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On the heat equation and the spectrum of the Dirichlet Laplacian for spiral regions in R^2.

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Abstract: We determine the asymptotic behaviour of trace $(e^{t\Delta_S})$ as $t \downarrow 0$ where Δ_S is the Dirichlet Laplacian for a class of spiral regions in R^2.

1. Introduction

The relation between the geometry of an open region $W \subset R^n$ and the asymptotic behaviour of the spectrum of the Dirichlet Laplacian Δ_W has received a lot of attention. The classical result of Weyl [1] states that if W is bounded and has volume $|W|$ and the volume $|\partial W|$ of the boundary ∂W is zero then the partition function

$$Z_W(t) = \text{trace}\,(e^{t\Delta_W}) , \quad t>0 \tag{1}$$

satisfies

$$\lim_{t\downarrow 0} Z_W(t) \, . \, t^{n/2} = \frac{|W|}{(4\pi)^{n/2}} \, . \tag{2}$$

The asymptotic distribution of the eigenvalues can be got from (2) by a standard Tauberian theorem. This was of great importance at the time since it settled in the affirmative the conjecture of Lorentz that the leading term in the asymptotic expansion of the eigenvalue distribution depends only on the volume of the region and not on its shape. Weyl used the theory of integral equations; insight into the result was given by Kac [2,14] who, using probabilistic techniques, showed that the diagonal element $K(\vec{x},\vec{x}:t)$ of the heat kernel at $\vec{x}$ associated, to $-\Delta_W + \frac{\partial}{\partial t}$ can be replaced by $\frac{1}{(4\pi t)^{n/2}}$ if $t^{1/2}/\text{dist}\,(x,\partial W)$ is much smaller than one. The asymptotic relation (2) follows by integrating $\frac{1}{(4\pi t)^{n/2}}$ with respect to $\vec{x}$ over W since most points in W satisfy this "principle of not feeling the boundary".

Rozenbljum [3] pointed out that there are regions S having infinite volume for which $Z_S(t)$ is nevertheless finite; the set $S = \{(x,y): |x|^{\mu} \cdot |y| < 1\}, \mu > 0$ in R^2 is an example of such a region. Rozenbljum, Simon and others [3-9] used analytical techniques to find $Z_S(t)$ for $t\downarrow 0$. It is clear that in two of the four horns of S most points "feel the boundary", so that if the probabilistic techniques of Kac are used for this problem stronger estimates are required. These were provided in [10] where the asymptotic behaviour of the partition function was determined for a class of horn-shaped regions in R^n. For example let

$$H = \{ (x,y) : 0 < y < f(x), \; x > 0 \} \quad , \tag{3}$$

where f(x) is a function on $(0,\infty)$ which decreases monotonically to zero and which is integrable at 0; for this region the Brownian bridge was decomposed into two independent bridges. The bridge in the x - direction obeys the principle of not feeling the boundary; in the y - driection the kernel at y for the operator $-\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}$ on the interval [0, f(x)] (with Dirichlet conditions) was computed explicitly. This gave for the diagonal element of the kernel at (x,y)

$$K_H(x,y;t) \sim \frac{1}{(4\pi t)^{1/2}} \cdot \sum_{k=1}^{\infty} \frac{2}{f(x)} \cdot e^{-\frac{t\pi^2 k^2}{f^2(x)}} \left(\sin \frac{\pi k y}{f(x)}\right)^2 , \; t \downarrow 0 \qquad (4)$$

so that

$$Z_H(t) = \int_0^\infty dx \int_0^{f(x)} dy . K_H(x,y;t) \sim \frac{1}{(4\pi t)^{1/2}} \int_0^\infty dx \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{f^2(x)}} , t\downarrow 0 \qquad (5)$$

provided the integral in (5) converges. The proof is to be found in [10].

Subsequently Davies [11] obtained geometrical conditions on a region V which are necessary and sufficient for $Z_V(t)$ to be finite for t>0.

In this paper we use probabilistic techniques to compute $Z_S(t)$ for spiral regions S which covers R^2 except for a set of measure 0. The boundary of S is given by (in polar coordinates)

$$\partial S = \left\{ (\phi, r(\phi)) : 0 \leq \phi < \infty , r(0)=0, r(\infty)=\infty , r(\phi) \text{ concave on } [0,\infty) \right\} . \qquad (6)$$

We will show that for a wide class of concave functions $r(\phi)$

$$Z_S(t) \sim \frac{1}{(4\pi t)^{1/2}} \int_0^\infty d\phi . r(\phi) . \sum_{k=1}^{\infty} e^{-\frac{tk^2}{4(\dot r(\phi))^2}} , t\downarrow 0 \qquad (7)$$

where $\dot{r}(\phi)$ denotes the derivative of $r(\phi)$. The corresponding horn-shaped region H(S) which we obtain by "unrolling" the spiral is of the form (in cartesian coordinates)

$$H(S) = \left\{ (x(\phi), y(\phi)) : x(\phi) = \int_0^{\phi} r(\tau)\, d\tau \;, \; 0 < y(\phi) < 2\pi \dot{r}(\phi),\; 0 < \phi < \infty \right\} . \tag{8}$$

We see from (5), (7) and (8) that

$$\lim_{t \downarrow 0} \; Z_S(t) \, / \, Z_{H(S)}(t) = 1. \tag{9}$$

Hence the Brownian bridge does not feel the curvature in the limit $t \downarrow 0$.

2. The Theorem

We will prove the following:

Theorem: Let r be a real-valued function on $[0,\infty)$ such that

1. $r(0)=0$, $r(\phi)$ is concave on $[0,\infty)$ and $\lim_{\phi \to \infty} r(\phi) = \infty$,
2. F(t) defined by

$$F(t) = \frac{1}{(4\pi t)^{1/2}} \int_0^{\infty} d\phi \, . \, r(\phi) \, . \sum_{k=1}^{\infty} e^{-\frac{t k^2}{4 (\dot{r}(\phi))^2}} \;, \tag{10}$$

is finite for all $t>0$,

3. $$\lim_{t \downarrow 0} F\left(\frac{r^2(\pi) t}{(r(\pi) - 2t^{1/2})^2} \right) \{F(t)\}^{-1} = 1 \;, \tag{11}$$

4. $$\int_0^{\infty} d\phi \, . \, r(\phi) \, e^{-\left\{ \frac{7}{12\pi} \cdot \frac{r(\phi)}{r(\pi)} \cdot \left(\log \frac{r(\phi)}{r(\pi)} \right)^{-1} \right\}^2} \quad \text{is finite,} \tag{12}$$

then

$$\lim_{t \downarrow 0} \frac{1}{F(t)} \, . \, \mathrm{trace} \left(e^{t \Delta_S} \right) = 1 \;, \tag{13}$$

where Δ_S is the Laplacian with Dirichlet conditions on the spiral given by (6).

Corollary: If $\partial S = \{(\phi, c\phi^\alpha) : 0 \leq \phi < \infty\}$ and the constants c and α satisfy $c \neq 0$ and $0<\alpha<1$ then

$$\lim_{t \downarrow 0} \operatorname{trace}\,(e^{t\Delta_S}) \cdot \left(\frac{t}{c^2}\right)^{\frac{1}{1-\alpha}} = \frac{1}{(1-\alpha)\,\pi^{1/2}} \cdot 2^{\frac{3\alpha-1}{1-\alpha}} \cdot \alpha^{\frac{1+\alpha}{1-\alpha}} \cdot \Gamma\left(\frac{1+\alpha}{2-2\alpha}\right) \cdot \zeta\left(\frac{1+\alpha}{1-\alpha}\right) . \tag{14}$$

Remark: Condition 3 in the theorem states that $r(\phi)$ should increase slower than $\dfrac{\phi}{(\log \phi)^{3/2}}$ as $\phi \to \infty$. Condition 4 in the theorem implies that $r(\phi)$ should increase fast enough e.g. like $(\log \phi)^{\frac{1}{2}+\varepsilon}$, $\varepsilon>0$ as $\phi \to \infty$.

In Lemma 1 we give a pointwise upperbound on the diagonal element bf the heat kernel (denoted by $K_S(\phi,r;\,t)$ associated to $-\Delta_S + \frac{\partial}{\partial t}$ using its representation as a conditional Wiener propability [12]. In Lemma 2 we give the lowerbound. Then we obtain upper and lowerbounds on $Z_S(t) = \operatorname{trace}\,(e^{t\Delta_S})$, which prove the theorem.

Lemma 1: For $\phi \geq 4\pi$, $t>0$ and $0<\delta<\frac{\pi}{2}$

$$K_S(\phi, r; t) \leq \frac{1}{(\pi t)^{1/2} q} \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{q^2}} \left(\sin \pi k \cdot \frac{r - r(\phi-\delta)\cos\delta}{q}\right)^2 + \frac{1}{2\pi t} e^{-\frac{p^2}{4t}} , \tag{15}$$

where

$$p = 2r(\phi-\delta) \sin \delta, \tag{16}$$

$$q = r(\phi + 2\pi + \delta) - r(\phi-\delta) \cos \delta. \tag{17}$$

Proof: Let P_1, P_2, P_3 and P_4 be the vertices of a rectangle with boundary ∂P such that (in polar coordinates) $P_1 = (\phi+\delta,\, r(\phi-\delta))$, $P_4 = (\phi-\delta,\, r(\phi-\delta))$, $\angle POP_2 = \angle P_3OP = \tan^{-1} \dfrac{p}{2r(\phi+2\pi+\delta)}$, dist $(P_1,P_2) = q$, dist $(P_1,P_4) = p$, $P = (\phi, r)$ and denote the straight lines through P_1P_2, P_2P_3, P_3P_4 and P_4P_1 by p_1, p_2, p_3

and p_4 respectively. We write $K_S(\phi,r;t)$ as a conditional Wiener probability and use the property that the paths $P(\tau)$, $0<\tau<t$ are continuous with probability 1. Let θ denote the empty set, then

$$K_S(\phi,r;t) = \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\partial S=\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$= \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\partial S=\theta,\ P(\tau)\cap\partial P=\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$+ \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\partial S=\theta,\ P(\tau)\cap\partial P\neq\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$\leq \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\partial P=\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$+ \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\partial S=\theta,\ P(\tau)\cap\{P_1P_2\cup P_3P_4\}\neq\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$\leq \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\{p_2\cup p_4\}=\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$+ \frac{1}{4\pi t}\,\mathrm{Prob}\left\{ P(\tau)\cap\{p_1\cup p_3\}\neq\theta,\ 0\leq\tau\leq t \mid P(0)=P(t)=P\right\}$$

$$= \frac{1}{(\pi t)^{1/2}\cdot q}\cdot\sum_{k=1}^{\infty} e^{-\frac{t\pi^2k^2}{q^2}}\left(\sin\pi k.\frac{r-r(\phi-\delta)\cos\delta}{q}\right)^2 + \frac{1}{4\pi t}\left\{1-\frac{4.(\pi t)^{1/2}}{p}\sum_{k=0}^{\infty} e^{-\frac{t\pi^2(2k+1)^2}{p^2}}\right\}, \quad (18)$$

from which Lemma 1 follows if we use the Poisson summation formula (see [13]) ∎

<u>Lemma 2</u>: For $\phi\geq 4\pi$, $t>0$, $0<\gamma<\frac{\pi}{2}$ and r such that $r(\phi+\gamma)\leq r\leq r(\phi+2\pi-\gamma)\cos\gamma$,

$$K_S(\phi,r;t)\geq \frac{1}{(\pi t)^{1/2}\cdot s}\sum_{k=1}^{\infty} e^{-\frac{t\pi^2k^2}{s^2}}\left(\sin\pi k.\frac{r(\phi+2\pi-\gamma)\cos\gamma-r}{s}\right)^2 - \frac{1}{2\pi t}\,e^{-\frac{u^2}{4t}}, \quad (19)$$

where

$$u = 2r(\phi + 2\pi - \gamma)\sin\gamma, \tag{20}$$

$$s = r(\phi + 2\pi - \gamma)\cos\gamma - r(\gamma+\phi). \tag{21}$$

Proof: Let Q_1, Q_2, Q_3 and Q_4 be the vertices of a rectangle with boundary ∂Q such that $Q_2 = (\phi + 2\pi + \gamma, r(\phi + 2\pi - \gamma))$, $Q_3 = (\phi + 2\pi - \gamma, r(\phi + 2\pi - \gamma))$, $\angle POQ_1 = \angle Q_4OP = \tan^{-1}\frac{u}{2r(\phi+\gamma)}$, dist $(Q_2, Q_3) = u$, dist $(Q_1, Q_4) = s$, $P = (\phi, r)$ and denote the straight lines through Q_1Q_2, Q_2Q_3, Q_3Q_4 and Q_4Q_1 by q_1, q_2, q_3 and q_4 respectively. We have

$$K_S(\phi,r;t) \geqslant \frac{1}{4\pi t}\,\mathrm{Prob}\left\{P(\tau)\cap\partial Q = \theta,\ 0\leq\tau\leq t \,\middle|\, P(0)=P(t)=P\right\}$$

$$\geqslant \frac{1}{4\pi t}\,\mathrm{Prob}\left\{P(\tau)\cap\{q_1\cup q_3\} = \theta,\ P(\tau)\cap\{q_2\cup q_4\} = \theta,\ 0\leq\tau\leq t \,\middle|\, P(0)=P(t)=P\right\}$$

$$\geqslant \frac{1}{4\pi t}\,\mathrm{Prob}\left\{P(\tau)\cap\{q_2\cup q_4\} = \theta,\ 0\leq\tau\leq t \,\middle|\, P(0)=P(t)=P\right\}$$

$$- \frac{1}{4\pi t}\left(1 - \mathrm{Prob}\left\{P(\tau)\cap\{q_1\cup q_3\} \neq \theta,\ 0\leq\tau\leq t \,\middle|\, P(0)=P(t)=P\right\}\right)$$

$$= \frac{1}{(\pi t)^{1/2}s}\sum_{k=1}^{\infty} e^{-\frac{t\pi^2k^2}{s^2}}\left(\sin\pi k.\frac{r(\phi+2\pi-\gamma)\cos\gamma - r}{s}\right)^2$$

$$- \frac{1}{4\pi t}\left(1 - \frac{4.(\pi t)^{1/2}}{p}\sum_{k=0}^{\infty} e^{-\frac{t\pi^2(2k+1)^2}{u^2}}\right), \tag{22}$$

from which (19) follows (see [13]) ■

Proof of Theorem: We will find bounds on $Z_S(t)$ for $0<t\leqslant r^2(\pi)$. Choose δ and γ in Lemma 1 and Lemma 2 both ϕ and t dependent:

$$\delta = \left\{\frac{3\pi^2 t}{4r^2(\phi-\frac{\pi}{2})}\log\frac{r(\phi-\frac{\pi}{2})}{r(\pi)}\right\}^{1/2}, \tag{23}$$

$$\gamma = \left\{ \frac{3\pi^2 t}{4 r^2(\phi + \frac{3\pi}{2})} \log \frac{r(\phi + \frac{3\pi}{2})}{r(\pi)} \right\}^{1/2} . \tag{24}$$

Let $\tilde{\phi}_t$ be the unique positive solution of

$$r(\tilde{\phi}) = \left\{ r^2(\pi)\, t\, F(t) \right\}^{1/4} , \tag{25}$$

and define

$$\phi_t = \max\left\{4\pi, \tilde{\phi}_t\right\}, \quad \psi_t = \max\left\{4\pi, \tilde{\phi}_t - 2\pi\right\} , \tag{26}$$

$$K = \left\{ \phi : \phi \geq \phi_t ,\ 2\pi\, \dot{r}(\phi - \tfrac{\pi}{2}) \leq \left(2\delta\, \dot{r}(\phi - \tfrac{\pi}{2}) + \frac{r(4\pi)}{r(\frac{7\pi}{2})} \cdot \frac{\delta^2}{2} \cdot r(\phi - \tfrac{\pi}{2}) \right) \cdot \frac{r(\pi)}{t^{1/2}} \right\} , \tag{27}$$

$$L = \left\{ \phi : \phi \geq \phi_t ,\ 2\pi\, \dot{r}(\phi - \tfrac{\pi}{2}) \geq \left(2\delta\, \dot{r}(\phi - \tfrac{\pi}{2}) + \frac{r(4\pi)}{r(\frac{7\pi}{2})} \cdot \frac{\delta^2}{2} \cdot r(\phi - \tfrac{\pi}{2}) \right) \cdot \frac{r(\pi)}{t^{1/2}} \right\} , \tag{28}$$

$$M = \left\{ \phi : \phi \geq \psi_t ,\ 2\pi\, \dot{r}(\phi + 2\pi) \leq \left(2\gamma\, \dot{r}(\phi + 2\pi) + \frac{\gamma^2}{2} \cdot r(\phi + 2\pi) \right) \cdot \frac{r(\pi)}{t^{1/2}} \right\} , \tag{29}$$

$$N = \left\{ \phi : \phi \geq \psi_t ,\ 2\pi\, \dot{r}(\phi + 2\pi) \geq \left(2\gamma\, \dot{r}(\phi + 2\pi) + \frac{\gamma^2}{2} \cdot r(\phi + 2\pi) \right) \cdot \frac{r(\pi)}{t^{1/2}} \right\} . \tag{30}$$

First we will use

$$0 \leqslant K_S(\phi,r;t) \leqslant \frac{1}{4\pi t}\ , \tag{31}$$

and then we will use Lemma 1.

$$Z_S(t) = \int_0^{2\pi} d\phi \int_0^{\infty} dr.\ r\ K_S(\phi, r; t)$$

$$= \int_0^{2\pi} d\phi \left\{ \int_0^{r(\phi+2\pi)} dr.\ r K_S(\phi, r; t) + \sum_{n=1}^{\infty} \int_{r(\phi+2\pi n)}^{r(\phi+2\pi n+2\pi)} dr.\ r.\ K_S(\phi, r; t) \right\}$$

$$\leqslant \int_0^{2\pi} d\phi \int_0^{r(4\pi)} dr.\ r.\ K_S(\phi, r; t) + \sum_{n=1}^{\infty} \int_{2\pi n}^{2\pi n+2\pi} d\phi \int_{r(\phi)}^{r(\phi+2\pi)} dr.\ r.\ K_S(\phi, r; t)$$

$$\leqslant \frac{r^2(4\pi)}{4t} + \frac{r^2(\phi_t)}{4t} + \int_{\phi_t}^{\infty} d\phi \int_{r(\phi)}^{r(\phi+2\pi)} dr.\ r.\ K_S(\phi, r; t)$$

$$\leqslant \frac{r^2(4\pi)}{4t} + \frac{r(\pi)}{4}\left(\frac{F(t)}{t}\right)^{1/2} + \int_{\phi_t}^{\infty} d\phi \int_{r(\phi)}^{r(\phi+2\pi)} dr.\ \frac{r}{(\pi t)^{1/2}\cdot q} \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{q^2}} \Big(\sin \pi k\ \frac{r - r(\phi-\delta)\cos\delta}{q}$$

$$+ \int_{\phi_t}^{\infty} d\phi \int_{r(\phi)}^{r(\phi+2\pi)} dr.\ \frac{r}{2\pi t}\ e^{-\frac{p^2}{4t}}$$

$$\equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\ . \tag{32}$$

Because of (16), (17) and condition 1 of the theorem we have for $\phi \geqslant 4\pi$

$$p \geqslant \frac{4\delta}{\pi}\ r(\phi-\delta) \geqslant \frac{4\delta}{\pi}\ r(\phi - \tfrac{\pi}{2})\ , \tag{33}$$

$$q \leqslant (2\pi+2\delta)\,\dot r(\phi-\delta) + \frac{\delta^2}{2}\ r(\phi-\delta) \leqslant (2\pi+2\delta)\,\dot r(\phi-\tfrac{\pi}{2}) + \frac{\delta^2 r(4\pi)}{2\,r(\frac{7\pi}{2})}\ .\ r(\phi-\tfrac{\pi}{2})\ . \tag{34}$$

By (23) and condition 1 of the theorem we get

$$\mathrm{II} \leqslant \int_{4\pi}^{\infty} d\phi \,.\, r(\phi+2\pi) \,.\, \frac{1}{2\pi t} \,.\, e^{-\frac{p^2}{4t}} \int_{r(\phi)}^{r(\phi+2\pi)} dr$$

$$\leqslant \frac{1}{t} \int_{4\pi}^{\infty} d\phi \,.\, \dot{r}(\phi) \,.\, r(\phi+2\pi) \,.\, e^{-\frac{4\delta^2}{\pi^2 t} r^2(\phi-\frac{\pi}{2})}$$

$$\leqslant \frac{1}{t} \int_{4\pi}^{\infty} d\phi \,.\, \dot{r}(\phi) \,.\, r(\phi+2\pi) \,.\left(\frac{r(\pi)}{r(\phi-\frac{\pi}{2})} \right)^3$$

$$\leqslant \frac{r^3(\pi)}{t} \int_{\frac{7\pi}{2}}^{\infty} d\phi \,.\, \frac{\dot{r}(\phi-\frac{\pi}{2})}{r^2(\phi-\frac{\pi}{2})} \,.\, \sup_{\phi \geqslant 4\pi} \,.\, \frac{r(\phi+2\pi)}{r(\phi-\frac{\pi}{2})}$$

$$= \frac{1}{t} \,.\, \frac{r(6\pi) \,.\, r^3(\pi)}{r(\frac{9\pi}{2}) \,.\, r(3\pi)} \,. \tag{35}$$

Furthermore

$$\mathrm{III} \leqslant \int_{\phi_t}^{\infty} d\phi \int_{r(\phi-\delta)\cos\delta}^{r(\phi+2\pi+\delta)} dr \,.\, r(\phi+2\pi) \,.\, \frac{2}{(\pi t)^{1/2} \,.\, q} \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{q^2}} \left(\sin \pi k \,.\, \frac{r - r(\phi-\delta)\cos\delta}{q} \right)^2$$

$$= \frac{1}{(4\pi t)^{1/2}} \int_{\phi_t}^{\infty} d\phi \,.\, r(\phi+2\pi) \,.\, \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{q^2}}$$

$$\leqslant \frac{1}{(4\pi t)^{1/2}} \int_{\phi_t}^{\infty} d\phi \,.\, r(\phi+2\pi) \,.\, \sum_{k=1}^{\infty} \exp - \frac{t\pi^2 k^2}{\left\{ (2\pi+2\delta)\, \dot{r}(\phi-\frac{\pi}{2}) + \frac{\delta^2 r(4\pi)}{2\, r(\frac{7\pi}{2})} \,.\, r(\phi-\frac{\pi}{2}) \right\}^2} \,. \tag{36}$$

Since for $\phi \geqslant \phi_t \geqslant 4\pi$

$$2\delta \,.\, \frac{r(\pi)}{t^{1/2}} \leqslant \pi \,.\, \frac{r(\pi)}{r(\phi-\frac{\pi}{2})} \,.\left\{ 3 \log \frac{r(\phi-\frac{\pi}{2})}{r(\pi)} \right\}^{1/2} < \pi \tag{37}$$

we see that if $\phi \in K$ then

$$(2\pi+2\delta)\, \dot{r}(\phi-\tfrac{\pi}{2}) + \frac{\delta^2\, r(4\pi)}{2\, r(\frac{7\pi}{2})} \,.\, r(\phi-\tfrac{\pi}{2}) \leqslant 2\,.\, \frac{r(4\pi).\, r(\pi)}{r(\frac{7\pi}{2}).\, t^{1/2}}\, \delta^2 .\, r(\phi-\tfrac{\pi}{2}) \tag{38}$$

If we denote the contribution from $\phi \in K$ in the right hand side of (36) by V and define

$$\alpha(\phi) = \left\{ \frac{2r(\frac{7\pi}{2}).\, r(\phi-\frac{\pi}{2})}{r(4\pi).\, r(\pi).\, 3\pi.\, \log\left(r(\phi-\frac{\pi}{2})/r(\pi)\right)} \right\}^2 \tag{39}$$

we get by (23) and (39)

$$\mathrm{V} \leqslant \int_{\phi_t}^{\infty} d\phi .\, r(\phi+2\pi) \,.\, \frac{1}{(4\pi t)^{1/2}} \sum_{k=1}^{\infty} e^{-\alpha(\phi)k^2} . \tag{40}$$

Since for $\alpha > 0$

$$\sum_{k=1}^{\infty} e^{-\alpha k^2} \leqslant e^{-\alpha}\left(1-e^{-3\alpha}\right)^{-1} , \tag{41}$$

and for $\phi \geqslant \pi$

$$\alpha(\phi) \geqslant \left(\frac{2}{3\pi} \cdot \frac{r(\frac{7\pi}{2})}{r(4\pi)}\right)^2 \geqslant \left(\frac{7}{12\pi}\right)^2 \tag{42}$$

we obtain finally

$$\mathrm{V} \leqslant \frac{1}{(4\pi t)^{1/2}} \cdot \frac{r(6\pi)}{r(4\pi)} \left\{ 1 - e^{-\frac{49}{48\pi^2}} \right\}^{-1} \int_0^{\infty} d\phi .\, r(\phi).\, \exp - \left\{ \frac{7 r(\phi)}{12\pi\, r(\pi)\, \log\left(r(\phi)/r(\pi)\right)} \right\}^2 . \tag{43}$$

Furthermore

$$r(\phi + 2\pi) \leq r(\phi - \tfrac{\pi}{2}).(1 + \frac{5\pi}{2\phi - \pi}), \tag{44}$$

so that the contribution from $\phi \in L$ in the right hand side (36) is smaller than

$$\int_{\phi_t}^{\infty} d\phi \,.\, r(\phi + 2\pi) . \frac{1}{(4\pi t)^{1/2}} \sum_{k=1}^{\infty} \exp - \frac{tk^2}{4\left\{r(\phi - \frac{\pi}{2}).(1 + \frac{t^{1/2}}{r(\pi)})\right\}^2}$$

$$\leq \left(1 + \frac{5\pi}{2\phi_t - \pi}\right) . \quad F\left(\frac{r^2(\pi)t}{(r(\pi) + t^{1/2})^2}\right) . \tag{45}$$

We note that since $tF(t) \uparrow \infty$ as $t \downarrow 0$ we have $\phi_t \uparrow \infty$ as $t \downarrow 0$. Because of (32), (35) (36), (43), (45), (11) and (12) we have proved $\limsup_{t \downarrow 0} Z_S(t) / F(t) = 1$. The next step is to obtain a lowerbound on $Z_S(t)$ for $0 < t \leq r^2(\pi)$ using Lemma 2 and (31)

$$\begin{aligned} Z_S(t) &\geq \int_{\psi_t}^{\infty} d\phi \int_{r(\phi)}^{r(\phi + 2\pi)} dr \,.\, r \,.\, K_S(\phi, r; t) \\ &\geq \int_{\psi_t}^{\infty} d\phi \,.\, r(\phi) \int_{r(\phi + \gamma)}^{r(\phi + 2\pi - \gamma)\cos\gamma} dr \,.\, K_S(\phi, r; t) \\ &\geq \frac{1}{(4\pi t)^{1/2}} \int_{\psi_t}^{\infty} d\phi \,.\, r(\phi) \sum_{k=1}^{\infty} e^{-\frac{t\pi^2 k^2}{s^2}} \\ &\quad - \frac{1}{2\pi t} \int_{\psi_t}^{\infty} d\phi . \left(r(\phi + 2\pi) - r(\phi)\right) . e^{-\frac{4\gamma^2}{\pi^2 t} r^2(\phi + \frac{3\pi}{2})} \equiv \mathrm{VI} - \mathrm{VII} . \end{aligned} \tag{46}$$

By (24) we get for VII

$$\mathrm{VII} \leq \frac{1}{2\pi t} \int_{\psi_t}^{\infty} d\phi \left(r(\phi + 2\pi) - r(\phi)\right) . \left(\frac{r(\pi)}{r(\phi + \frac{3\pi}{2})}\right)^3 \leq \frac{r^3(\pi)}{t\, r(4\pi)} . \tag{47}$$

Since

$$s \geqslant (2\pi - 2\gamma)\,\dot r(\phi + 2\pi) - \frac{\gamma^2}{2}\, r(\phi+2\pi) \quad , \tag{48}$$

we obtain

$$\mathrm{VI} \geqslant \frac{1}{(4\pi t)^{1/2}} \int_N d\phi \,.\, r(\phi) \sum_{k=1}^{\infty} \exp - \frac{t\pi^2 k^2}{\left\{(2\pi - 2\gamma)\,\dot r(\phi+2\pi) - \frac{\gamma^2}{2} r(\phi+2\pi)\right\}^2}$$

$$\geqslant \frac{1}{(4\pi t)^{1/2}} \int_N d\phi \,.\, r(\phi) \sum_{k=1}^{\infty} \exp - \frac{tk^2}{\left\{2\,\dot r(\phi+2\pi)\,(1 - t^{1/2}/r(\pi)\right\}^2}$$

$$\geqslant \frac{1}{(4\pi t)^{1/2}} \int_{\psi_t}^{\infty} d\phi \,.\, r(\phi+2\pi)\left(1 - \frac{2\pi}{\phi}\right) \sum_{k=1}^{\infty} \exp - \frac{tk^2}{\left\{2\dot r(\phi+2\pi)\,.\,(1 - t^{1/2}/r(\pi))\right\}^2}$$

$$- \frac{1}{(4\pi t)^{1/2}} \int_M d\phi \,.\, r(\phi) \sum_{k=1}^{\infty} \exp - \frac{tk^2}{\left\{2\dot r(\phi+2\pi)(1 - t^{1/2}/r(\pi)\right\}^2} \equiv \mathrm{VIII} - \mathrm{IX} \,. \tag{49}$$

We have

$$\mathrm{VIII} \geqslant F\left(\frac{r^2(\pi)t}{(r(\pi) - t^{1/2})^2}\right) - \frac{2\pi}{\psi_t} F(t) - \frac{1}{4\pi t}\int_{-2\pi}^{\psi_t} \dot r(\phi+2\pi)\, r(\phi+2\pi)\, d\phi$$

$$= F\left(\frac{r^2(\pi)t}{(r(\pi) - t^{1/2})^2}\right) - \frac{2\pi}{\psi_t} F(t) - \frac{r(\pi)}{4}\cdot\left(\frac{F(t)}{t}\right)^{1/2} - \frac{r^2(6\pi)}{4t} \,. \tag{50}$$

Since $2\gamma \dfrac{r(\pi)}{t^{1/2}} \leqslant \pi$ we have

$$2\pi\, \dot r(\phi+2\pi) \leqslant \gamma^2\, r(\phi+2\pi)\,.\,\frac{r(\pi)}{t^{1/2}} \quad , \qquad \phi \in M,\ 0 < t \leqslant r^2(\pi) \,, \tag{51}$$

so that

$$\mathrm{IX} \leq \frac{1}{(4\pi t)^{1/2}} \int_M d\phi \, . \, r(\phi) \sum_{k=1}^{\infty} \exp -\left\{ \frac{tk\pi}{r(\pi) \; \gamma^2 \; r(\phi+2\pi)} \right\}^2 . \tag{52}$$

If we put

$$\beta(\phi) = \left\{ \frac{2\pi t}{r(\pi)\gamma^2 r(\phi+2\pi)} \right\}^2 = \left\{ \frac{4 \; r^2(\phi+\frac{3\pi}{2})}{3\pi \, r(\pi) \, r(\phi+2\pi)} \left(\log \, r(\phi+\tfrac{3\pi}{2})/r(\pi)\right)^{-1} \right\}^2 \tag{53}$$

then

$$\beta(\phi) \geq \left\{ \frac{11 \, r(\phi+\frac{3\pi}{2})}{9\pi \, r(\pi) \; \log \, r(\phi+\frac{3\pi}{2})/r(\pi)} \right\}^2 \geq \left(\frac{11}{9\pi} \right)^2 , \tag{54}$$

so that by (41)

$$\mathrm{IX} \leq \frac{1}{(4\pi t)^{1/2}} \left\{ 1 - e^{-\frac{121}{27\pi^2}} \right\}^{-1} \int_0^{\infty} d\phi \, . \; r(\phi) \exp -\left\{ \frac{11 \, r(\phi)}{9\pi \, r(\pi) \; \log \, (r(\phi)/r(\pi))} \right\}^2 . \tag{55}$$

Since $\psi_t \uparrow \infty$ as $t \downarrow 0$ and because of (46), (47), (49), (50), (55), (11) and (12) we have proved $\liminf\limits_{t \downarrow 0} Z_S(t) \,/\, F(t) = 1$. This completes the proof of the Theorem.

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