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BISTABLE FLOW DRIVEN BY COLOURED GAUSSIAN NOISE:

A CRITICAL CASE STUDY

by

P. Hanggi[+], F. Marchesoni[§], P. Grigolini[+]

[+]Dipartimento di Fisica, Università di Pisa, and
GNSM del CNR, P.za Torricelli 2, I-56100-Pisa (Italy)

[§]Dublin Institute for Advanced Studies,
10 Burlington Road, Dublin 4, (Ireland).

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## ABSTRACT

A one-dimensional flow driven by additive Gaussian noise is considered. Possible pitfalls in the use of Langevin equations of non-Markovian processes are identified. In particular, the stationarity of the random force in a non-Markovian Langevin equation does not imply that the dynamics of the non-Markovian process is consistent with an initial stationary preparation scheme. The small relaxation time Fokker-Planck approximation schemes put forward in the recent literature are applied and it is pointed out that those schemes may lead to unreliable predictions. The results for the activation rate as evaluated from the approximative Fokker-Planck schemes does not coincide in leading order in $\tau$ (Arrehnius factor) with a computer simulation for small noise intensity: thus showing that the wings of the stationary probability $\bar{p}(x)$ are in leading order in $\tau$ not recovered from the approximative Fokker-Planck schemes.

## 1. Introduction

There has been recent interest in non-linear systems subjected to external noise with a finite correlation time. In many situations the influence of a finite correlation time $\tau$ on the dynamics of a macroscopic variable plays a minor role such that an approximative Markovian theory, e.g. a Fokker-Planck description, modelling the statistical macroscopic flow is justified|1,2|. On the other hand there exist cases where the influence of the bath on the macroscopic flow of an order parameter must be modelled with a coloured noise source|3-6|. A well-known example of this kind of situation is the phenomenon of motional narrowing in magnetic resonance. Kubo|4,7| has shown that a very short correlation time of the fluctuating magnetic field yields a vanishing effect on the motion of the spin; on the contrary, if the fluctuations of the field are large and correlated over a long time scale, the motion of the spin is greately modified. Another important example is the relevant influence of the correlated noise on the activation rates in equilibrium systems|8-11| and in driven non-equilibrium systems|12|.

Generally the finite correlation of the noise will have an effect on the form of the stationary dynamics. This fact has been exploited in recent studies of the so-called coloured noise induced transitions|13-17|. Because the underlying dynamics are governed by a non-Markovian process, the exact master equation can be obtained in special cases only|5,12-14,18|. Thus one generally must invoke an approximation procedure such as the small relaxation time Fokker-Planck approximation scheme put forward by the Barcelona group|14-15,19-20|. Moreover it has been pointed out previously on several occasions|1,5,21-23| that within a non-Markovian dynamics the initial preparation procedure as reflected in the statistical properties of the correlated noise|23|, is of equal importatnce as the macro-dynamical law generated from an initial probability $p_0$ of the macro-variables. Therefore caution must be exercised in interpreting correctly the statistical information of statistical quantities calculated by means of a generalized Langevin equation or master equation dynamics|5,24|.

Some of those pitfalls will be investigated in the following Sections for the example of overdamped particle motion in a symmetric double-well driven by a coloured Gaussian noise source. In Section 2 we investigate the problem of initial preparation more closely and clear up possible misconceptions about content of statistical information in the use of the non-Markovian

Langevin equation. Section 3 contains the results of the Fokker-Planck approximation schemes of the Barcelona group|14,15,19| and possible shortcomings of those approximation schemes are pointed out. The activation rate is considered in Section 4. The activation rate is evaluated by employing the perturbation Fokker-Planck approximation schemes and it is compared with the exact results of a computer simulation.

## 2. Effects of preparation: The example of a bistable stochastic flow.

In what follows we consider a stochastic flow of a one-dimensional order parameter x(t). We assume a symmetric bistable flow modelled by the set of stochastic differential equations:

$$\dot{x} = a x - b x^3 + \xi \qquad a,b > 0 \tag{2.1a}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi + \eta(t) \quad . \tag{2.1b}$$

$\eta$ (t) is a stationary Gaussian white noise source of zero mean and correlation function

$$\langle \eta(t)\,\eta(s)\rangle = \frac{2D}{\tau^2}\,\delta(t-s) \quad . \tag{2.1c}$$

Integration of (2.1b) yields with $\xi(t_0=0) = \xi_0$

$$\xi(t) = \xi_0\, e^{-t/\tau} + \int_0^t \exp[-(t-s)/\tau]\,\eta(s)\,ds \quad . \tag{2.2}$$

Assigning to the first two moments of $\xi_0$ the values given by the equations

$$\langle \xi_0 \rangle = 0 \tag{2.3a}$$

$$\langle \xi_0^2 \rangle = D/\tau \quad , \tag{2.3b}$$

we find

$$\langle \xi(t) \rangle = 0 \tag{2.4}$$

and for the auto-correlation function the time-homogeneous result

$$\langle \xi(t)\, \xi(s) \rangle = \frac{D}{\tau} \exp\left[ -|t-s|/\tau \right] \tag{2.5}$$

Moreover $\xi(t)$ is <u>Gaussian and stationary only if</u> prepared in Gaussian form consistent with (2.3a) and (2.3b), i.e.

$$\rho(\xi) = \left( \frac{\tau}{2\pi D} \right)^{1/2} \exp\left[ - \frac{\xi_0^2}{(2D/\tau)} \right] \quad . \tag{2.6}$$

Thus the system of differential equations in (2.1) is equivalent to a non-Markovian Langevin equation driven with additive Gaussian correlated noise

$$\dot{x} = ax - bx^3 + \xi(t) \tag{2.7}$$

and $\xi(t)$ obeying the properties in (2.4) and (2.5). Because $\langle \xi(t) \rangle = 0$, the deterministic limit $(D \to 0)$ of (2.7) is clearly given by

$$\dot{x} = ax - bx^3 = f(x) \tag{2.8}$$

which is also derivable from a potential $\dot{x} = -\frac{\partial}{\partial x} V(x)$ with

$$V(x) = -\frac{a}{2} x^2 + \frac{b x^4}{4} . \tag{2.9}$$

Next let us address the question of initial preparation|1,5,21-23| of the total system at time $t_0=0$. Let $\rho_0^T(x, \xi_0)$ denote the initial probability for the total Markovian system in (2.1). Then the initial probability $p_0(x)$ of the macro-variable is given by

$$p_0(x) = \int \rho_0^T(x, \xi_0)\, d\xi_0 . \tag{2.10}$$

In virtue of (2.6) we have on the other hand <u>independent of $p_0(x)$</u>

$$\int \rho_0^T(x, \xi_0)\, dx = \rho_0(\xi_0) \equiv \rho(\xi_0) \tag{2.11}$$

In other words the (conditional) probability $W(\xi_0 | x(0)=x_0)$|23| for the bath variable $\xi_0$ , given a fixed macroscopic value $x(0)=x_0$, equals

$$W(\xi_0 | x(0)=x_0) = \frac{\rho_0^T(x, \xi_0)}{p_0(x)} = \rho(\xi_0) \tag{2.12}$$

and is normalized on the surface $S(x(0)=x_0)$, i.e.

$$\int dx\, d\xi_o\, W(\xi_o \mid x(0) = x_o)\, \delta(x - x_o) = 1. \qquad (2.13)$$

From (2.12) we immediately derive that the chosen initial preparation procedure is consistent with (2.11) and being independent of $p_o(x)$ implies the factorization

$$\rho_o^{I}(x, \xi_o) = p_o(x)\, \rho(\xi_o), \qquad (2.14)$$

i.e. a correlation-free preparation procedure|23|. The 'stationary preparation class', on the other hand, is characterized by $W_s(\xi_o \mid x(0)=x_o)$

$$W_s(\xi_o \mid x(0) = x_o) = \frac{\bar{\rho}(x, \xi_o)}{\bar{p}(x)}, \qquad (2.15)$$

where $\bar{\rho}$ and $\bar{p}$ denote the stationary probabilities of the dynamics $(x(t), \xi(t))$ and $x(t)$, respectively. Now it is readily seen from the corresponding Fokker-Planck dynamics of (2.1) that $\bar{p}(x, \xi_o)$ does not factorize. For example, setting $b=0$ and $a < 0$ one finds|25| (Z denotes the normalization)

$$\bar{p}(x, \xi_o; b=0) = Z^{-1} \exp\left[ -\frac{x^2}{2} \left( \frac{D}{a(a\tau+1)} \right)^{-1} \right] \cdot$$

$$\cdot \exp\left[ -\frac{(\xi_o - ax)^2}{2} \left( \frac{D}{\tau(a\tau+1)} \right)^{-1} \right] \qquad (2.16)$$

which does not factorize. Therefore (2.15) and (2.12) define different initial preparation procedures. In particular, setting $p_0(x)=\bar{p}(x)$, i.e. $\rho_0^T(x,\xi_0)=\rho(\xi_0)\bar{p}(x)$, the process generated from the non-Markovian dynamics (2.7) is not stationary.

Moreover initial correlation functions given by the equation

$$\langle g(x(t))\, f(x(0))\rangle = \int \bar{p}(x)\,\langle g(x(t))\,|\, f(x(0))=f(x)\rangle\, dx \qquad (2.17)$$

are not to be identified with the stationary correlation functions given by

$$\langle g(x(t))\, f(x(0))\rangle_s = \langle g(x(t+s))\, f(x(s))\rangle_s \quad . \qquad (2.18)$$

These equal the initial correlation functions generated with $p_0=\bar{p}(x)$ as determined by the stationary preparation scheme of eq.(2.15). With the correlation free preparation, (2.18) is obtained in the limit of long times only, $s\to\infty$ . This possible pitfall in the abuse of approximative Fokker-Planck schemes has been recently discussed by the Barcelona group|24| from a different point of view. We recast such remark in an alternative language because we think it important to persuade the reader out of the erroneous feeling|26| that preparation is irrelevant to the <u>steady state dynamics</u> of the non-Markovian processes.

## 3. Approximative Fokker-Planck dynamics

It is well-known |5,14-16| that the exact closed form master equation for a non-linear flow driven by coloured Gaussian noise, of the type in (2.7) with $b \neq 0$ and $a > 0$ (whose structure does not depend on the choice of initial macroprobability $p_0(x)$) cannot be derived. Neglecting transients we see that the master equation for the flow in (2.7) has the structure

$$\dot{p}_t(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\partial}{\partial x}\right)^n K_n(x)\, p_t(x) \tag{3.1}$$

Because we have neglected transients, (3.1) can be utilized only in the evaluation of quantities determined by asymptotic long time dynamics such as the stationary probability|20| or a mean first passage time(MFPT) at weak noise. Assuming that the correlation time $\tau$ of the Gaussian noise source $\xi(t)$, (2.5), is small the Barcelona group put forward a short relaxation time Fokker-Planck approximation (SRTFPA)|14,15,19|. If we apply the SRTFPA to our flow in (2.7), we find

SRTFPA: $$\dot{p}_t(x) = -\frac{\partial}{\partial x}(ax - bx^3)\, p_t(x) +$$

$$+ D \frac{\partial^2}{\partial x^2} \left( 1 + \tau [a - 3bx^2] \right) P_t(x) \quad . \tag{3.2}$$

This approximation can be improved by summing up the terms $D\tau^n$ which occur in a small time expansion|14,15| of the functional derivative relation|5| which in our case reads

$$\frac{\delta x(t)}{\delta \xi(t')} = \theta(t-t') \left\{ 1 + \int_{t'}^{t} ds \, [a - 3b\, x^2(s)] \frac{\delta x(s)}{\delta \xi(t')} \right\} \tag{3.3a}$$

$$= \theta(t-t') \exp \int_{t'}^{t} ds \, [a - 3bx^2(s)] \quad . \tag{3.3b}$$

Summing up all term $D\tau^n$, $n \geqslant 0$, one obtains the 'best Fokker-Planck approximation' (BFPA) of the Barcelona school|15| which in our case reads:

$$\text{BFPA:} \quad \dot{P}_t(x) = - \frac{\partial}{\partial x} (ax - bx^3) P_t(x) + D \frac{\partial^2}{\partial x^2} H(x,\tau) P_t(x) \tag{3.4}$$

With $f(x)=ax-bx^3$, $H(x,\tau)$ is given by

$$H(x,\tau) = f(x) \left( 1 + \tau f(x) \frac{\partial}{\partial x} \right)^{-1} \frac{1}{f(x)} =$$

$$= f(x) \left( 1 - \tau f(x) \frac{\partial}{\partial x} + \tau^2 f(x) \frac{\partial}{\partial x} f(x) \frac{\partial}{\partial x} + \dots \right) \frac{1}{f(x)} = \tag{3.5}$$

$$= 1 + \tau f'(x) + \tau^2 \left[ (f'(x))^2 - f(x) f''(x) \right] + O(\tau^3) \quad .$$

At the extrema $\bar{x}=\{x_1, x_u, x_2\}$ of the deterministic flow (2.8) one finds (prime denotes derivation after x)

$$H(\bar{x}) = 1 + f'\tau + (f')^2\tau^2 + (f')^3\tau^3 + \dots$$
$$= (1 - f'(\bar{x})\tau)^{-1} . \tag{3.6}$$

Therefore at the locally stable states $x_1$, $x_2$, $x_{1,2} = \mp(a/b)^{1/2}$, we have

$$H(x_1) = H(x_2) = (1 + 2a\tau)^{-1} \tag{3.7}$$

and at the locally unstable state $x_u=0$

$$H(x_u) = (1 - a\tau)^{-1} . \tag{3.8}$$

The equations (3.2) and (3.4) have the structure of a Fokker-Planck equation. However the diffusion term in (3.2), (3.4) is not necessarily positive for all the values of x |15|. The stationary solution $\bar{p}(x)$ of (3.2), (3.4) must of course always stay positive. If we set $\bar{p}(x)=0$ in the region of possible negative diffusion, H(x), the stationary probability $\bar{p}(x)$ of the BFPA (3.4) reads

$$\bar{p}(x) = \frac{Z^{-1}}{H(x)} \left\{ \exp - \frac{1}{D}\int_0^x \frac{f(y)}{H(y)}\, dy \right\} \theta\{H(x)\} , \tag{3.9}$$

where the Heaviside step function $\theta\{H(x)\}$ guarantees a positive support of $\bar{p}(x)$. Here and in the following we assume that the correlation time $\tau$ is small enough such that $\bar{p}(x)$ is nonvanishing within the interval $|x_1, x_2|$ of the bistable region.

A crucial question to be asked in this context is the following: Does (3.4) give good approximation to the (unknown) stationary probability $\bar{p}(x)$ as determined via (3.1)? In other words, is $\bar{p}(x)$ in (3.9) actually accurate up to first order terms in the correlation time $\tau$ ? Clearly $\bar{p}(x)$ in (3.9) cannot be correct up to terms of order $\tau^2$ and higher, because term of order $O(\tau^2)$ contribute to the Kramers-Moyal moments $K_n$, $n \geqslant 1$, in (3.1). In this context a recent paper written on this issue contains an incorrect statement|27|: In ref.15 it has been claimed that the coefficients $K_1$ and $K_2$ in (3.1) 'contain only terms with coefficients $D\tau^n$, $n \geqslant 0$'. From (3.3a) however we have for example

$$\frac{d^2}{ds^2} \left. \frac{\delta x(t)}{\delta \xi(s)} \right|_{s \to t} = \left\{ (f'(x(t)))^2 - f(x(t))\, f''(x(t)) \right\} - f''(x(t))\,\xi(t) \quad (3.10)$$

If inserted into small $\tau$-expansion|15|, we obtain for the master operator in (3.1) a term

$$D\tau^2 \frac{\partial^2}{\partial x^2} \Big\{ \left[ (f'(x))^2 - f(x)\, f''(x) \right] P_t(x) \qquad (3.11)$$

$$+ \frac{D}{\tau} f''(x) \frac{\partial}{\partial x} \int_0^\infty \left\langle \frac{\delta x(t)}{\delta \xi(s)}\, \delta(x(t) - x) \right\rangle \exp\left(-\frac{|t-s|}{\tau}\right) ds \Big\}$$

Approximating the term $\delta x(t) / \delta \xi(s)$ occurring in (3.11) by 1 - see (3.3) - one finds from the last term in (3.11)

$$D^2 \tau^2 \frac{\partial^2}{\partial x^2} f''(x) \frac{\partial}{\partial x} P_t(x) \quad . \tag{3.12}$$

This term obviously contributes both to $K_1(x)$ and $K_2(x)$ in (3.1) if the differential operators $\left(\frac{\partial}{\partial x}\right)$ are rearranged into the Kramers-Moyal form of eq.(3.1). There is generally an infinite hierarchy of infinite many terms, $D^m \tau^n$ $(m \leq n,\ n \geq 2)$, contributing to $K_1(x)$, $K_2(x)$ and higher order Kramers-Moyal moments. Only if we neglect simply all terms which yield also a contribution to a Kramers-Moyal moment $K_n(x)$ of order $n > 2$ are the coefficients $K_1(x)$, $K_2(x)$ solely determined by the terms $D\tau^n$, yielding (3.4).

Assuming knowledge of the full master operator in (3.1), one would look for a long time Fokker-Planck approximation to the master equation dynamics. Such a scheme has recently been put forward for the case that (3.1) describes a Markovian master equation dynamics |28|. An important finding has been that a Fokker-Planck approximation to the long time master equation dynamics must contain information of higher order Kramers-Moyal moments $K_n(x)$, $n > 2$. Thus it is not clear a priori to what extent the approximation in (3.4), (3.9) gives a correct description of the tails of $\bar{p}(x)$ if viewed in function of the correlation time $\tau$ . The influence of the infinite many series with terms $D^m \tau^n$ , $m \leq n$, $n > 1$, (not a single series!) together with the influence of higher order Kramers-Moyal moments $K_n(x)$, $n > 2$, likely will yield renormalized first and second Kramers-Moyal moments with a different small relaxation behaviour. The approximation in (3.7), (3.9) has been checked in special cases|14| (transformations of Gaussian processes) and also in refs.15 and 20 by use of numerical and analogue simulations respectively. The approximation yields satisfactory results for quantities like stationary moments, location of maxima, etc.|14,15|, very much like the truncated Kramers-Moyal Fokker-Planck approximation to a master equation can yield satisfactory results despite the fact that $\bar{p}(x)$ is not exact.

A physical quantity which sensitively probes the form of the stationary probability $\bar{p}(x)$, in particular in the region where $\bar{p}(x)$ is very small, is the escape rate. By use of a transport theory approach|12,28| to the master equation dynamics in (3.1), the Arrehnius factor of the rate is solely determined by the leading order of the stationary probability|12,28,29|, i.e. details of the jump-statistics and boundary conditions reflect themselves only in the prefactor and not in the Arrehnius factor of the rate expression.

4. Activation rates

The activation rates of bistable flows present interesting physical quantities which depend crucially on the detailed form of the stationary probability $\bar{p}(x)$. In particular the minimum of $\bar{p}(x)$ around $x=x_u=0$ determines essentially the Arrehnius factor of the activation rates. Most naturally one would like to evaluate the rate via a transport theory approach of the type used for dichotomic Markov noise|12,29b|. In the absence of an exact master equation (3.1) modelling the long time behaviour of x(t), this approach is of no use here. Alternatively we could evaluate the MFPT at weak noise of the underlying two-dimensional Fokker-Planck dynamics in (2.1)|30,31|. However, because a detailed balance does not hold for (2.1), the standard methods|32-34| fail and the more general method of refs.30 and 31 is rather cumbersome because the stationary probability $\bar{\rho}(x, \xi)$ must first be determined perturbatively. If T denotes the MFPT to reach the barrier top, the activation rate is estimated as

$$r = 1/2T \tag{4.1}$$

where the factor 1/2 takes into account that the random walker has equal chance to either continue to the adjacent stable state or return to the old stable state.

The approximations in (3.2), (3.4) of the Barcelona group are meant to be useful small relaxation time approximations for the stationary dynamics. In what follows we then look upon (3.2), (3.4) as a Fokker-Planck approximation to the long time dynamics of the (unknown) master equation dynamics (3.1). Then, within the assumption of a small enough relaxation time $\tau$, yielding a positive diffusion $D(x)=DH(x)$ within the bistable region $|x_1, x_2|$, the MFPT $T(x)$ can be readily evaluated|35,36|. If $x=-\infty$ is a (natural) reflecting boundary and $x=x_u=0$ an absorbing state, one finds |35,36| for the MFPT $T(x)$ of a walker which started out at $x(0)=x < 0$

$$T(x) = \int_x^0 \frac{dy}{\bar{p}(y)\, D(y)} \int_{-\infty}^{y} \bar{p}(z)\, dz \ . \tag{4.2}$$

$\bar{p}(x)$ denotes the stationary probability of the corresponding Fokker-Planck equation (3.4), (3.2) respectively and $D(x)$ is the corresponding diffusion coefficient, i.e. $D(x)=DH(x)$ or $D(X)=D|1+ \tau(a-3bx^2)|$ if (3.2) is utilized. For a weak noise, i.e. $D < a^2/b$, we can evaluate (4.2) by use of the method of steepest descendent. With the BFPA, (3.4), one obtains

$$T = \frac{\pi}{a\sqrt{2}} \left( \frac{1 + a\tau}{1 - 2a\tau} \right)^{1/2} \exp\left( \Delta\phi / D \right) \qquad (4.3a)$$

where with (3.5)

$$\Delta\phi = \int_0^{x_1} \frac{f(y)\,dy}{H(y)} = \int_0^{-(a/b)^{1/2}} f(y)\left[1 - \tau f'(y) + \tau^2 f'(y) f''(y) + O(\tau^3)\right] dy$$

$$= \frac{a^2}{4b}\left(1 - a^2\tau^2\right) + O(\tau^3). \qquad (4.3b)$$

Most importantly the term linear in $\tau$ vanishes exactly. By use of (3.2) one finds instead

$$T = \frac{\pi}{a\sqrt{2}} \left( \frac{1 + 2a\tau}{1 - a\tau} \right)^{1/2} \exp\left( \Delta\phi / D \right) \qquad (4.4a)$$

with

$$\Delta\phi = \int_0^{x_1} \frac{f(y)\,dy}{1 + \tau f'(y)} = \frac{a^2}{4b}\left(1 - a^2\tau^2/2\right) + O(\tau^3). \qquad (4.4b)$$

Because (3.2) takes into account only the term of order $D\tau$, the term of order $\tau^2$ in (4.4b) is, of course, meaningless. Again the term of order $\tau$ in $\Delta\phi$ vanishes. For $\tau=0$ both results (4.3) and (4.4) coincide and r=1/2T equals the well-known Smoluchowski rate |35,36|.

Most importantly we note that the Arrehnius factor of T(x)

$$\exp\left( \Delta\phi / D \right) \qquad (4.5)$$

does not exhibit a correlation time dependence in first order in $\tau$. Based on the SRTFPA in (3.2) and ref.16, the Barcelona group constructs an approximation for the stationary probability of (3.2) of the form|15|

$$\bar{p}(x) = p_o(x) + \tau\, p_1(\tau) + O(\tau^2) \tag{4.6}$$

where $p_o(x)$ is the white noise stationary probability $p_o(x)$, which in our case reads

$$p_o(x) = Z^{-1} \exp\left(-\frac{-ax^2/2 + bx^4/4}{D}\right) \tag{4.7}$$

By use of their result in (2.23) of ref.15, one obtains in our case (2.7)

$$\bar{p}(x) = p_o(x)\left\{1 + \tau\left[c - (a - 3bx^2) - \frac{1}{2D}(ax - bx^3)^2\right]\right\} \tag{4.8}$$

where

$$c = -\frac{1}{2D}\int_{-\infty}^{+\infty} (ax - bx^3)^2\, p_o(x)\, dx < 0 . \tag{4.9}$$

This approximation for $\bar{p}(x)$ is not necessarily positive. Considering (4.8) as a short-time approximation in first order, the bracket in (4.8) is now exponentiated|15|. This *ad hoc*|15| exponentiation guarantees a positive $\bar{p}(x)$ which explicitly reads

$$\bar{p}(x) = p_o(x) \exp\left(- \frac{D\tau \left[|c| + (a - 3bx^2) + \frac{1}{2D}(ax - bx^3)^2\right]}{D}\right). \quad (4.10)$$

If we take (4.10) seriously, we would obtain an Arrehnius factor of T given by

$$\exp\left(\Delta\phi/D\right) = \exp \frac{a^2/4b - 2D\tau a}{D} = \exp(-2\tau a)\cdot\exp\left(\frac{a^2/4b}{D}\right) \quad (4.11)$$

In this case the Arrehnius factor does exhibit a dependence on $\tau$, but in the form of a mere prefactor correction, $\exp(-2a\tau)$. The results in (4.3), (4.4) and (4.11) are in clear contrast to a result found for symmetric dichotomic noise where the Arrehnius factor increases with increasing correlation time $\tau$ |12|. In view of the absence of a term proportional to $\tau$ in the Arrehnius factors (4.3), (4.4), (4.11) we performed a numerical simulation for T based on the bistable flow (2.7). The results are given in fig.1. In contrast to our forecastings in (4.3), (4.4) and (4.11) $\Delta\phi$ is increasing with increasing correlation time $\tau$. The increase is proportional to first order in $\tau$ and is not dependent on the small noise parameter D. These results imply the following conclusions:

(i) Because the calculated Arrehnius factors disagree with the simulation, the Fokker-Planck approximation schemes in (3.2), (3.4) cannot be correct in leading order in $\tau$ if viewed as a long time approximation to the master equation dynamics

(3.1). In other words the stationary probability $\bar{p}(x)$, (3.9), of (3.4) cannot be equal to the exact stationary probability $\bar{p}(x)$ in leading order in $\tau$ as determined from (2.7) or, equivalently, from the (unknown) master equation dynamics (3.1).

(ii) The approximation in (4.11) based on the ad hoc exponentiation scheme of (4.6) can also not be correct in leading order in $\tau$ if compared with the exact probability. This fact is not remedied if (4.6) is used in connection with (3.4) instead of (3.2); the correlation time $\tau$ is merely substituted by a 'renormalized' correlation time $\tau_R$ |15|.

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# FIGURE CAPTION

Fig. 1 - $\Delta\phi$, defined in eq.(4.4a), versus the noise auto-correlation time $\tau$. The computer simulation of eqs.(2.1) has been carried out by applying the numerical algorithm of ref.15 with an integration step of 0.01. The values of the parameters are a=b=1, D=0.1( ● ) and D=0.05( ▲ ). $T$ is the average over 1000 first passage times occurred from the initial conditions (2.6) for $\xi_0$ and $p(x,0)=\delta(x-x_{\pm})$ for $x$. The maximum error bar in our numerical simulation is estimated to be about 10%.

Fig 1
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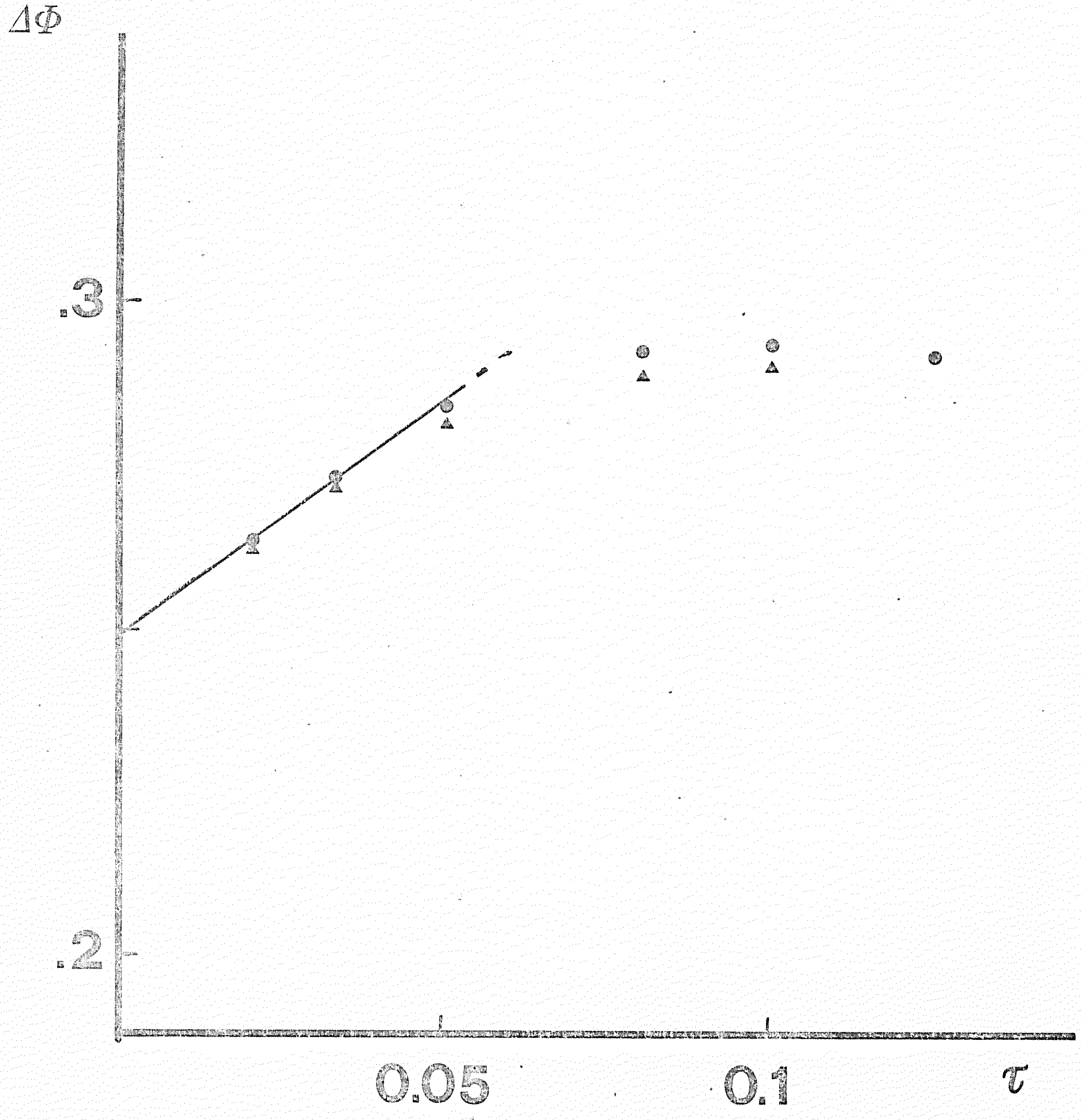

Peter Hänggi,
Dept. of Physics,
Polytechnic Institute of New York,
333 Jay Street,
Brooklyn N.Y. 11201,
(U.S.A.).

Fabio Marchesoni,
Dipartmento di Fisica,
Università di Perugia,
Via Elce di Sotto,
I-06100 Perugia,
(Italy).

Paolo Grigolini,
Dipartmento di Fisica,
and GNSM - CNR,
P.za Torricelli 2,
I-56100, Pisa,
(Italy).