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TIME-DEPENDENT VORTICES AND MONOPOLES*)

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ABSTRACT

We discuss the relevance of global existence proofs. The underlying mathematical theory is outlined, and it is shown how the additional problems in the case of vortices and monopoles can be overcome. Ways of building on the existence proofs are indicated.

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INTRODUCTION

One of the most impressive achievements in the study of classical topolog-ically nontrivial solutions has been the construction of static multi-monopole solutions. Since Forgács, Horvath and Palla[1), Ward[2) and Nahm[3) have shown, using different techniques, how to construct multi-monopole solutions, the theory of static monopole solutions has been completed by filling in missing details and unifying different aspects.[4)

With the theory of static monopoles in satisfactory shape it is natural to turn to time-dependent monopoles. For the special case of slowly-moving monopoles a particularly interesting idea to describe a time-dependent process has been put forward by Manton [5). Manton suggested that the scattering of slowly-moving monopoles in the Prasad-Sommerfield limit should be studied by finding the metric and the geodesics in the parameter space of static multi-monopole configurations. Atiyah and Hitchin [6), by an indirect method, have found the metric for the two-monopole solution. The most interesting result of their work is that monopoles can get converted into dyons.

Another possible step beyond static solutions consists in giving existence proofs, both locally and if possible globally. We have taken this step for vortices [7) as well as for monopoles [8). Our work involves no approximations and should therefore be relevant as an underpinning to all approximation techniques. As the example of general relativity shows, where regular initial data can develop a singularity (black hole), a global existence proof, which guarantees that this does not happen, is also interesting in itself.

SEGAL'S THEOREM

The global existence proofs for vortices and monopoles are based on Segal's existence theory for semi-linear evolution equations.[9) Segal, first, considers the integral equation

$$u(t) = W(t,0)u_o + \int_0^t W(t,s)\, K_s(u(s))ds, \tag{1}$$

and shows that, under suitable conditions,

$$u_{n+1}(t) = W(t,0)u_o + \int_0^t W(t,s)\, K_s(u_n(s))ds \tag{2}$$

is a Cauchy sequence, and the limit satisfies eq. (1). The conditions (i) $u \in B$, where B is a Banach space, (ii) W(t,s) is a linear continuous propagator on B, and (iii) $K_t(u): [0,\infty) \times B \to B$ is continuous and satisfies the Lipschitz condition, are sufficient to guarantee a unique continuous solution locally. The difficult part of the local proof is to show that $K_t(u) \in B$ and that $K_t(u)$ satisfies the Lipschitz condition.

To guarantee the existence of a global solution one has to show that $||u(t)||$ does not diverge at any finite time t. This is the most difficult part of our proofs. After it has been achieved it is simple to go back to the differential equation

$$\frac{d}{dt}u = Au + K_t(u), \tag{3}$$

corresponding to the integral equation (1). All one has to do is to prove that $K_t(u)$ is a C^1 map which is easier than to prove that $K_t(u)$ is a C^0 map when $K_t(u)$ is a polynomial in u. Then, assuming sufficient smoothness for the initial data, global existence of a unique solution of eq. (3) follows.

ϕ^4 THEORY

Instead of applying the general theory immediately to the cases we are really interested in and be faced right away with complex technical difficulties, let us study part of the problems which have to be overcome using ϕ^4 theory as a simple model. If there is no symmetry breaking,

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}K(u(s))ds \tag{4}$$

with

$$u = \begin{pmatrix} \phi \\ \pi=\partial_t\phi \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \partial_x^2-m^2 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 \\ m^2\phi-\phi^3 \end{pmatrix}, \tag{5}$$

is the integral equation to be solved in 1+1 space-time dimensions.

For any integral equation, we are guided in our choice of Banach space by the known results for the associated linear equation (K=0). It is known that A given in (5) generates a 1-parameter semigroup on each Sobolev space $H_{s+1} \times H_s$ $(s \geq 0)$, where $f \in H_s$ if f and all its derivatives up to order s are square integrable. Hence, $H_1 \times L^2$ (s=0) should be our first choice and we have to show that

$$||u||^2 = ||\phi||^2_{L^2} + ||\partial_x\phi||^2_{L^2} + ||\pi||^2_{L^2} < \infty \qquad (6)$$

implies

$$||K(u)|| < \infty \qquad (7)$$

and

$$||K(u) - K(\mu)|| \leq C\, ||u - \mu||. \qquad (8)$$

For K given in (5), the inequalities (7) and (8) hold true if

$$||\phi||_{L^6} \leq K\, ||\phi||_{H_1} \quad \text{and} \quad ||\phi||_{L^3} \leq K\, ||\phi||_{H_1} \qquad (9)$$

hold. The inequalities are special cases of the Nirenberg-Gagliardo inequalities,

$$||\phi||_{f^p} \leq K\, ||D^m f||^a_{L^r} ||f||^{1-a}_{L^q} , \qquad (10)$$

where $1/p = a[(1/r) - (m/n)] + (1-a)(1/q)$, $0 \leq a \leq 1$, (if $m-n/r$ is a non-negative integer, only $a < 1$ is allowed) and $f : R^n \longrightarrow R^k$. This completes the local existence proof for the integral equation (4), and assuming sufficient smoothness for the initial value data, the local existence proof for the corresponding differential equation, as well.

That the local proof works already for $H_1 \times L^2$ (s=0) simplifies the global existence proof sonsiderably. Energy conservation,

$$\frac{d}{dt}\int_{-\infty}^{\infty} \{\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_x\phi)^2 + \tfrac{1}{4}\phi^4\}dx = 0 , \qquad (11)$$

alone guarantees that $||\pi||_{L^2}$ and $||\partial_x\phi||_{L^2}$ do not blow up. What is left is to study the L^2-norm of ϕ. To do this define

$$\bar{E} = E + \tfrac{1}{2}\int_{-\infty}^{\infty} \phi^2 dx , \qquad (12)$$

and calculate

$$\frac{d\bar{E}}{dt} = \int_{-\infty}^{\infty} \dot{\phi}\pi dx \leq C\sqrt{\bar{E}}. \tag{13}$$

The inequality (13) implies

$$\bar{E} \leq (C_0 + \tfrac{1}{2}Ct)^2 , \tag{14}$$

which is a global bound on the L^2-norm of ϕ.

In the case of spontaneous symmetry breaking,

$$\partial_t^2\phi = \partial_x^2\phi + \phi(1 - \phi^2) \tag{15}$$

is the relevant differential equation, and the solutions with asymptotic behaviour

$$\phi \longrightarrow \pm 1 \quad \text{for } x \longrightarrow \pm\infty \tag{16}$$

are the ones we are interested in. Because of this asymptotic behaviour, ϕ is not in L^2, and we cannot repeat the proof given above literally.

The natural remedy in this situation is to subtract a background field ϕ^o:

$$\phi = \phi^o + f \;,\; \partial_t\phi^o = 0 \;,\; \phi^o \longrightarrow \pm 1 \quad \text{for } x \longrightarrow \pm\infty, \tag{17}$$

and work with the field f. On ϕ^o, we try to impose enough conditions to prove existence for f, making sure at the same time that the topologically nontrivial configurations we are most interested in are still allowed as fields ϕ. This can be done successfully for the model (15). Furthermore, we can keep control of the topology because $f \varepsilon H_1$ implies $f \longrightarrow 0$ for $x \longrightarrow \pm\infty$, and therefore

$$\lim_{x \to \pm\infty}\phi(t) = \lim_{x \to \pm\infty}\phi(0). \tag{18}$$

VORTICES AND MONOPOLES

The Landau-Ginsburg model in 2+1 space-time dimensions poses a number of serious additional problems. Fortunately, the most difficult ones of these can be solved by imitating Moncrief's proof for the Maxwell-Klein-Gordon theory.[11)] In the Landau-Ginzburg model we have to solve the differential equation (3) for

$$u^T = (a_o, \partial_t a_o, a_1, \partial_t a_1, a_2, \partial_t a_2, f, \pi = \partial_t f + ia_o f) , \tag{19}$$

where a_μ and f are the fields after we have subtracted the topologically nontrivial background, whereas in the Maxwell-Klein-Gordon theory we can work with the fields A_μ and ϕ themselves. The choice of Banach space, however, is the same.

In both cases, the operator A (see references for this and other formulae to complicated to present here) generates a 1-parameter semigroup on each Sobolev space $(H_{s+1} \times H_s)^4$ $(s \geq 0)$. Therefore, our first choice would be $(H_1 \times L^2)^4$. However, with this choice of Banach space we have to show, in particular, that if a_o, $f \in H_1$, then $K_7 = -ia_o f \in H_1$, which, in general, is not true. Since a_o, $f \in H_2$ implies $a_o f \in H_2$, we can, however, choose $(H_2 \times H_1)^4$, and show that the local existence proof goes through. The price we pay for working with s = 1 instead of s = 0 is a major complication in the global existence proof because the energy or pseudo-energies of the same order are not good enough to bound the higher-order norms. A pseudo-energy of higher order has to be used to prove global existence.

In a gauge theory, one, of course, also has to cope with the gauge freedom. In our proof, we fix the gauge by imposing the Lorentz condition, i.e., the Lorentz condition and the Gauss equation

$$\Delta a_o - \partial_t(\partial_1 a_1 + \partial_2 a_2) = i[(\phi^o + f)\overline{(\pi + ia_o\phi^o)} + c.c.] \tag{20}$$

are the initial value constraints. To complete the proof one must show that these constraints are propagated by the evolution equation.

For time-dependent monopoles, one goes again beyond the simplest choice of space which works for the corresponding linear problem. Eardley and Moncrief (ref.12) have shown that without a symmetry breaking potential (i.e., without monopoles) putting the potentials A_μ into H_2 and the fields $F_{\alpha\beta}$ into H_1 is a successful strategy. Eardley and Moncrief work in the A_o gauge and use the formal solution of the constraint,

$$F^c_{oi} = -\frac{1}{4\pi}\partial_i \int d^3x' \; \frac{\rho(x')}{|x-x'|}, \tag{21}$$

with

$$\rho = [F_{oi}, A_i] - \langle D_o\phi, T_a\phi \rangle T_a, \quad i = 1,2,3, \tag{22}$$

(T_a: generators of the gauge group).

This technique only works if the formal solution of the constraint is in H_2 if the potentials and fields are in H_2 and H_1, respectively. Because this is no longer true in the case of topologically nontrivial configrations, we put all the following fields into H_2:

$$a_i,\ \partial_t a_i,\ b_i := \varepsilon_{ijk}(\partial_j a_k + a_j a_k + [A^o_j, a_k]),$$
$$f,\ \partial_t f,\ \omega_i := \partial_i f + a_i f + a_i \phi^o + A^o_i f. \qquad (23)$$

We than extend the technique of Segal[13] and of Ginibre and Velo[14] to give a local existence proof.

For the global existence proof we switch back to the Eardley-Moncrief technique, extending this technique to an order of differentiability necessary to match our local proof. This involves deriving a priori L^∞-bounds on $F_{\alpha\beta}$ and $D_\mu\phi$, which follow form the equations of motion and energy-momentum conservation integrated over a finite part of the past light cone. From the L^∞-bounds one can derive L^2 and H_1-bounds using the energy. Using the higher-derivative pseudo-energies one can then push the bounds on the H_s-norms step by step up to higher s. Since our local proof works in H_2, two steps are enough to complete the global proof.

OUTLOOK

With the question of global existence of solutions settled we can turn our attention to the asymptotic properties of these solutions. One important question to answer rigorously is whether in a scattering process of vortices or monopoles some of the energy radiates out along the light cone. A positive answer would be proof that vortices and monopoles are no solutions in the strict sense. A possible technique to find this answer could be the one used by Glassey and Strauss[15]. Glassey and Strauss have studied extensively the conservation laws which also play an important role in the global existence proof to detect radiation along the light cone.

An even more ambitious program for further studies centers around the role of Sobolev imbeddings for classical existence proofs as well as for renormalizability. As the inequalities (9) and (10) show, imbeddings play a crucial role in our existence proofs. They also play a crucial role in the existence proofs for other classical equations. It can be shown,[16] e.g., that the equation

$$-\Delta u = |u|^{p-1}u - u, \quad u : \mathbb{R}^n \to \mathbb{R}, \quad n \geq 3, \tag{24}$$

has no nontrivial solution if $p \geq (n+2)/(n-2)$. On the other hand, it is known that ϕ^k theory in d space-time dimensions is renormalizable if $k \leq (d+2)/(d-2)+1$. That Sobolev inequalities play a role for renormalizability is not so surprising if we notice that the inequality reflects the fact that the free theory (L^2-norm of $\partial_\mu \phi$) controls the nonlinearity ϕ^k. To uncover a deep connection between the existence of classical solutions and renormalizability would be a very interesting result.

REFERENCES

1. Forgács, P., Horvath, Z. and Palla L., Phys. Lett. 99B, 232 (1981).
2. Ward, R.S., Commun. Math. Phys. 79, 317 (1981).
3. Nahm, W., Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard and W. Nahm (Singapore: World Scient. Publ. 1982) p. 87.
4. Donaldson, S.K., Commun. Math. Phys. 96, 387 (1984), and references therein.
5. Manton, N.S., Phys. Lett. 110B, 54 (1982); Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard and W. Nahm (Singapore : World Scient. Publ. 1982) p. 95.
6. Atiyah, M.F. and Hitchin, N.J., Phys. Lett. 107A, 21 (1985); Phil. Trans. R. Soc. Lond. A315, 459 (1985).
7. Burzlaff, J. and Moncrief, V., J. Math. Phys. 26, 1368 (1985).
8. Burzlaff, J. and O'Murchadha, N., Commun. Math. Phys. 105, 85 (1986).
9. Segal, I., Ann. Math. 78, 339 (1963).
10. Nirenberg, L., Ann. Scuola Norm. Sup. Pisa 13, 115 (1959); Gagliardo, E., Ric. Mat. 8, 24 (1959).
11. Moncrief, V., J. Math. Phys. 21, 2291 (1980).
12. Eardley, D.M. and Moncrief, V., Commun. Math. Phys. 83, 171 and 193 (1982).
13. Segal, I., J. Funct. Anal. 33, 175 (1979).
14. Ginibre, J. and Velo, G., Commun. Math. Phys. 82, 1 (1981).
15. Glassey, R.T. and Strauss, W.A., Commun. Math. Phys. 67, 51 (1979).
16. Pohozaev, S.I., Sov. Math. Doklady 5. 1408 (1965).