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BOSE-EINSTEIN CONDENSATION OF FREE PHOTONS

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Abstract: A grand-canonical description of a photon gas implies Bose-Einstein condensation above Planck's mean photon number density of black body radiation. For a finite reflecting cavity approximately all excess photons would occupy the Dirichlet ground state thereby forming a monochromatic radio wave.

1. Introduction: It is common to describe black body radiation in terms of a canonical ensemble with unconstrained number of particles. One can ask the question what, in the absence of a black body, would be the implications of a grand-canonical photon equilibrium where the particle number density could be fixed independent of the temperature? As a consequence Bose-Einstein condensation is inescapable if one chooses the second approach, in contrast to the first one. For a rigorous discussion of Einstein condensation in a general free Bose gas we refer to van den Berg et al. [1].

We consider an increasing sequence of finite smooth cavities C(R), with volume V(R), spheres or parallelepipeds say, labelled by some characteristic length R (radius, edge length). We assume Dirichlet boundary conditions (reflecting walls). The photon hamiltonian H(R), relevant for thermal equilibrium, is [2]

$$H(R) = \sum_p \hbar\omega_p b_p^* b_p, \quad [b_p, b_{p'}^*] = \delta_{pp'}, \quad [b_p, b_{p'}] = 0;$$

p=(k,e) runs over the energy states k of the cavity C(R), and the two helicity states e. Let $\varepsilon(R,p) = \varepsilon(1,p)/R > 0$ be the eigenvalues of the single-photon hamiltonian h(R), the restriction of H(R) to the one-particle subspace of the Fock space. The integrated spectral density F_R of h(R) is uniquely given by

$$\Phi_R(\beta) := \sum_p e^{-\beta\lambda(R,p)} = \int_0^\infty e^{-\beta\lambda}\, dF_R(\lambda), \quad \lambda(R,p) := \varepsilon(R,p) - \varepsilon(R,1);$$

asymptotically it is $F_R(\lambda) = (\lambda/\hbar c)^3/(3\pi^2) + O(\frac{\lambda^2}{R})$ [3].

2. Grand-canonical approach: We calculate the infinite volume limit where the inverse temperature β is given, and the grand-canonical mean particle number density

$$\rho(R,\beta,\mu) = \sum_p (e^{\beta(\lambda(R,p)-\mu)} - 1)^{-1} / V(R)$$

is assigned a fixed value $\bar{\rho}$, thus making μ dependent on β, $\bar{\rho}$, and R: $\rho(R,\beta,\mu(R,\beta,\bar{\rho})) := \bar{\rho}$. To use the results in [1], we define

$$\gamma(R,\beta) := \sum_p e^{-\beta V(R)\lambda(R,p)} \quad ; \quad \rho^1(R,\beta,x) := 2\,(e^{-\beta x} - 1)^{-1}/V(R),\ -x \in \mathbb{R}^+;$$

$$\rho_m(\beta) := \lim_{x\to\infty} \lim_{R\to\infty} \int_{x/V(R)}^{\infty} (e^{\beta\lambda} - 1)^{-1}\, dF_R(\lambda);$$

$$\rho_c(\beta) := \lim_{R\to\infty} \int_0^{\infty} (e^{\beta\lambda} - 1)^{-1}\, dF_R(\lambda).$$

Since

$$\lim_{R\to\infty} \Phi_R(\beta) = (2/\pi^2)(\beta\hbar c)^{-3}, \quad \lim_{R\to\infty} \gamma(R,\beta) = 2,$$

$$\rho_c(\beta) = \rho_m(\beta) = (2/\pi^2)(\beta\hbar c)^{-3} g_3(1), \quad g_\alpha(z) := \sum_{n=1}^{\infty} z^n/n^\alpha,$$

the following theorem is an immediate consequence of [1]:

Theorem: Given $\bar{\rho}, \beta \in \mathbb{R}^+$. Define $\mu(\beta,\bar{\rho})$ to be zero for $\bar{\rho} > \rho_c(\beta)$, and to be the unique real root of

$$(2/\pi^2)(\beta\hbar c)^{-3} g_3(e^{\beta\mu}) = \bar{\rho}, \quad \text{for } \bar{\rho} \leq \rho_c(\beta).$$

Then:

(i) $\lim_{R\to\infty} \mu(R,\beta,\bar{\rho}) = \mu(\beta,\bar{\rho});$

(ii) the limit of the grand-canonical pressure is

$$(1/\beta)(2/\pi^2)(\beta\hbar c)^{-3} g_4(e^{\beta\mu(\beta,\bar{\rho})});$$

(iii) the ground state occupation density is given by

$$\lim_{R\to\infty} \rho^1(\beta,\mu(R,\beta,\bar{\rho})) = (\bar{\rho} - \rho_c(\beta))^+,$$

where $(x)^+$ is the positive part of x.

3. Discussion: Above the critical number density $\rho_c(\beta)$ the excess photon density $\bar{\rho} - \rho_c(\beta)$ occupies the ground state. Since, in the thermodynamical limit, the "condensate" does not contribute to the pressure which is one third of the energy density, there is a paradox. - The thermodynamic limit provides an approximative description of a photon gas in a finite region. Condensation in this situation would mean that, for a cubic cavity with 1 m edge length say, it is possible to form a 2 m radio wave in the cavity by an arbitrary large amount of photons without disturbing appreciately the temperature of the photon gas. It would be interesting to know if this has any consequences which can be detected experimentally.

References:

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