful starting point for a discussion of the further developments in the nonlinear theory. Further, this will be useful in the explication of the stochastic metallicity evolution of the system, since it is the fluctuations of the stellar and gas fraction in any part of the system that will drive the alteration of the metal abundance of the disk. Provided the system is well mixed, which is fine for the laboratory but not necessarily for a spiral galaxy, there will be a predictable spread in the abundances derivable from the evolution equations for the system.

The extension of these models to two dimensions, a prerequisite for realistic models of spiral galaxies, can be accomplished by using stochastic methods for justification. A diffusion equation for the stellar population including birth and death terms was first asserted by Shore (74) and also recently emploted by Nozakura and Ikeuchi (103). It is possible, however, to derive this equation from first principles provided the spatial distribution for the stellar velocities has a random component as well as that due to the differential rotation of the galaxy.

Assume that the distribution of orbital eccentricities is a random function of space. Then at any position in the galaxy, there will be stars on either inbound or outbound legs of their orbits, distributed about the mean motion at that galactocentric radius. The master equation for the stellar population of that region of the disk will be given, then, by:

$$\begin{aligned} P_t(x,t) &= A_+P(x-dx,t-dt)+A_-P(x+dx,t-dt)-B_+P(x,t)-B_-P(x,t) \\ &\quad -rP(x,t)+f((P(x,t-dt)(1-P(x,t))+P(x,t)(1-P(x,t-dt)) \end{aligned} \qquad (4-19)$$

where the coefficients $A_{\pm}$ and $B_{\pm}$ refer to the motion of stars through the region without changes in the composition of the stellar component of the system and r and f are the death and stimulated birthrates. It is assumed that the gas fraction g is given by 1-s, as before. It is then simple to see that this equation reduces, in the continuum limit, to the diffusion equation:

$$\frac{\partial s}{\partial t} = \eta\nabla^2 s - \Omega\frac{\partial s}{\partial \phi} + as(1-s) - rs \qquad (4-20)$$

which is given by Shore (74). The key reason for the diffusion term appearing is that the Master Equation couples the population at x-dx with that at x+dx, thereby giving rise to a second derivative. It is also the case that the diffusion is dominated by the tendency, due to the random distribution of orbital eccentricities, to appear like a small diffusive variation in the stellar population in the zone superimposed on the convective derivative due to differential rotation of the centroid. Clearly, this model can be generalized to three dimension, depending upon the choice of the vertical structuring of the rotation law, since for the halo polulation the orbits form a more or less spherical distribution about the galactic plane. The two population can, in fact, be coupled through the stellar distribution function P(x,t), which can be broken into subcomponents (subpopulations) and which can then change the birth and death terms (since, for instance, there is no current formation of halo stars while these become supernova or form planetary nebulae during their passage through the disk).

In the one zone picturee, the metallicity evolution can be solved using the coupled star-gas evolution equations and the same is true for this case. If we assume that the terms in the metallicity function are only spatially dependent through the stellar population evolution equation then it is possible to solve explicitly for the metallicity as a function of position in the galaxy.

One recent attempt at a phenomenological model for evolution of the metallicity of the galactic disk has been presented by White and Audouze(104). Their picture assumes that gas is re-cycled, possibly non conservatively, between the interstellar medium and stellar interiors, where nuclear processing alters the abundances and increases the metallicity. The material, on being returned to the interstellar medium, is mixed so that subsequent generations of stars will draw from this polluted source of matter. They proced as follows. Take f to be the fraction of the disk locked up in long-lived stars and g to be the remaining fraction in gas, which can be polluted by an amount ΔZ in its heavy metal abundance (the species of element being considered will remain momentarily unspecified). The yield of the medium is defined to be:

$$\tilde{Y}_1 = (f^{-1}-1)g\Delta Z_1 \qquad (4\text{-}21)$$

The probability that any parcel of gas will not have been incorporated into stars in N events, and will finally find itself inside a long-lived star is:

$$P(N)=f(1-f)^N \qquad (4\text{-}22)$$

while the probability that this parcel will have been enriched n times out of N will be the conditional probability :

$$P(n|N) = \binom{N}{n} g^n (1-g)^{N-n} \qquad (4\text{-}23)$$

If the Poisson process of enrichment is assumed (the gas is completely mixed and randomly enriched). Then, using the Chapman-Kolmogorov equation, we have:

$$P(n) = \sum_{N=n}^{\infty} P(n|N)P(N) \qquad (4\text{-}24)$$

for the probability of n enrichments. Therefore, the metallicity increase is the expectation value:

$$<Z_1> = \Delta Z_1 \ \Sigma \, nP(n) \qquad (4\text{-}25)$$

This is simply, for the gas-star model we have chosen:

$$\begin{aligned} <Z_1> &= (a^{-1} -1)\ \Delta Z_1 \\ \sigma^2 &= <Z_1^2>\ (1-a)^{-1} \end{aligned} \qquad (4\text{-}26)$$

The rate of increase of metals is therefore dependent upon:

$$a = f/(g+f-fg) \qquad (4-27)$$

which is the probability that the parcel will be in the gas phase at the time of sampling. We note that this system is assumed to be open, since in the case of f=1-g, we see that the system is unbounded if g=1, and that therefore this is not a realistic model for the system. The inherent stochastic dispersion of the model results from the fact that the infalling material will not be contaminated until it is incorporated into the disk material and enriched from the star formation occurring in the disk.

Altough the model has been elaborated by White and Audouze to study the variations among different elements, the essentials of the model remain unchanged We have described it as phenomenological because it does not include the effects of the alteration of the conditions of the galactic disk on the star formation rate or the spectrum of mass and processing of the stellar component of the system. This absence of feedback which is essential to an understanding of the time-development of the metallicity, is also one of the basic features which we have covered with our models. This paper is however, noteworthy in being the first attempt to understand both the variation of metal abundance and the cosmic spread associated with the star formation and enrichment processes in the disk. The conclusion of this paper is that infall is indeed necessary, as discussed by Lynden-Bell (97) and Shore(74), and also by Thuan and Ostriker (86). The lack of consideration of the change in the star formation characteristics as a function of the change of the composition of the disk do, however, limit severely the use of this model to real galaxies.

5 - MATHEMATICAL TREATMENT OF A SIMPLE ASTROPHYSICAL MODEL

In the foregoing Sections we reviewed some stochastic models for the stellar formation and galactic evolution widely employed in the literature. In the present Section we shall discuss explicitly a simplified astrophysical model where the stochastic processes introduced for mimicking the complexity of the relevant interactions, are dealt with by recourse to the analytical tools of the previous articles (notably Grigolini and Marchesoni, from now on referred to as GM).

Even though we focus on the detailed treatment of a single example, we make some preliminary remarks: (i) all the approaches mentioned above which adopt the Langevin or the Fokker-Planck formalism are to be regarded as merely phenomenological in their nature. The corresponding Langevin (or Fokker-Planck) equations can not be derived from a proper global Hamiltonian description, no matter which restrictions are imposed. That is mainly due to our scarce knowledge of the intimate structure of the systems under study. Generally our Langevin phenomenological equations are written down by two steps. First of all we must recognize and characterize the relevant constituents whose interplay is likely to determine the time evolution of the observed physical quantities (i.e. the choice of the variables and the deterministic terms). Thereafter, we try to account for the presence of the remainder on assuming that some parameters, first introduced as deterministic, undergo stochastic fluctuations with statistics that can not be readly referred to the actual dynamics of the system. (ii) This procedure in spite of the possible criticism as arbitrary, is useful for testing the stability of more refined but purely dynamical models. Ferrini et al (90,92) showed that small fluctuations of the parameters of nonlinear models can affect dramatically both the time evolution and the stationary equilibrium state of the system. This effect can be explained in terms of basic mechanisms such the so-called noise induced phase transitions introduced in great detail by Faetti et al. (present Volume). The problem of the exact Langevin equations lies outside the limits of our discussion (see Marchesoni Part 2). (iii) Most of the mathematical techniques reviewed in the present volume provide subsidiary tools to the authors addressing the stochastic methods in astrophysics. The reduced model approach (GM) can be successfully employed for treating delay equations like those involved in the metallicity problem. In this case we can suggest a sensible criterion for adding new macro-variables to a simpler starting model (Ferrini et al (93)). On the contrary, an abiabatic elimination scheme (like the AEP in GM) is of great use when we need to simplify a set of stochastic equations without loosing relevant information. An example of this reduction technique for a specific astrophysical model is worked out in the following. Finally, the CFP by Grosso and Pastori-Parravicini (present Volume) is certainly a flexible numerical algorithm for explicitly computing the time dependence of related statistical quantities of astrophysical interest.

Let us choose as a starting point for a stochastic galactic evolutionary model, the set of deterministic equations (4-4). Such a simplified version of a two phase picture of the galactic medium assumes s(t) and g(t) as the relevant constituents of the system. If we imagine that the matter exchange with the galactic halo is a random process, the rate of infall f is now being given by $f + \eta(t)$. For simplicity we assume $\eta(t)$ a white Gaussian noise with zero mean value and correlation:

$$\langle \eta(t)\, \eta(0) \rangle = 2D\, \delta(t) \qquad (5\text{-}1)$$

The two phase stochastic model we address now reads:

$$\begin{aligned} \dot{s}(t) &= -r\, s(t) + a\, s(t)\, g(t) \\ \dot{g}(t) &= r' s(t) - a' s(t)\, g(t) + f + \eta(t) \end{aligned} \qquad (5\text{-}2)$$

Here r is the rate of star decay, r' is the rate of return of mass to gas, a is the rate of induced star formation, a' is the rate of star breeding.

In spite of its simplicity, application of AEP to the Langevin system (5-2) is not straightforward. Most notably we must slightly improve the procedure summarized in GM so as to deal with systems like this one, where the distinction between fast and slowly relaxing variables is not clearcut. Let us start by studying the fixed point structure of (5-2). In the absence of stochastic terms ($\eta(t)=0$) we find only one fixed point (s_o, g_o),

$$s_o = \frac{fa}{ra' - ar'} \quad ; \quad g_o = \frac{r}{a} \qquad (5\text{-}3)$$

with $s_o > 0$ when $\frac{r}{a} > \frac{r'}{a'}$

This inequality corresponds to the circumstance that stars burn gaseous matter with a positive rate. The energy production by stellar combustion is not accounted for by our model. It is noteworthy that if s(t) is positive at a fixed time, it will be bounded in the positive axis for all the times. We can prove this statement by noting that all time derivatives of s vanish when s vanishes. Indeed, if at $t = t_o$ $s(t_o)=0$ then $\dot{s}(t_o)=0$. From eq.(5-2) we also write down a general expression for the n-th derivative :

$$\overset{(n+1)}{s}(t) = a\left(s(t) + g(t)\right)^{n} \quad ; \quad n \geq 1 \qquad (5\text{-}4)$$

where the r.h.s. is to be expanded formally as the n-th power of a binomial, but the powers denote the derivative order. By induction, at $t = t_0$ $\overset{(n)}{s}(t_0) = 0$ for any n. This property holds in the presence of external fluctuations as well. Let us change variables as follows:

$$\begin{aligned} g &\longrightarrow G = g - g_o \\ s &\longrightarrow S = s - s_o \\ t &\longrightarrow \tau = a t \end{aligned} \qquad (5\text{-}5)$$

The set of stochastic equations (5-2) can be rewritten as:

$$\dot{S} = S G + s_0 G$$
$$\dot{G} = -\frac{f}{s_0 a} S - \frac{a'}{a} s_0 G - \frac{a'}{a} S G + \frac{\eta(t)}{a} \qquad (5-6)$$

On changing notation: $x = \frac{S}{s_0}$ $\quad v = G$ $\quad \gamma = \frac{a'}{a} s_0$

eqs.(5-6) show a form resembling the explanatory systems studied in GM :

$$\dot{x} = x v + v$$
$$\dot{v} = -\frac{f}{a} x - \gamma v - \gamma x v + \frac{\eta(t)}{a} \qquad (5-7)$$

The corresponding Fokker-Planck equation reads:

$$\frac{\partial}{\partial \tau} \rho(x,v;\tau) = \Gamma \rho(x,v;\tau) = \qquad (5-8)$$
$$= \left\{ -v \frac{\partial}{\partial x} x - v \frac{\partial}{\partial x} + \frac{f}{a} x \frac{\partial}{\partial v} + \gamma x \frac{\partial}{\partial v} v + \gamma \frac{\partial}{\partial v} v + \frac{D}{a^2} \frac{\partial^2}{\partial v^2} \right\} \rho(x,v;\tau)$$

where $\rho(x,v;\tau)$ denotes the probability distribution of (x,v) at time τ/a.

In the following we apply the AEP in the case of large viscosity γ and small fluctuations intensity D . We determine explicitly the range of parameter values for which our perturbation approach is valid . Our strategy now consists of considering $v(t)$ as a fast relaxing variable and $x(t)$

-related to the star population- as the observable of interest. We notice that in the present case the usual prescription for writing down the Langevin equation corresponding to the first order perturbation approximation is very suspect. Indeed, the Smoluchowski approximation is often obtained by putting $\dot{V}=0$ in eq(5-7) so that we obtain:

$$\dot{x} = -\frac{f}{a\gamma}x + \frac{\eta(t)}{a\gamma} \qquad (5\text{-}9)$$

The corresponding Fokker-Planck equation is given by:

$$\frac{\partial}{\partial\tau} P(x;\tau) = \frac{1}{\gamma}\frac{\partial}{\partial x}\left[\frac{f}{a}x + \langle v^2\rangle \frac{\partial}{\partial x}\right] P(x;t) \qquad (5\text{-}10)$$

where $\langle v^2\rangle = D/a^2\gamma$

and $P(x;t)$ is the reduced probability distribution of the observable x. The equilibrium distribution is:

$$\bar{P}(x) = \mathcal{N}\exp\left(-\frac{f}{a}\frac{x^2}{2\langle v^2\rangle}\right) \qquad (5\text{-}11)$$

where $\mathcal{N}$ is a normalization constant. From eq.(5-5) we know that

$$s(t) = s_o\left(x(t) - 1\right) \qquad (5\text{-}12)$$

It implies that eventhough the equilibrium distribution (5-11) is centered around x=0 (i.e. $s=s_0$), negative values of s(t) are allowed. This is in contrast with the exact result proven above.

The naive approximation on eq.(5-9) fails because $v(t)$ can not be considered so a fast relaxing variable with respect to

$x(t)$ as to assume $\dot{v}=0$ in eq.(5-7). The presence of a mixing term, $-\gamma x v$, would lead us to employ AEP with some caution. We can get rid of $v(t)$ as promised but only when D can be considered small. The condition of a large γ as imposed by Smoluchowski, is no longer enough for treating our system.

In order to apply the AEP of GM we separate the Fokker-Planck operator Γ on eq.(5-8) into an unperturbed part Γ_0, and a perturbation part Γ_1, so that

$$\Gamma = \Gamma_0 + \Gamma_1 \tag{5-13}$$

where

$$\Gamma_0 = \gamma \left[\frac{\partial}{\partial v} v + \langle v^2 \rangle \frac{\partial^2}{\partial v^2} \right] \tag{5-14}$$

and

$$\Gamma_1 = -v \frac{\partial}{\partial x} x - v \frac{\partial}{\partial x} + \frac{f}{a} x \frac{\partial}{\partial v} + \gamma x \frac{\partial}{\partial v} v \tag{5-15}$$

The three relevant parameters γ, $\frac{f}{a}$, $\langle v^2 \rangle$ have been suitably defined above. The result of our perturbative approach is a Fokker-Planck equation for the reduced probability distribution $P(x;\tau)$ of the form:

$$\frac{\partial}{\partial \tau} P(x;\tau) = \sum_{r=0}^{\infty} \Gamma_r P(x;\tau) \tag{5-16}$$

where Γ_r are the perturbation terms of the Fokker-Planck operator of order r-th with respect to the perturbation parameter $1/\gamma$. However, such a counting rule is not reliable in the present case because of the last term Γ_1 is proportional to

γ .

If we proceed further disregarding such a warning, we easily find the explicit expression for $\Gamma_1 + \Gamma_2$ $(\Gamma_0 = 0)$:

$$\Gamma_1 + \Gamma_2 = \frac{1}{\gamma}\frac{\partial}{\partial x}\left\{ Q_1(x) + \langle v^2 \rangle \frac{\partial}{\partial x} Q_2(x) \right] \qquad (5\text{-}17)$$

with

$$Q_1(x) = \frac{f}{a}\left(x - \frac{x^3}{\gamma}\right) + 3\langle v^2\rangle x(1+x) \qquad (5\text{-}18)$$

and

$$Q_2(x) = 1 - 3x^2 + 2x^3 \qquad (5\text{-}19)$$

We adopted GM notation.

The diffusion coefficient $Q_2(x)$ exhibits the following properties: (i) $Q_2(-1)=0$; (ii) $Q_2'(-1)=0$ (prime denotes derivative after x); (iii) $Q_2(1)<0$. According to eq.(5-12), $x=-1$ corresponds to s=0 and x=1 to s= $2s_0$. On the other hand, the (stationary) equilibrium distribution, $\bar{P}(x)$, of a standard Fokker-Planck equation of the type (5-17) is given by:

$$\bar{P}(x) = \frac{\mathcal{N}_c}{Q_2(x)} \exp\left[- \int_c^x \frac{Q_1(x')}{Q_2(x')} \, dx' \right] \qquad (5\text{-}20)$$

where $\mathcal{N}_c$ is a suitable normalization constant. We can easily check that $\bar{P}(x)$ vanishes in x=-1 as it should be, but becomes meaningless (i.e. negative) around x=1. Moreover, on calculating

Γ_2 we would notice that the term $\gamma x \frac{\partial}{\partial v} V$ in Γ_1 is responsible for producing contributions proportional to $1/\gamma$ while those should be wholly accounted for by Γ_1. Such a mechanism works at higher perturbation orders as well, so that our perturbation criterion has to be completely restated.

Let us assume that

$$\gamma \gg 1 \qquad ; \qquad \langle v^2 \rangle \ll \frac{f}{a} \tag{5-21}$$

so that $Q_1(x)$ can be approximated by:

$$Q_1(x) = \frac{f}{a} x \tag{5-22}$$

Restrictions (5-21) are not enough to make all terms coming from Γ_r (with $r > 2$) negligible for determining $Q_2(x)$. On the contrary, we must sum all contribution to each Γ_r in order to pick up the terms proportional to $\langle v^2 \rangle / \gamma$. For large γ, i.e. $(f/a)^{1/2}$; $\langle v^2 \rangle \ll \gamma$ the remainder is certainly comparatively small.

In Section 3 of GM is shown that Γ_r can be written as a sum of terms $\mathcal{D}_n^{(m,0)}$ with m+n=r and of their products $\mathcal{D}_{n_1}^{(m_1,0)} \cdots\cdots \mathcal{D}_{n_k}^{(m_k,0)}$ with $m_1 + \cdots + m_k + n_1 + \cdots + n_k = r$. Among these only $\mathcal{D}_1^{(m,0)}$ can give rise to contributions proportional to $\langle v^2 \rangle / \gamma$; the others generate terms of order $1/\gamma^2$ or higher.

Let us now focus on any single pair $\mathcal{D}_1^{(2n,0)}$ and $\mathcal{D}_1^{(2n+1,0)}$, $n \geq 0$. Following the basic rules of our AEP (GM, Section 4), we note that:

(i) Each $\mathcal{D}_1^{(2n,0)}$ is the integral product of 2n+2 Γ_1 ;

(ii) Γ_1 consists of terms 'odd' in v, but the last one which is the cause of our difficulties and is 'even' in v;

(iii) The first Γ_1 in any product $\mathcal{D}_1^{(m,0)}$ contributes by means of two 'odd' components , $-v\frac{\partial}{\partial x}x - v\frac{\partial}{\partial x}$, only;

(iv) Since the global balance of the powers of v must be even and non-negative, another Γ_1 at least contributes to the product through its 'odd' components;

(v) The other 2n Γ_1 in $\mathcal{D}_1^{(2n,0)}$ can contribute of a factor $\gamma x \frac{\partial}{\partial v} v$ each. The formal perturbation order (2n+1) is then decreased at most by 2n: these are precisely the terms $1/\gamma$ which are to be resummed.

(vi) On counting the derivative order after x, we note that only the components of Γ_1 which are proportional to v (i.e. 'odd') contain an x-derivative. Since we are interested in the contributions displaying only two of such components, (v), and factors like $\gamma x \frac{\partial}{\partial v} v$ do not affect the final power of v, we conclude that the corresponding terms are order $\langle v^2 \rangle/\gamma$, contain a second-order x-derivative, and therefore contribute to $Q_2(x)$;

(vii) We readily prove that the numerical factors coming from the internal integrals cancel out those obtained when moving $\frac{\partial}{\partial v}$ towards the left exactly.

Table II helps the reader to visualize our resummation rules. From (vii) we prove immediately that the contributions to

TABLE II

$Q_2(x)$ from the (2n+1) terms in (b) are identical. Contributions from the pair $\mathcal{D}_1^{(2n,0)} + \mathcal{D}_1^{(2n+1,0)}$ can be reordered as in (c). The signs are as in Table II. The operatorial part of the integral factors in square brakets can be rewritten as follows($\partial = \partial/\partial x$):

$$(A) = \partial\,(1-x-2x^2)$$

$$(B) = 2\,\partial\,(1+x)x \qquad (5-23)$$

Since any circle corresponds to the operator $\partial\,(1+x)$ and any cross to a power of x, finally we obtain

$$Q_2(x) = \sum_{n=0}^{\infty} x^{2n}\left[(2n+1)\,Q_2^{(0)}(x) + 2n\,x(1+x)^2\right] \qquad (5-24)$$

where $Q_2^{0}(x)$ is given in eq.(5-19). If we truncate the series on eq.(5-24) for n=0, we recover the naive approximation (5-17)-(5-19).

The series (5-24) can be easily resummed on employing the properties of the geometrical series:

$$Q_2(x) = \begin{cases} \infty & |x|>1 \\ 1 & |x|\leq 1 \end{cases} \qquad (5-25)$$

The corresponding equilibrium distribution is determined by substituting eqs.(5-22) and (5-25) into eq.(5-20) (Stratonovitch, 1967):

$$\bar{P}(x) = \begin{cases} 0 & |x|>1 \\ \mathcal{N}'\exp\left(-\frac{a}{f}\,\frac{x^2}{2\langle v^2\rangle}\right) & |x|\leq 1 \end{cases} \qquad (5-26)$$

After a very complicated elimination procedure, we recovered a Fokker-Planck equation (and the corresponding equilibrium distribution) which closely resembles the ingenuous approximation in eq.(5-9). The stochastic observable now ranges within the interval [-1,1] so that the normalization constant $\mathcal{N}'$ in eq.(5-26) is not to be mistaken with $\mathcal{N}$ in eq.(5-11). This implies that the astrophysical quantity s(t) assumes values between 0 and $2s_0$.

The upper-bound $2s_0$ is to be regarded as an artefact of the perturbation criterion we adopted for calculating $Q_2(x)$ in eq.(5-25). In order to check the reliability of our treatment, we carried out a numerical simulation of the stochastic system (5-2). A detailed description of the numerical algorithm is available elsewhere (Ferrini et al. (93)). The comparison between the analytical expression for $\bar{p}(x)$ and the result of our simulation is illustrated in Fig.3. We note that the agreement with our predictions is fairly close. The lower-bound for s(t) is correctly recovered while a long tail lingers over the limiting value $2s_0$. Such a constraint is expected to disappear when proceeding further with our perturbation method.

FIGURE 3

We finally summarize the restrictions under which the procedure described above is reliable. Taking into account definition (5-3), we rewrite conditions (5-21) as follows:

$$0 < \left(\frac{r}{a} - \frac{r'}{a'}\right) \ll \frac{f}{a} \ll \left(\frac{f}{a}\right)^2 \frac{a^2}{D} \qquad (5\text{-}27)$$

The conditions previously stated now read:

$$0 < \left(\frac{r}{a} - \frac{r'}{a'}\right) \ll \left(\frac{f}{a}\right)^{1/2} ; \left(\frac{f^2}{D}\right)^{1/2} \qquad (5-28)$$

We notice that these inequalities can be satisfied for many different choices of the physical parameters. We recall that in our perturbation approach we defined only three effective parameters, f/a, γ and $\langle v^2 \rangle$.

TABLE CAPTION

Table I – Langevin systems presented in the literature for the modelling of galactic evolution.

Table II – The resummation procedure is visualized. (a) notation; (b) the counting rule for reckoning $\mathcal{D}_1^{(2n,0)}$ terms contributing to $Q_2(x)$; (c) diagrammatic expression of $Q_2(x)$ (see text).

FIGURE CAPTION

Fig. 1 – s, g, Z vs. t, in the case of constant infall f . s and g are in arbitrary units; unit of time is 10^8 years.

Fig. 2 – s, g, Z vs. t, in the case of exponentially decreasing infall.

Fig.3 – (Stationary) equilibrium distribution for $\bar{p}(s)$. (✱) refer to the result of our numerical simulation for f/a=1, $\langle v^2 \rangle$ =.1, γ=5. The accuracy of our data is evaluated to be about 10%. The solid line represents our theoretical prediction.

TABLE I

	Model	Comments	Ref.
1-	$s(t)+g(t)=1$ $\dot{s}(t)=as(t)g(t)=as(t)[1-s(t)]$	The meaning of s, g, c is in the te the text as well as that of the various rates. Model 4 includes the effects of lag between generations of stars.	88
2-	$\dot{s}(t)=-rs(t)+as(t)g(t)$ $\dot{g}(t)=-dg(t)+f$		
3-	$\dot{s}(t)=-rs(t)+as(t)c(t)+bg(t)^n$ $\dot{c}(t)=-dc(t)-a's(t)c(t)+eg(t)^m$ $\dot{g}(t)=rs(t)+f-eg(t)^m-bg(t)^n$		
4-	$\dot{s}(t)=-\gamma s(t)+\beta s(t)g(t-\Delta t)$ $\dot{g}(t)=\gamma' s(t- \ t)+\mu-\beta s(t- \ t)g(t)$		
5-	$\dot{s}(r,t)=-ds(r,t)+as(r,t)g(r,t)+\eta\nabla^2 s(r,t)$ $g(r,t)=1-s(r,t)$	A diffusion (nonlocal) and a rotation term are introduced.	74
6-	$(\frac{\partial}{\partial t}-\eta\nabla^2)s(\vec{r},t)=-ds(\vec{r},t)+as(\vec{r},t)[1-s(\vec{r},t)]-\Omega\frac{\partial}{\partial\phi}s(\vec{r},t)$ $g(\vec{r},t)=1-s(\vec{r},t)$		
7-	$\dot{s}(t)=-rs(t)+as(t)g(t)$ $\dot{g}(t)=-dg(t)+r's(t)-hs(t)^n+f_o+F(t)$	The model contains spontaneous star formation and stochastic infall.	90

TABLE I (continued)

	MODEL	Comments	Ref.
8-	$\dot{s}(t)=-rs(t)+as(t)g(t)$ $\dot{g}(t)=-dg(t)-ls(t)^{k}+f_{o}+F(t)$	Detailed numerical simulations are performed on these models.	92
9-	$\dot{s}(t)=-rs(t)+as(t)c(t)+bg(t)^{n}$ $\dot{c}(t)=-dc(t)-a's(t)c(t)+eg(t)^{m}$ $\dot{g}(t)=rs(t)-eg(t)^{m}-bg(t)^{n}+f_{o}+F(t)$		
10-	$\dot{s}(t)=-rs(t)+as(t)c(t)$ $\dot{c}(t)=-dc(t)+rs(t)-a's(t)c(t)+f_{o}+F(t)$		
11-	$\frac{\partial}{\partial t}X(t)=aX(t)-X(t)[bY(t)-cY(t)^{2}+dY(t)^{3}]$ $\frac{\partial}{\partial t}Y(t)=-\alpha Y(t)+X(t)[\beta+\gamma Y(t)-\delta Y(t)^{2}]+D\nabla^{2}Y(t)$	This model, as the following ones, is a model for the interstellar medium. X is the density of the interstellar gas, Y the density of interstellar diffusive energy.	75
12-	$\frac{\partial}{\partial t}X(r,t)+v\frac{\partial}{\partial r}X(r,t)=aX(r,t)-X(r,t)[bY(r,t)-cY(r,t)^{2}+dY(r,t)^{3}]$ $\frac{\partial}{\partial t}Y(r,t)+v\frac{\partial}{\partial r}Y(r,t)=-\alpha Y(r,t)+X(r,t[\beta+\gamma Y(r,t)-\delta Y(r,t)^{2}]+D\nabla^{2}Y(r,t)$		105

TABLE I (continued)

	MODEL	Comments	Ref.
13-	$\frac{dX_c}{dt} = AX_w - BX_c X_h^2 + D_c \nabla^2 X_c + \Omega(r)\frac{\partial X_c}{\partial \theta}$ $\frac{dX_h}{dt} = BX_c X_h^2 - X_h X_w + D_h \nabla^2 X_h + \Omega(r)\frac{\partial X_h}{\partial \theta}$ $\frac{dX_w}{dt} = X_h X_w - AX_w + D_w \nabla^2 X_w + \Omega(r)\frac{\partial X_w}{\partial \theta}$	X_c refers to cold clouds, X_w to warm gas and X_h to hot gas.	103

TABLE II

a) $\Sigma = v\frac{\partial}{\partial x}(1+x)$ $\qquad\qquad$ $\Pi = \gamma x\frac{\partial}{\partial v}v$

b)

$$\left.\begin{array}{l} \Sigma\ \Sigma\ \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \\ \Sigma\Pi\Pi\ \Sigma\ \Pi\ \overset{2n-1}{\cdots\cdots}\ \Pi \\ \cdots\cdots\cdots\cdots\cdots \\ \Sigma\ \Pi\ \overset{2n}{\cdots\cdots}\ \Pi\ \Sigma \end{array}\right\} \quad 2n+1 \text{ terms} \propto \frac{\langle v^2\rangle}{\gamma}$$

c)

$$\left\{\begin{array}{l} \Sigma\left[\begin{array}{cc} \Sigma & \\ -\Sigma & \Pi \\ -\Pi & \Sigma \end{array}\right]\begin{array}{l} \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \\ \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \\ \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \end{array} \end{array}\right\} \quad x(2n+1)+$$

$$+\left\{\begin{array}{l} \Sigma\left[\begin{array}{cc} \Sigma & \Pi \\ \Pi & \Sigma \end{array}\right]\begin{array}{l} \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \\ \Pi\ \overset{2n}{\cdots\cdots}\ \Pi \end{array} \end{array}\right\} \quad x2n$$

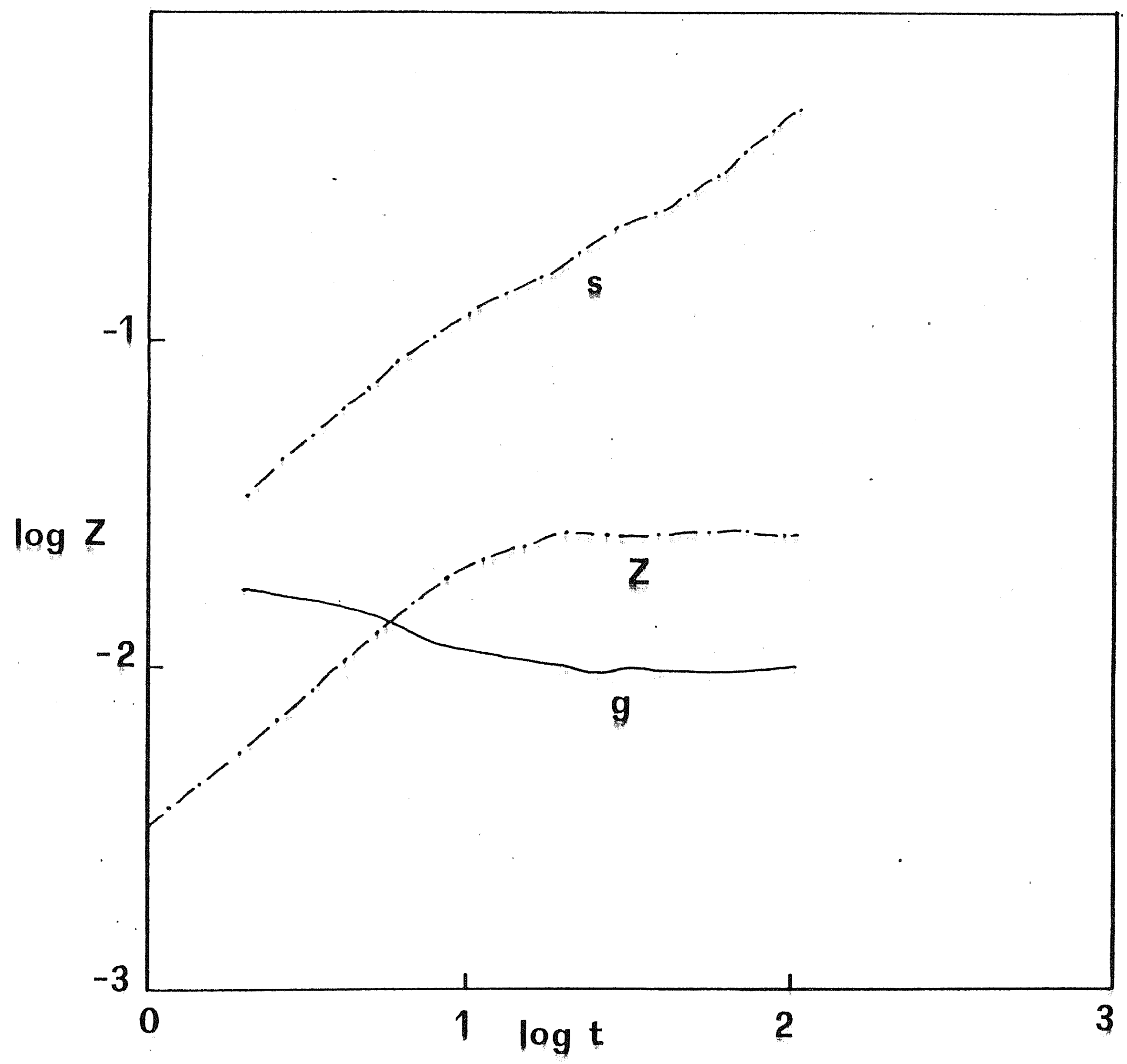

Figure 1

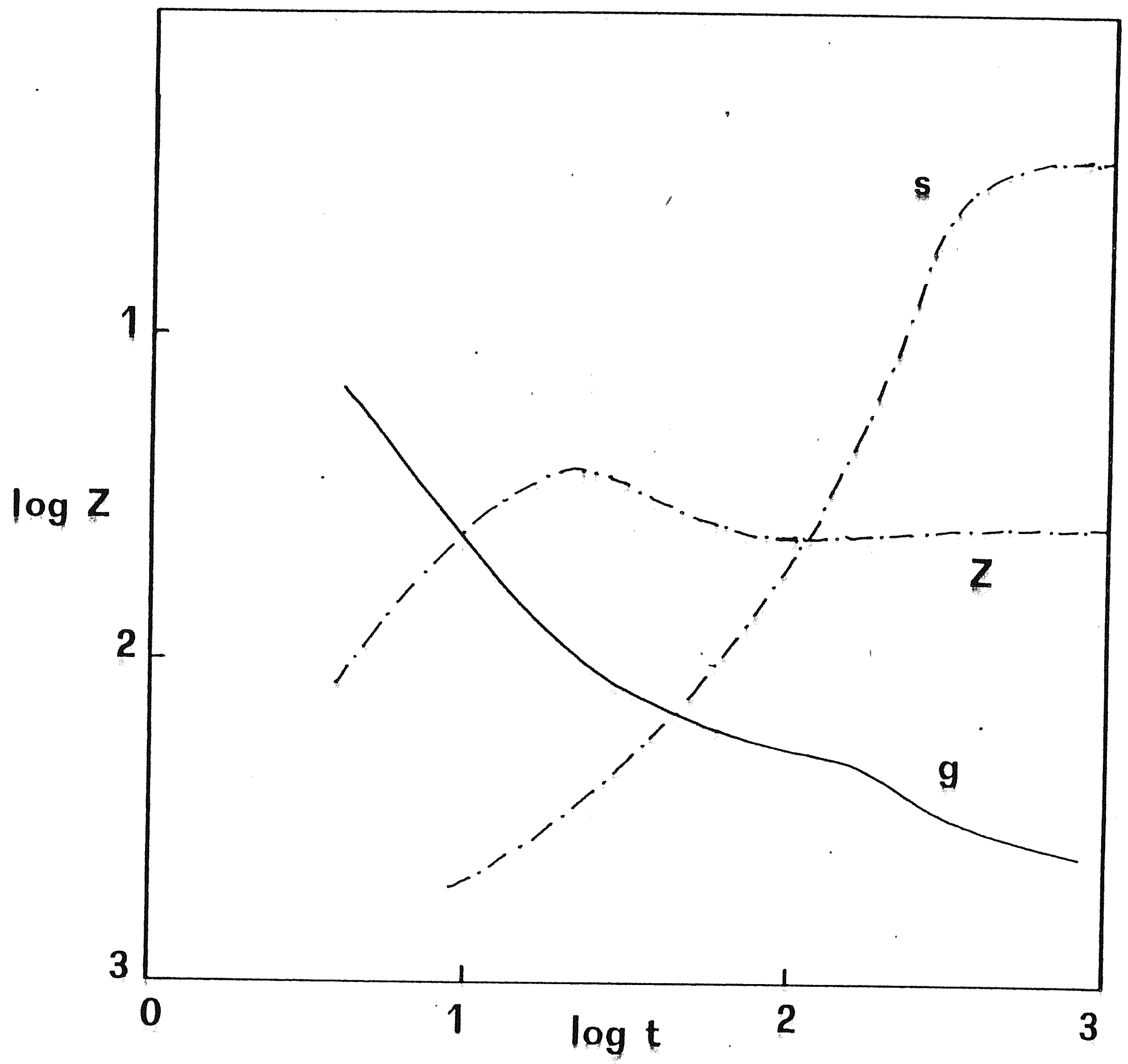

Figure 2

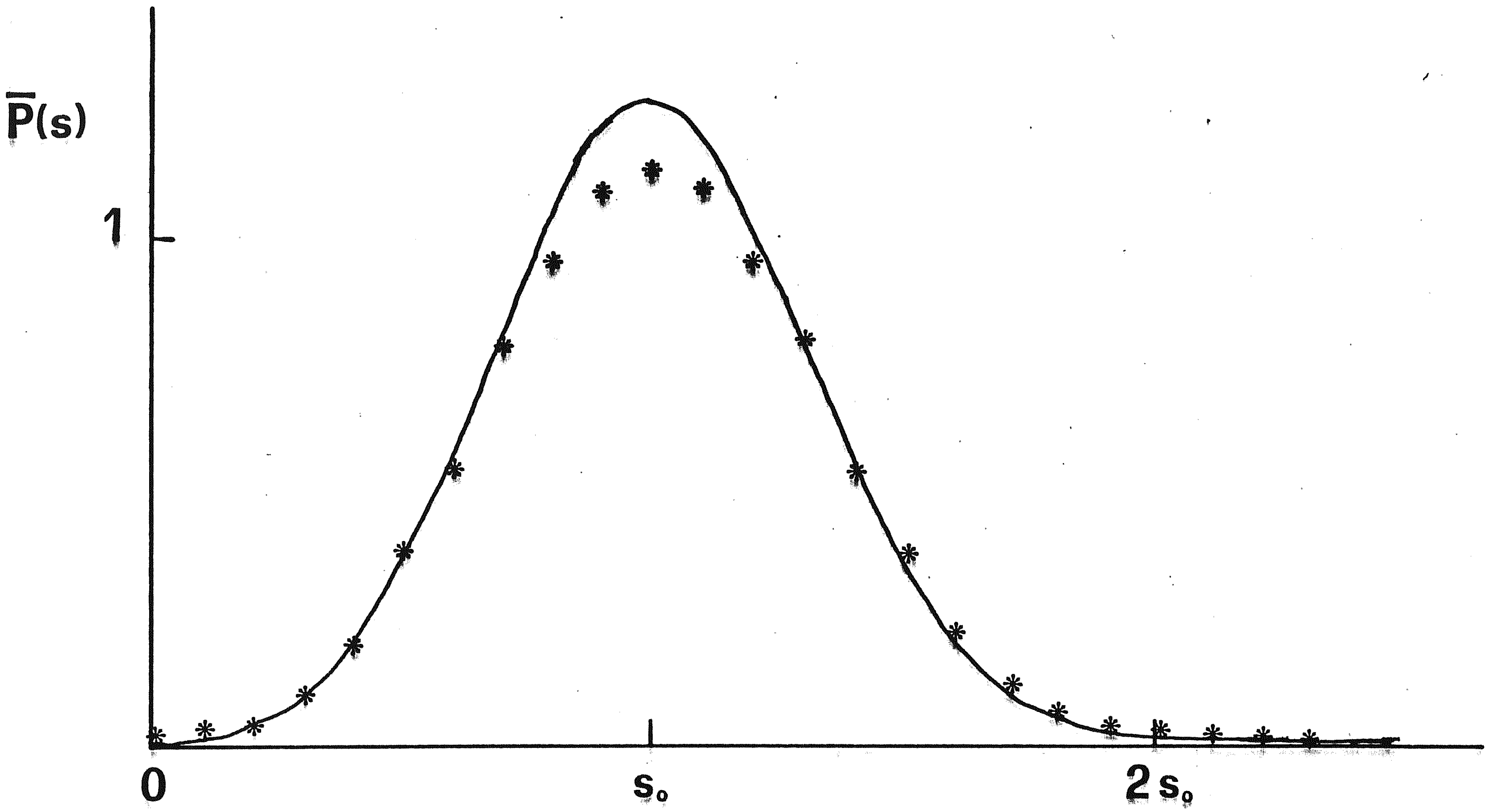

Figure 3

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