DIAS-STP-86-41

AM/86/08

An SU(8) Model for the Unification of Superconductivity, Charge and Spin Density Waves

Allan I. Solomon
Faculty of Mathematics, Open University
Milton Keynes, MK7 6AA, U.K.

and

Joseph L. Birman
Department of Physics, City College of the City University
of New York, NY 10031 USA

DIAS-STP-86-41

An SU(8) Model for the Unification of Superconductivity, Charge and Spin Density Waves*

Allan I. Solomon[†]
and
Joseph L. Birman[††]

Institute for Scientific Interchange
10133 Torino, Italy

PACS: 03.65Fd; 05.30,Fk; 74.20-Z

* Address correspondence and reprint requests to the authors' permanent addresses: [†] Faculty of Mathematics, Open University, Milton Keynes, MK7 6AA, U.K.

[††] Department of Physics, City College of the City University of New York, NY 10031 USA

17 September 1986

Abstract

We analyze a model Hamiltonian for a many-electron system which unifies superconductivity, charge density waves and spin density waves. We show that the spectrum generating algebra for this system is su(8), and identify all 63 generators of this Lie algebra as symmetry operators which are broken in transition to the condensed state, together with 56 order operators, whose expectations give the order parameters of the various phases present in the model. We tabulate the discrete symmetry properties of these operators. We construct a chain of subalgebras of sub-models with corresponding decoupled phases. We finally indicate how the finite temperature Green's Functions may be obtained and used to solve the problem of self-consistency of the order parameters in the model.

1. Introduction

The pioneering experiments of Sooryakumar and Klein [1] on the coexistence of superconductivity and charge density wave phases, and many subsequent investigations, both theoretical and experimental [2], have sparked interest in those systems for which the coexistence of these and other phases, such as ferro- and anti-ferromagnetic, are possible. In this paper we give a purely theoretical description, based on the approach of Lie algebras, to a system which is capable of embracing the phenomena of superconductivity and density waves. The model we analyze incorporates conventional homogeneous singlet superconductivity - and, perforce as a consequence of algebraic consistency, homogeneous triplet superconductivity. The density wave phenomena are those of charge density waves and spin density waves (antiferromagnetism); within the same algebraic framework it is also possible to include ferromagnetic effects.

The approach we adopt is that of the spectrum-generating Lie algebra (SGA). Our model will be described by a Hamiltonian H given in terms of fermion creation and annihilation operators $a^{\dagger}_{k\sigma}$, $a_{k'\sigma'}$ for electrons constituting the electron gas in the system. Under suitable approximations, which we detail, H becomes a sum of bilinears in these operators; and so the terms of H generate a compact Lie algebra, the SGA of the model. For a model sufficiently general to include the physical phenomena noted above, the algebra is su(8).

The advantages of this algebraic approach are manifold. Firstly, the various phenomena are synthesized into a single structure in which their relationships are transparent. The most striking example of this is the relationship between the existence of singlet superconductivity and density waves on the one hand, and triplet superconductivity on the other [3]. Another example is the description of the large number of "order operators" - these are operators whose expectations give the order parameters - which it would otherwise be difficult to classify. Secondly, although such a complex system does not lend itself easily to explicit calculation, the existence of low-dimensional faithful representations (8x8 matrices in the case of the full system, smaller matrices in the case of subsystems) simplifies explicit calculation of such physical quantities as spectra and phase coexistence boundaries, as we have previously illustrated in the simpler superconductivity - charge density wave su(4) case [4], as well as selection rules [5] for various transition processes.

Thirdly, this model may be regarded as unifying a variety of submodels, obtainable as subalgebras of su(8), which describe interesting physical systems of one or more phases, many of which have been previously treated separately in the literature [6]. Finally, within the context of mean field theory, where our model is firmly situated, finite temperature effects may be treated using the thermal Green's Function method, and problems of self-consistency may also be tackled in this manner. We touch upon these questions in the final section of this paper.

2. Model Hamiltonian

Our starting Hamiltonian is a conventional sum of contributions from kinetic energy, superconducting and density wave terms, thus:

$$H = H_{KE} + H_{SC} + H_{DW} \tag{2.1}$$

where

$$H_{KE} = \sum \varepsilon(k)\ a^{\dagger}_{k\sigma}\ a_{k\sigma} \tag{2.2}$$

$$H^{O}_{SC} = \sum \Delta^{*}_{0}(k)\ a_{k\uparrow}\ a_{-k\downarrow} + h.c. \tag{2.3}$$

$$H_{DW} = \sum \gamma_{\mu}(k)\ a^{\dagger}_{k+Q}\ \sigma_{\mu}\ a_{k} + h.c. \tag{2.4}$$

In the above, $a^{\dagger}_{k\sigma}$ is the fermion creation operator for an electron in the Bloch state labelled by wave-vector k with spin σ, and energy $\varepsilon(k)$. We have the anticommutation rule

$$\{a_{k\sigma},\ a^{\dagger}_{k'\sigma'}\} = \delta_{kk'}\ \delta_{\sigma\sigma'} \tag{2.5}$$

with other anticommutators zero. The BCS parameter $\Delta_0(k)$ may be taken complex, as may the density wave coupling constant $\gamma_{\mu}(k)$. Here $Q \equiv 2k_F$ is the characteristic wave-vector of antiferromagnetic order, where k_F is the Fermi level. We have implicitly summed over the spin indices (understood) in H_{DW}, and over the index $\mu = 0,1,2,3$; we include $\mu = 0$ corresponding to a γ_0 charge-density wave coupling, while γ_i $(i = 1,2,3)$ is the spin-density wave term.

In principle, the summations in the above terms are over all k-values. However, we now effect a considerable simplification, which leads to a decoupling and eventual algebraic solvability, by assuming that our model is quasi-one-dimensional, with no contributions from terms for which $|k| > Q$. The first two terms (2.2), (2.3) may then be rearranged by use of the identity

$$\sum_{-Q}^{Q} f(k) = \sum_{0}^{k_F} \{f(k) + f(-k) + f(\bar{k}) + f(-\bar{k})\}$$

where $\bar{k} \equiv k - Q$; and a similar reduction of (2.4) leads to the model Hamiltonian $H = \sum_{k=0}^{k_F} H(k)$, where

$$\begin{aligned} H(k) = {} & \varepsilon(k)(a^{\dagger}_{k\sigma}a_{k\sigma} + a^{\dagger}_{-k\sigma}a_{-k\sigma}) + \varepsilon(\bar{k})(a^{\dagger}_{\bar{k}\sigma}a_{\bar{k}\sigma} + a^{\dagger}_{-\bar{k}\sigma}a_{-\bar{k}\sigma}) \\ & + \Delta_0^{*}\, a_{k\uparrow}a_{-k\downarrow} + \Delta_0^{*}\, a_{-k\uparrow}a_{k\downarrow} + \Delta_0'^{*}\, a_{\bar{k}\uparrow}a_{-\bar{k}\downarrow} + \Delta_0'^{*}\, a_{-\bar{k}\uparrow}a_{\bar{k}\downarrow} + \text{h.c.} \\ & + \gamma_\mu\, a^{\dagger}_{k\alpha}\,\sigma_\mu^{\alpha\beta} a_{\bar{k}\beta} + \gamma_\mu\, a^{\dagger}_{-\bar{k}\alpha}\,\sigma_\mu^{\alpha\beta}\, a_{-k\beta} + \text{h.c.} \end{aligned} \qquad (2.6)$$

Here, as throughout the paper, we sum over repeated indices. We allow a k-dependence of the BCS singlet gap parameter Δ_0, and so write Δ_0 for $\Delta_0(k)$, and Δ_0' for $\Delta_0(\bar{k})$.

We note that $[H(k), H(k')] = 0$ for $k,k' \in [0,k_F]$ so we have decoupled the Hamiltonian into a direct sum. As in reference [7], where we treated H_{DW} in more detail, we now define the set $\{B_i(k)\}$ $(i = 1,2,\ldots, 8)$, by

$$\{B_i(k)\} = \{a_{k\uparrow},\ a^\dagger_{-k\downarrow},\ a_{\bar{k}\uparrow},\ a^\dagger_{-\bar{k}\downarrow};\ a_{k\downarrow},\ a^\dagger_{-k\uparrow},\ a_{\bar{k}\downarrow},\ a^\dagger_{-\bar{k}\uparrow}\} \tag{2.7}$$

From (2.5) we have $\{B_i, B_j^\dagger\} = \delta_{ij}$ whence the operators $X_{ij} \equiv B_i^\dagger B_j$ generate the Lie algebra $g\ell(8)$ with commutation relations $[X_{ij}, X_{k\ell}] = \delta_{jk} X_{i\ell} - \delta_{i\ell} X_{kj}$. The Hamiltonian $H(k)$ in (2.6) is a linear sum of hermitian combinations and has trace zero since $\varepsilon(k) = \varepsilon(-k)$; therefore $H(k)$ may be considered as an element of $su(8)$. The spectrum generating algebra (SGA) of the model Hamiltonian H is thus a subalgebra

$$\oplus_k\, g_{(k)} \subset \oplus_k\ su(8)_{(k)}\ ,$$

with each $g_{(k)}$ isomorphic to a fixed Lie algebra g (which we shall call the SGA of our model). We shall determine g later; we show that the presence of singlet superconductivity and spin density waves is sufficient to generate the whole $su(8)$ algebra. This very rich rank-7 algebra possesses, in a Cartan basis, seven mutually commuting operators which we interpret as conserved quantities (above the transition temperatures) which are no longer conserved in the various phases present in the model below the appropriate transition temperatures; and 56 other basis elements which are putative order operators, whose expectations are order parameters for the corresponding phases [8].

The bulk of this paper will be devoted to exploiting the algebraic consequences of this system of operators. We commence by introducing some notation. Define the Pauli matrices

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the 4x4 matrices

$$S_i = \frac{1}{2} \tau_0 \times \tau_i, \ T_i = \frac{1}{2} \tau_1 \times \tau_i,$$

$$U_i = \frac{1}{2} \tau_2 \times \tau_i, \ W_i = \frac{1}{2} \tau_3 \times \tau_i,$$

$$E_i = \frac{1}{2} \tau_i \times \tau_0; \qquad (i = 1,2,3). \tag{2.8}$$

The set (2.8) provides a basis for su(4) [5]. The basis for su(8) that we shall use is given by

$$\{\underline{S} \times \tau_\mu, \ \underline{T} \times \tau_\mu, \ \underline{U} \times \tau_\mu, \ \underline{W} \times \tau_\mu, \ \underline{E} \times \tau_\mu, \ I \times \underline{\tau}\}, \ (\mu = 0,1,2,3). \tag{2.9}$$

Here I is the 4 x 4 identity matrix. This is effectively a triple Nambu representation. The algebra $su(8)_{(k)}$ is generated by $\{\sum_{ij} B_i^\dagger(k)\, M_{ij}^r\, B_j(k)\}$ (r=1,..., 63) where M_{ij}^r is one of the 63 hermitian matrices defined in (2.9).

If we take the standard representation of the gl(8) algebra generated by $X_{ij}(k) \equiv B_i^\dagger(k)B_j(k)$, $\hat{X}_{ij}(k) = e_{ij}$, where

$$(e_{ij})_{\ell m} = \delta_{i\ell}\,\delta_{jm} \qquad (i,j,\ell,m = 1,2,\ldots 8)$$

then (2.9) is a basis for a representation of $su(8)_{(k)}$; we shall consistently denote this representation by a circumflex ^. In this representation the number operator $N = \sum N(k)$, where

$$N(k) = \sum_{\alpha=\uparrow,\downarrow} (a^\dagger_{k\alpha}a_{k\alpha} + a^\dagger_{-k\alpha}\,a_{-k\alpha} + a^\dagger_{\bar{k}\alpha}\,a_{\bar{k}\alpha} + a^\dagger_{-\bar{k}\alpha}\,a_{-\bar{k}\alpha})$$

is given by

$$\hat{N}(k) = I \times \tau_3, \text{ where I is the 4 x 4 unit matrix.} \tag{2.10}$$

The spin operator $\sum_k \underline{\sigma}(k)$, where

$$\underline{\sigma}(k) = \sum_{\alpha,\beta} (a^\dagger_{k\alpha}\underline{\sigma}^{\alpha\beta}a_{k\beta} + a^\dagger_{-k\alpha}\underline{\sigma}^{\alpha\beta}a_{-k\beta} + a^\dagger_{\bar{k}\alpha}\underline{\sigma}^{\alpha\beta}a_{\bar{k}\beta}$$

$$+ a^\dagger_{-\bar{k}\alpha}\,\underline{\sigma}^{\alpha\beta}a_{-\bar{k}\beta})$$

is given by $(\hat{\sigma}_1(k), \hat{\sigma}_2(k), \hat{\sigma}_3(k)) = (E_1 \times \tau_3, E_2 \times \tau_3, E_3 \times \tau_0)$.

(2.11)

(The spin matrices σ_μ are defined as usual by $\sigma_\mu \equiv \frac{1}{2}\tau_\mu$.)

Introduce the operator $S(k) = \frac{1}{2} \sum_{\alpha=\uparrow,\downarrow} \left[a^\dagger_{k\alpha} a_{k\alpha} + a^\dagger_{-k\alpha} a_{-k\alpha}) - (a^\dagger_{\bar{k}\alpha} a_{\bar{k}\alpha} + a^\dagger_{-\bar{k}\alpha} a_{-k\alpha}) \right]$

represented by

$$\hat{S}(k) = S_3 \times \tau_3 . \tag{2.12}$$

We may now rewrite the Hamiltonian (2.6) as

$$H(k) = \frac{1}{2} (\varepsilon + \varepsilon') N(k) + (\varepsilon - \varepsilon') S(k)$$

$$- \Delta_0 D_0(k) - \Delta'_0 D'_0 (k) + h.c.$$

$$+ \gamma_\mu \Gamma_\mu (k) + h.c. \tag{2.13}$$

In (2.13) we have introduced a scalar, complex superconducting order operator

$$D_0(k) = a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} + a^\dagger_{-k\uparrow} a^\dagger_{k\downarrow} \qquad [\underline{\sigma}(k), D_0(k)] = 0 \tag{2.14}$$

with a similar expression for $D'_0(k)$ in which k is replaced by $\bar{k}$. We have also introduced a complex charge-spin density wave order operator $\Gamma_\mu(k)$, defined by

$$\Gamma_\mu(k) = a^\dagger_{k\alpha}\,\sigma^{\alpha\beta}_\mu\, a_{\bar{k}\beta} + a^\dagger_{-\bar{k}\alpha}\,\sigma^{\alpha\beta}_\mu\, a_{-k\beta} \tag{2.15}$$

The $\mu = 0$ scalar component is the charge density part, while the $\mu = 1,2,3$ vector components refer to the spin density wave. The real and imaginary parts of $\Gamma_\mu(k)$ are two of a quartet of density wave order operators $\Gamma^{(\alpha)}_\mu$, fully defined in Chapter 3, which satisfy

$$[\underline{\sigma},\Gamma^{(\alpha)}_0] = 0; \quad [\sigma_\ell,\Gamma^{(\alpha)}_m] = ie_{\ell mn}\,\Gamma^{(\alpha)}_n \tag{2.16}$$

where $e_{\ell mn}$ is the permutation symbol on $\ell,m,n = 1,2,3;\ \alpha = 1,2,3,4.$

In the representation with basis (2.9) and number and spin operators represented by (2.10) and (2.11) respectively, these order operators are given by

$$\hat{D}_0 = (E_3 + W_3) \times \frac{1}{2}\,(\tau_1 + i\tau_2) \tag{2.17}$$

$$\hat{D}'_0 = (E_3 - W_3) \times \frac{1}{2}\,(\tau_1 + i\tau_2)$$

and

$$\{\hat{\Gamma}_0,\hat{\Gamma}_1,\hat{\Gamma}_2,\hat{\Gamma}_3\} = \{\frac{1}{2}\,(S_1+iS_2)\times\tau_3,\ \frac{1}{2}\,(T_1+iT_2)\times\tau_3,$$

$$\frac{1}{2}\,(U_1+iU_2)\times\tau_3,\ \frac{1}{2}\,(W_1+iW_2)\times\tau_0\}. \tag{2.19}$$

We may now rewrite our starting Hamiltonian (2.6) in the representation with basis (2.9) as

$$\hat{H} = \hat{H}_{KE} + \hat{H}_{SC} + \hat{H}_{SDW} + \hat{H}_{CDW} \tag{2.20}$$

$$\hat{H}_{KE} = (\varepsilon+\varepsilon')\ (\tfrac{1}{2}\ I \times \tau_3) + (\varepsilon-\varepsilon')\ S_3 \times \tau_3 \tag{2.21}$$

$$\hat{H}_{SC} = -(\alpha+\alpha')(E_3 x\tau_1) + (\alpha'-\alpha)(W_3 x\tau_1) + (\beta+\beta')\ (E_3 x\tau_2) + (\beta-\beta')\ (W_3 x\tau_2) \tag{2.22}$$

$$\hat{H}_{SDW} = Rl\gamma_1\ (T_1 x\tau_3) + Rl\gamma_2\ (U_1 x\gamma_3) + Rl\gamma_3\ (W_1 x\tau_0) - Im\gamma_1\ (T_2 x\tau_3) - Im\gamma_2\ (U_2 x\tau_3) - Im\gamma_3\ (W_2 x\tau_0) \tag{2,23}$$

$$\hat{H}_{CDW} = Rl\gamma_0\ (S_1 x\tau_3) - Im\gamma_0\ (S_2 x\tau_3) \tag{2.24}$$

In (2.22) $\Delta_0 = \alpha+i\beta$, $\Delta_0' = \alpha'+i\beta'$. The expressions (2.23) and (2.24) give the spin density wave and charge density wave terms respectively. The operators in (2.21) - (2.24) are only part of a full system of order operators for this model. We define and examine the full system of order operators in the next section. In the meanwhile we write down for reference the matrix for the Hamiltonian (2.20) in the basis (2.9)

$$\hat{H} = \left[\begin{array}{c|c|c|c|c|c|c|c}
\varepsilon & -\Delta_0 & \frac{1}{2}(\gamma_0+\gamma_3) & 0 & 0 & 0 & \frac{1}{2}(\gamma_1-i\gamma_2) & 0 \\
-\Delta_0^* & -\varepsilon & 0 & -\frac{1}{2}(\gamma_0-\gamma_3) & 0 & 0 & 0 & -\frac{1}{2}(\gamma_1-i\gamma_2) \\
\frac{1}{2}(\gamma_0^*+\gamma_3^*) & 0 & \varepsilon' & -\Delta_0' & \frac{1}{2}(\gamma_1^*-i\gamma_2^*) & 0 & 0 & 0 \\
0 & -\frac{1}{2}(\gamma_0^*-\gamma_3^*) & -\Delta_0'^* & -\varepsilon' & 0 & -\frac{1}{2}(\gamma_1^*-i\gamma_2^*) & 0 & 0 \\
0 & 0 & \frac{1}{2}(\gamma_1+i\gamma_2) & 0 & \varepsilon & \Delta_0 & \frac{1}{2}(\gamma_0-\gamma_3) & 0 \\
0 & 0 & 0 & -\frac{1}{2}(\gamma_1+i\gamma_2) & \Delta_0^* & -\varepsilon & 0 & -\frac{1}{2}(\gamma_0+\gamma_3) \\
\frac{1}{2}(\gamma_1^*+i\gamma_2^*) & 0 & 0 & 0 & \frac{1}{2}(\gamma_0^*-\gamma_3^*) & 0 & \varepsilon' & \Delta_0' \\
0 & -\frac{1}{2}(\gamma_1^*+i\gamma_2^*) & 0 & 0 & 0 & -\frac{1}{2}(\gamma_0^*+\gamma_3^*) & \Delta_0'^* & -\varepsilon'
\end{array}\right]$$

3. The Order Operators

We now analyze the Lie algebra su(8) with basis (2.9). This rank-7 algebra has 7 Cartan (diagonal) elements, and 56 off-diagonal elements. If h is a Cartan, and e a typical non-diagonal element satisfying the canonical

$$[h,e] = \lambda e \qquad (\lambda \neq 0)$$

we see that in an eigenstate $|\rangle$ of h, $\langle|e|\rangle = 0$. The root vectors e, and linear combinations of such root vectors, are order operators for eigenstates of h. Their expectations are the order parameters which vanish in states for which h is a conserved operator. The eight Cartan elements for the u(8) algebra generated by the $B_i^\dagger B_j$ of (2.7) may simply be written $B_i^\dagger B_i$, (i=1,...,8); or more physically $n_{K\sigma}$, the number operator for K,σ ($K = \pm k, \pm \bar{k}$; $\sigma = \uparrow,\downarrow$). In terms of the basis (2.9), the Cartan elements are

$$\hat{N} = I \times \tau_3,\ \hat{P} = 2S_3 \times \tau_0,\ \hat{S} = S_3 \times \tau_3,\ \hat{F} = 2E_3 \times \tau_0 \ (= 2\hat{\sigma}_3)$$

$$E_3 \times \tau_3,\ W_3 \times \tau_0,\ W_3 \times \tau_3 \tag{3.1}$$

We have already introduced the number operator N, the difference of k, $\bar{k}$ number $S \equiv \frac{1}{2}(N_k - N_{\bar{k}})$ and the third component of spin σ_3 in Section 2. (This last plays the role of a ferromagnetic order parameter F.) The matrix $\hat{P}$ represents the momentum operator.

(In the case of u(8) we would have additionally the unit matrix $I \times \tau_0$.)

We now illustrate a useful algebraic method for obtaining the order operators Q_i corresponding to a given quantum observable h. The operator h is assumed to be one of the operators conserved in the lower symmetry phase; we take it to be one of the elements of the Cartan subalgebra, and therefore diagonal in our representation. From the above remarks, the $\hat{Q}_i$ are the elements of the Cartan basis which do not commute with $\hat{h}$. Defining the centralizer of $\hat{h}$ as

$$C_{su(8)}(\hat{h}) = \{x \in su(8): [x,\hat{h}] = 0\}$$

we see that the set of order operators we seek is precisely the complement in su(8) of this centralizer, $C'_{su(8)}(\hat{h})$. In addition, one may readily obtain such centralizers by the following method [9]: Let the matrix M in the defining representation of the group U(n) be diagonal, with eigenvalue multiplicities $m_1, m_2, \ldots m_S$ where $m_1+m_2+\ldots+m_S = n$. The little group of M is $U(m_1) \otimes U(m_2) \otimes \ldots \otimes U(m_S)$. Translating this result to the present Lie algebra context, if the diagonal matrix $\hat{h}$ has eigenvalues with multiplicities $m_1, m_2, \ldots, m_S$ where $m_1+m_2+\ldots+m_S = 8$, then

$$C_{u(8)}(\hat{h}) = u(m_1) \oplus u(m_2) \oplus \ldots \oplus u(m_S).$$

For the case of su(8) the corresponding result is

$$C_{su(8)}(\hat{h}) = s(u(m_1) \oplus \ldots \oplus u(m_s))$$

$$\sim u(1) \oplus su(m_1) \oplus \ldots \oplus su(m_s).$$

As an example, take for our quantum observable h the number operator N. This is represented by the matrix $\hat{N} = I \times \tau_3$ (equation (2.10)); we have $m_1 = m_2 = 4$, and so

$$C_{su(8)}(\hat{N}) = u(1) \oplus su(4) \oplus su(4).$$

Taking the complement, we find by this means 32 N-non-conserving operators $C'(\hat{N})$, which split into 16 superconducting D operators $C'(\hat{N}) \cap C(\hat{P})$, and 16 anomalous A operators $C'(\hat{N}) \cap C'(\hat{P})$. There are 16 density-wave Γ operators $C(\hat{N}) \cap C'(\hat{P})$, and finally 8 ferromagnetic F operators $C(\hat{N}) \cap C(\hat{P}) \cap C'(\hat{F})$. The first three sets of operators divide naturally into scalar plus vector quartets as follows:

Superconducting Order Operators

$$\hat{D}_\mu^{(1)} = (E_3 \times \tau_1, -E_2 \times \tau_1, E_1 \times \tau_1, \tfrac{1}{2} I \times \tau_2)$$

$$\hat{D}_\mu^{(2)} = (E_3 \times \tau_2, -E_2 \times \tau_2, E_1 \times \tau_2, -\tfrac{1}{2} I \times \tau_1)$$

$$\hat{D}_\mu^{(3)} = (-W_3 \times \tau_1, U_3 \times \tau_1, -T_3 \times \tau_1, -S_3 \times \tau_2)$$

$$\hat{D}_\mu^{(4)} = (-W_3 \times \tau_2, U_3 \times \tau_2, -T_3 \times \tau_2, S_3 \times \tau_1) \qquad (3.2)$$

Charge-Spin Density Wave Operators

$$\hat{\Gamma}_{\mu}^{(1)} = (-S_2 x\tau_3,\ T_1 x\tau_3,\ U_1 x\tau_3,\ W_1 x\tau_0)$$

$$\hat{\Gamma}_{\mu}^{(2)} = (S_1 x\tau_3,\ T_2 x\tau_3,\ U_2 x\tau_3,\ W_2 x\tau_0)$$

$$\hat{\Gamma}_{\mu}^{(3)} = (S_1 x\tau_0,\ T_2 x\tau_0,\ U_2 x\tau_0,\ W_2 x\tau_3)$$

$$\hat{\Gamma}_{\mu}^{(4)} = (-S_2 x\tau_0,\ T_1 x\tau_0,\ U_1 x\tau_0,\ W_1 x\tau_3) \qquad (3.3)$$

Anomalous Order Operators

$$\hat{A}_{\mu}^{(1)} = (W_2 x\tau_2,\ -U_1 x\tau_1,\ T_1 x\tau_1,\ S_1 x\tau_2)$$

$$\hat{A}_{\mu}^{(2)} = (W_1 x\tau_1,\ -U_2 x\tau_2,\ T_2 x\tau_2,\ -S_2 x\tau_1)$$

$$\hat{A}_{\mu}^{(3)} = (-W_2 x\tau_1,\ -U_1 x\tau_2,\ T_1 x\tau_2,\ -S_1 x\tau_1)$$

$$\hat{A}_{\mu}^{(4)} = (-W_1 x\tau_2,\ -U_2 x\tau_1,\ T_2 x\tau_1,\ S_2 x\tau_2) \qquad (3.4)$$

Ferromagnetic Order Operators

The eight ferromagnetic order parameters

$$\{E_1,\ E_2,\ T_3,\ U_3\}\ x\ \{\tau_0,\ \tau_3\} \qquad (3.5)$$

are simply the off-diagonal elements of the ferromagnetic sub-algebra which is their closure, namely

$$\{E_1,\ E_2,\ E_3,\ T_3,\ U_3,\ W_3\}\ \ x\ \ \{\tau_0, \tau_3\}\ . \qquad (3.6)$$

This algebra is the su(2) ⊕ su(2) ⊕ su(2) ⊕ su(2) generated by

$$\{a^{\dagger}_{k}\,\underline{\sigma}\,a_{k},\ a^{\dagger}_{-k}\,\underline{\sigma}\,a_{-k},\ a^{\dagger}_{\bar{k}}\,\underline{\sigma}\,a_{\bar{k}},\ a^{\dagger}_{-\bar{k}}\,\underline{\sigma}\,a_{-\bar{k}}\},$$

four independent spin algebras. Corresponding to four linearly independent combinations of these spins, we may define the operators $\underline{\sigma}$ (2.11) and $\underline{\sigma}^{(1)}$, $\underline{\sigma}^{(2)}$, $\underline{\sigma}^{(3)}$ (5.10).

4. Discrete Symmetries

(i) Parity-Inversion π

This is defined by $\pi a_{k\sigma}\pi^{\dagger} = a_{-k\sigma}$, where π is a unitary, linear operator. Acting on the B-basis (2.7) we have

$$\pi\ (B_1, B_2, B_3, B_4)\ \pi^{\dagger} = (B_6^{\dagger}, B_5^{\dagger}, B_8^{\dagger}, B_7^{\dagger}).$$

We may represent this action as an 8x8 matrix

$$\pi\ B_i\ \pi^{\dagger} = \sum_j A_{ij}\ B_j^{\dagger}$$

where

$$A = \tau_1 \times \tau_0 \times \tau_1 .$$

The action on a bilinear in B, $\sum_{ij} m_{ij}\ B_i^{\dagger}\ B_j$ (with tr m = 0) is easily calculated to be

$$\pi\ (B^{\dagger}\ m\ B)\pi^{\dagger} = -\ B^{\dagger}\ A\tilde{m}A\ B$$

where $\tilde{m}$ is the transpose of the 8x8 matrix m. Thus, in the 8x8 representation of Section 2 (2·8 et seq.) parity inversion corresponds to

$$m \rightarrow A\tilde{m}A .$$

(ii) Time Inversion T

This is defined by $Ta_{k\sigma}T^{\dagger} = \sum_{\sigma'} (i\tau_2)_{\sigma\sigma'}\ a_{-k\sigma'}$, where T is a unitary, anti-linear operator. Acting on the B-basis of (2.7) we have

$$\mathcal{T}\ (B_1,B_2,B_3,B_4;\ B_5,B_6,B_7,B_8)\ \mathcal{T}^{\dagger}$$

$$= (\ B_2^{\dagger}, -B_1^{\dagger},\ B_4^{\dagger}, -B_3^{\dagger};\ -B_6^{\dagger},\ B_5^{\dagger}, -B_8^{\dagger},\ B_7^{\dagger})\ .$$

We may represent this action as an 8x8 matrix,

$$\mathcal{T}\ B_i\ \mathcal{T}^{\dagger} = \sum_j T_{ij}\ B_j^{\dagger}$$

where

$$T = \ i\tau_3 \times \tau_0 \times \tau_2\ .$$

The action on a bilinear $\sum m_{ij}\ B_i^{\dagger} B_j$ (with tr m = 0) is readily evaluated to give

$$\mathcal{T}(B^{\dagger} m B)\ \mathcal{T}^{\dagger} = B^{\dagger} T\ m^{\dagger}\ T\ B$$

In our 8x8 representation, time reversal corresponds to

$$m \ \rightarrow T\ m^{\dagger}\ T.$$

(iii) Charge Conjugation C

From the action $\psi_\sigma(x) \rightarrow C\ \psi_\sigma(x)\ C^{\dagger} = \psi_\sigma^{\dagger}(x)$, we define charge conjugation to act on the electron destruction operator $a_{k\sigma}$ by $C\ a_{k\sigma}\ C^{\dagger} = a_{-k\sigma}^{\dagger}$ where C is a unitary, linear operator. On the B-basis, we have $C\ (B_1,B_2,B_3,B_4)\ C^{\dagger} = (B_6,B_5,B_8,B_7)$ whence

$$C\ B_i\ C^{\dagger} = \sum A_{ij}\ B_j$$

where A is the same matrix as in part (i). The action on a bilinear in B_i is therefore given by

$$C\ (B^\dagger\ m\ B)\ C^\dagger = B^\dagger\ (A\ m\ A)\ B;$$

in our representation the effective action of charge conjugation corresponds to

$$\ddot{m} \to AmA\ .$$

We append a table of the discrete transformation properties of the 15 scalars and 48 vectors of this model (Table 1).

Table 1: Parity, Time Reversal and Charge Conjugation Properties

Scalars

	$D_0^{(1)}$	$D_0^{(2)}$	$D_0^{(3)}$	$D_0^{(4)}$	;	$\Gamma_0^{(1)}$	$\Gamma_0^{(2)}$	$\Gamma_0^{(3)}$	$\Gamma_0^{(4)}$	;	$A_0^{(1)}$	$A_0^{(2)}$	$A_0^{(3)}$	$A_0^{(4)}$	;	N	S	P
π	+	+	+	+	;	−	+	−	+	;	−	+	−	+	;	+	+	−
T	+	−	+	−	;	+	+	−	−	;	−	+	+	−	;	+	+	−
C	−	+	−	+	;	−	−	+	+	;	+	−	−	+	;	−	−	+

Vectors

	$\underline{D}^{(1)}$	$\underline{D}^{(2)}$	$\underline{D}^{(3)}$	$\underline{D}^{(4)}$	;	$\underline{\Gamma}^{(1)}$	$\underline{\Gamma}^{(2)}$	$\underline{\Gamma}^{(3)}$	$\underline{\Gamma}^{(4)}$	;	$\underline{A}^{(1)}$	$\underline{A}^{(2)}$	$\underline{A}^{(3)}$	$\underline{A}^{(4)}$	;	$\underline{\sigma}$	$\underline{\sigma}^{(1)}$	$\underline{\sigma}^{(2)}$	$\underline{\sigma}^{(3)}$
π	−	−	−	−	;	+	−	+	−	;	−	+	−	+	;	+	−	−	+
T	−	+	−	+	;	−	−	+	+	;	−	+	+	−	;	−	+	+	−

5. Commutators

In (3.2), (3.3) and (3.4), writing $X_\mu = D_\mu^{(\alpha)}$, $A_\mu^{(\alpha)}$ or $\Gamma_\mu^{(\alpha)}$, the zero-component operators are the scalar quantities satisfying

$$[\underline{\sigma}, X_0] = 0. \tag{5.1}$$

Thus $D_0^{(\alpha)}$ (α= 1,2,3,4) are the ordinary superconducting singlet order operators occurring in the (2.22), while the $\Gamma_0^{(\alpha)}$ are the charge-density order operators, of which the two even-time-reversal scalars appear in (2.24). The triplet operators satisfy

$$[\sigma_i, X_j] = i\, e_{ijk} X_k \tag{5.2}$$

$$[X_i, X_j] = i\, e_{ijk} \sigma_k \tag{5.3}$$

so that, for example, $\Gamma_j^{(\alpha)}$ are spin-density order operators, of which the two odd-time-reversal triplets appear in (2.23). The operators in (3.2), (3.3) and (3.4) satisfy

$$[D_0^{(\alpha)}, \Gamma_0^{(\alpha)}] = i\, A_0^{(\alpha)} \tag{5.4}$$

and

$$[D_i^{(\alpha)}, \Gamma_j^{(\alpha)}] = i\, e_{ijk}\, A_k^{(\alpha)} \tag{5.5}$$

with two similar sets of commutators obtained by cyclic permutation.

The N, P and S operators (3.1) move one quartet of order operators to the next, for example:

$$[\tfrac{1}{2}N, D_\mu^{(1)}] = i\, D_\mu^{(2)} \qquad [\tfrac{1}{2}N, D_\mu^{(3)}] = i\, D_\mu^{(4)}$$

$$[S, D_\mu^{(2)}] = i\, D_\mu^{(3)} \qquad [S, D_\mu^{(4)}] = i\, D_\mu^{(1)} \tag{5.6}$$

The analogous commutators for $\Gamma_\mu^{(\alpha)}$ and $A_\mu^{(\alpha)}$ are:

$$[\tfrac{1}{2}P, \Gamma_\mu^{(1)}] = i\, \Gamma_\mu^{(2)} \qquad [\tfrac{1}{2}P, \Gamma_\mu^{(4)}] = i\, \Gamma_\mu^{(3)}$$

$$[S, \Gamma_\mu^{(1)}] = i\, \Gamma_\mu^{(3)} \qquad [S, \Gamma_\mu^{(4)}] = i\, \Gamma_\mu^{(2)} \tag{5.7}$$

$$[\tfrac{1}{2}N, A_\mu^{(1)}] = i\, A_\mu^{(3)} \qquad [\tfrac{1}{2}N, A_\mu^{(4)}] = i\, A_\mu^{(2)}$$

$$[\tfrac{1}{2}P, A_\mu^{(3)}] = i\, A_\mu^{(2)} \qquad [\tfrac{1}{2}P, A_\mu^{(1)}] = i\, A_\mu^{(4)} \tag{5.8}$$

The singlet and triplet components of the order operators are related as follows:

$$[D_0^{(\alpha)}, \underline{D}^{(\alpha)}] = i\, \underline{\sigma}^{(1)}$$

$$[\Gamma_0^{(\alpha)}, \underline{\Gamma}^{(\alpha)}] = i\, \underline{\sigma}^{(2)}$$

$$[A_0^{(\alpha)}, \underline{A}^{(\alpha)}] = i\, \underline{\sigma}^{(3)} \qquad \text{(no sum over } \alpha) \tag{5.9}$$

These 'pseudo-spin' triplets are represented by

$$\hat{\underline{\sigma}}^{(1)} = (E_1 \times \tau_0,\ E_2 \times \tau_0,\ E_3 \times \tau_3)$$

$$\hat{\underline{\sigma}}^{(2)} = (T_3 \times \tau_0,\ U_3 \times \tau_0,\ W_3 \times \tau_3)$$

$$\hat{\underline{\sigma}}^{(3)} = (-T_3 \times \tau_3,\ -U_3 \times \tau_3,\ -W_3 \times \tau_0). \tag{5.10}$$

These triplets have the following commutation relations:

$$[\sigma_i^{(\alpha)}, \sigma_j^{(\alpha)}] = i\, e_{ijk}\, \sigma_k$$

$$[\sigma_i, \quad \sigma_j^{(\alpha)}] = i\, e_{ijk}\, \sigma_k^{(\alpha)}$$

$$[\sigma_i^{(\alpha)}, \sigma_j^{(\beta)}] = -i\, e_{ijk}\, e^{\alpha\beta\gamma}\, \sigma_k^{(\gamma)}$$

$$(i,j,k;\ \alpha,\beta,\gamma = 1,2,3) \tag{5.11}$$

The $\sigma_j^{(\alpha)}$ connect triplet components with singlet, for example:

$$[\underline{\sigma}^{(1)}, D_0^{(\alpha)}] = i\, \underline{D}^{(\alpha)}$$

and

$$[\underline{\sigma}^{(1)}, \bar{\underline{D}}^{(\alpha)}] = i\, D_0^{(\alpha)} \tag{5.12}$$

with similar relations for $\Gamma_\mu^{(\alpha)}$ and $A_\mu^{(\alpha)}$.

6. The Spectrum Generating Algebra

We may write our starting Hamiltonian (2.20) in terms of the order operators (3.2), (3.3) as:

$$\hat{H} = \frac{1}{2}(\varepsilon+\varepsilon')\,\hat{N} + (\varepsilon-\varepsilon')\,\hat{S} + \Delta_0^{(\alpha)}\hat{D}_0^{(\alpha)} + \gamma_\mu^{(1)}\hat{\Gamma}_\mu^{(1)} + \gamma_\mu^{(2)}\hat{\Gamma}_\mu^{(2)} \tag{6.1}$$

(with summation over μ and α)

where

$$\{\Delta_0^{(1)}, \Delta_0^{(2)}, \Delta_0^{(3)}, \Delta_0^{(4)}\} \equiv \{-\mathrm{Re}(\Delta_0+\Delta_0'),\ \mathrm{Im}(\Delta_0+\Delta_0'),$$

$$\mathrm{Re}(\Delta_0-\Delta_0'),\ \mathrm{Im}(\Delta_0'-\Delta_0)\}$$

and

$$\gamma_\mu^{(1)} = \{\mathrm{Im}\,\gamma_0,\ \mathrm{Re}\,\underline{\gamma}\}\ ,\ \gamma_\mu^{(2)} = \{\mathrm{Re}\,\gamma_0,\ -\mathrm{Im}\,\underline{\gamma}\} \tag{6.2}$$

From the form of the Hamiltonian given in (6.1), using the commutation relations of the previous section, it is a straightforward matter to determine the spectrum generating algebra g for this system; that is, the algebra generated by the elements of (6.1). Since these are all elements of su(8), the SGA must be a subalgebra of su(8). In fact, we now demonstrate that the algebraic closure g of the operators occurring in (6.1) is all of su(8). This has the consequence that all of the 63 operators of the theory will appear in the time evolution of the order operators already present in the Hamiltonian (6.1). Whether they give rise to physical phases will depend on an evaluation of their expectations in the eigenstates of (6.1) or on a self-consistent analysis.

The generation of all su(8) from (6.1) may be seen in the following stages:

(i) Since $\Gamma_\mu^{(1)}$, $\Gamma_\mu^{(2)}$ ε H and S ε H, using (5.7) we have that all $\Gamma_\mu^{(\alpha)}$ ε g.

(ii) Also $[\Gamma_i^{(1)}, \Gamma_j^{(2)}] = i\, e_{ijk}\, \sigma_k$, so $\underline{\sigma}\ \varepsilon$ g.

(iii) Evaluating $[\Gamma_i^{(1)}, \Gamma_j^{(4)}] = i\, e_{ijk}\, \sigma_k^{(1)}$ gives $\underline{\sigma}^{(1)}$ ε g.

(iv) From $D_0^{(\alpha)}$, $\underline{\sigma}^{(1)}$ ε g, using (5.12) gives all $D_\mu^{(\alpha)}$ ε g.

(v) Using (5.4), (5.5) we see that $D_\mu^{(\alpha)}$, $\Gamma_\mu^{(\alpha)}$ generate $A_\mu^{(\alpha)}$.

(vi) As in (iii) $[D_i^{(1)}, D_j^{(3)}] = i\, e_{ijk}\, \sigma_k^{(3)}$ and $[A_i^{(1)}, A_j^{(2)}] = i\, e_{ijk}\, \sigma_k^{(2)}$ imply that all $\underline{\sigma}^{(\alpha)}$ ε g. The 60 operators $D_\mu^{(\alpha)}$, $\Gamma_\mu^{(\alpha)}$, $A_\mu^{(\alpha)}$, $\underline{\sigma}^{(\alpha)}$, $\underline{\sigma}$ together with the three remaining Cartan operators S, N and $P = \frac{2}{i}[\Gamma_\mu^{(1)}, \Gamma_\mu^{(2)}]$ exhaust su(8).

We may note at this point that the commutation relation

$$[D_0^{(3)}, \underline{\Gamma}^{(2)}] = -\, i\, \underline{A}^{(3)}$$

generates an odd-parity odd-time inversion anomalous triplet term from singlet superconductivity (T = - 1) and a spin-density term (T = - 1). The production of such an anomalous term has been previously noted in the literature [10].

However, even more striking is the generation of conventional

(Q=0) triplet superconductivity from the interaction of singlet superconductivity and density waves. A simplified model [3] exhibiting this phenomenon may be obtained from (6.1) by choosing $\Delta_0=\Delta_0'$ (real) and $\gamma_\mu^{(2)}=0$ in (6.2). It is also sufficient to choose axes so that only $\underline{\gamma}^{(2)} \to \gamma_3^{(2)}$. It may be shown that the SGA of this submodel is so(4) ⊕ so(4). The even-time-reversal triplet superconductivity order operator $D_3^{(3)}$ is generated as a second-order effect of the interaction between the singlet superconductivity, and the charge and spin density waves; it has non-zero expectation in the ground state of the Hamiltonian and may therefore be considered as an observable phase [3].

7. Subalgebras and Submodels

It is a fairly straightforward matter to obtain the spectrum generating algebras corresponding to submodels of the Hamiltonian (6.1). These algebras are generated by subsets of the 63 su(8) operators, (3.1), (3,2), (3.3), (3.4) and (3.5). The components of the algebras generated by the order operator terms may most easily be calculated by taking centralizers; to these one must add the other terms of the Hamiltonian (such as kinetic energy N,S). We illustrate this method by obtaining the spectrum generating algebras of some previously noted submodels.

(i) Superconducting Models

The order operators for superconducting systems are defined as those which conserve momentum, but do not conserve number. As in Section 3, we obtain the set $\{D_\mu^{(\alpha)}\}$, represented by the matrices (3.2). These may be succinctly written as

$$\{\underline{E}, T_3, U_3, W_3, S_3, I\} \times \{\tau_1, \tau_2\} \tag{7.1}$$

and in this form we see that they generate the subalgebra

$$\{\underline{E}, T_3, U_3, W_3\} \times \tau_\mu \cup \{S_3, I\} \times \underline{\tau} \tag{7.2}$$

which is isomorphic to su(4) ⊕ su(4). (This algebra is the semi-simple component of the centralizer of momentum P in su(8);

$$C_{su(8)}(P) \sim u(1) \oplus su(4) \oplus su(4).)$$

As this su(4) ⊕ su(4) algebra also contains the appropriate kinetic energy terms N and S, this is the spectrum generating algebra corresponding to a two-component $(k,\bar{k})$ superconducting fermion system. Each su(4) corresponds to a mixed triplet-singlet superconductor as previously obtained [11] - one for k and the other for $\bar{k}$. This may be made explicit as follows; define

$$\tau_{\uparrow} = 1/2\ (\tau_0+\tau_3) \qquad \tau_{\downarrow} = 1/2\ (\tau_0-\tau_3).$$

Then the two commuting su(4) algebras are

$$k - \text{component: su}(4) \sim \{\underline{\tau}\times\tau_{\uparrow}\times\tau_{\mu}, \tau_0\times\tau_{\uparrow}\times\underline{\tau}\}$$
$$\bar{k} - \text{component: su}(4) \sim \{\underline{\tau}\times\tau_{\downarrow}\times\tau_{\mu}, \tau_0\times\tau_{\downarrow}\times\underline{\tau}\} \tag{7.3}$$

Conventional singlet superconductivity may be obtained as the centralizer of the spin operator in either of the above su(4) models, thus

$$C_{su(4)}(\underline{\sigma}) = \{\tau_3\times\tau_{\uparrow}\times\tau_1,\ \tau_3\times\tau_{\uparrow}\times\tau_2,\ \tau_0\times\tau_{\uparrow}\times\tau_3\} \sim so(3) \tag{7.4}$$

which is the spectrum-generating algebra of the singlet superconductor. In the notation of the previous section, the $so(3)_{(k)} \oplus so(3)_{(\bar{k})}$ singlet subalgebra has basis $\{N,S,D_0^{(\alpha)}\}$. The spin-one, pure triplet case corresponds to the $so(5)_{(k)} \oplus so(5)_{(\bar{k})}$ subalgebra with basis $\{N,S,\underline{D}^{(\alpha)},\underline{\sigma},\underline{\sigma}^{(3)}\}$, in the notation of the previous section. Each so(5) algebra is also the SGA for superfluid Helium three [12,13], or a spin-1 superconductor.

(ii) Density Wave Models

The order operators for density wave systems are defined as those which conserve number, but do not conserve momentum. As in Section 3, we obtain the set $\{\Gamma_\mu^{(\alpha)}\}$, represented by the matrices (3.3). We may rewrite this set as

$$\tau_\mu \times \{\tau_1,\tau_2\} \times \{\tau_0,\tau_3\} \tag{7.5}$$

As in (i) above, under commutation these generate

$$\underline{\tau} \times \tau_\mu \times \{\tau_0,\tau_3\} \cup \tau_0 \times \underline{\tau} \times \{\tau_0,\tau_3\} \tag{7.6}$$

which is again isomorphic to an su(4) ⊕ su(4) algebra. To obtain the SGA (spectrum generating algebra) of the density wave Hamiltonian containing the $\Gamma_\mu^{(\alpha)}$ order operators we must adjoin the number operator N, which is not present in (7.6). Thus the SGA for a mixed spin and charge density wave model is u(1) ⊕ su(4) ⊕ su(4), as previously obtained [7]. This algebra may be most simply obtained as the centralizer of the number operator N in su(8),

$$C_{su(8)}(N) \sim u(1) \oplus su(4) \oplus su(4) \tag{7.7}$$

The centralizer of spin $\underline{\sigma}$ in (7.7) gives the CDW algebra generated by $\{P,S,N,\Gamma_0^{(\alpha)}\}$ which is u(1) ⊕ so(4); we have previously obtained this directly from a model charge-density wave Hamiltonian [14].

The spin-1 part of (7.7), the pure spin density wave algebra, is generated by the non-spin conserving elements of (7.5), and has for basis $\{P,S,N,\ \underline{\Gamma}^{(\alpha)},\underline{\sigma},\underline{\sigma}^{(1)}\}$. This is the algebra $u(1) \oplus so(5) \oplus SO(5)$, as calculated previously from a specific density wave Hamiltonian [7].

If we consider only the parity-invariant elements of the above subalgebras, where the parity operator π is as defined in Section 4 (i), then we obtain subalgebras as follows:

$$\text{CDW} \qquad \{N,S,\ \Gamma_0^{(2)},\ \Gamma_0^{(4)}\} \quad \sim \quad u(2)$$

$$\text{SDW} \qquad \{N,S,\ \underline{\Gamma}^{(1)},\ \underline{\Gamma}^{(3)},\ \underline{\sigma}\} \sim \quad u(1) \oplus so(5) \tag{7.8}$$

These are the spectrum generating algebras for the model Hamiltonian (6.1) in the absence of superconductivity taking the coupling constants (6.2) real, and considering pure scalar and pure vector respectively.

(iii) Singlet Model

We obtain a spin-zero model by taking the centralizer of the spin operator $\underline{\sigma}$ in our su(8) algebra; thus

$$C_{su(8)}(\underline{\sigma}) \sim su(4)$$

with basis

$$\{I\times\tau_3,\ \bar{\underline{S}}\times\tau_0,\ \underline{S}\times\tau_3,\ \underline{W}\times\tau_1,\ \underline{W}\times\tau_2,\ E_3\times\tau_1,\ E_3\times\tau_2\} \tag{7.9}$$

This su(4) is isomorphic to that obtained previously for the spectrum generating algebra of a model Hamiltonian exhibiting the coexistence of superconductivity and charge density waves [4]. To see this isomorphism more readily, write the set (7.9) in the form

$$\tau_0 x \tau_\mu x\{\tau_0, \tau_3\} \cup \tau_3 x \tau_\mu x\{\tau_1, \tau_2\} \tag{7.10}$$

(where in (7.10) we have actually considered $C_{u(8)}(\underline{\sigma}) \sim u(4)$ for simplicity; we can always discard the central element $\tau_0 x \tau_0 x \tau_0$ later). The set (7.10) is clearly isomorphic to

$$\{\tau_0 x \tau_0,\ \tau_0 x \tau_3,\ \tau_3 x \tau_1,\ \tau_3 x \tau_2\} \quad x\ \tau_\mu$$

which in turn is isomorphic to $\tau_\mu x \tau_\mu$. This is the set of 15 generators $\{\underline{E}, \underline{S}, \underline{T}, \underline{U}, \underline{W}\}$ of su(4) (together with the unit element $\tau_0 x \tau_0$) of reference [4].

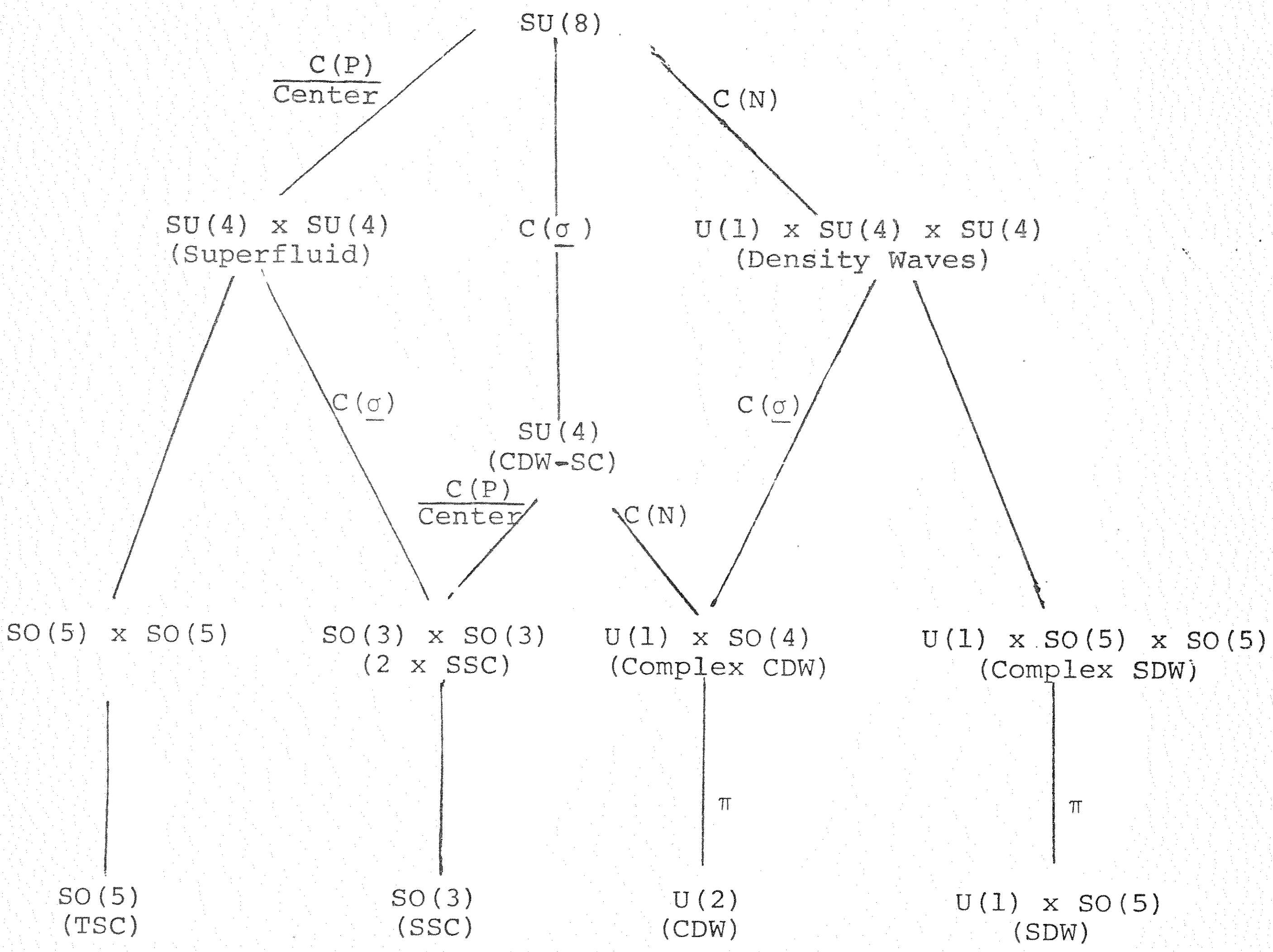
(iv) Spin Models

The eight spin order operators (3.5) generate the algebra with basis $\{\underline{\sigma}^{(\alpha)}\}$ (α = 1,2,3,4; $\underline{\sigma}^{(4)} \equiv \underline{\sigma}$); as remarked in Section 3 this is equivalent to four independent su(2) algebras. We may obtain the spectrum generating algebras for spin model Hamiltonians by adjoining the kinetic energy terms N and S. For example, the even parity spins give an algebra $\{N, S, \underline{\sigma}, \underline{\sigma}^{(3)}\}$. This splits up into $\{\tau_0 x \tau_\uparrow x \tau_3,\ \tau_1 x \tau_\uparrow x \tau_3,\ \tau_2 x \tau_\uparrow x \tau_3,\ \tau_3 x \tau_\uparrow x \tau_0\} \sim u(2)$, and $\{\tau_0 x \tau_\downarrow x \tau_3,\ \tau_1 x \tau_\downarrow x \tau_3,\ \tau_2 x \tau_\downarrow x \tau_3,\ \tau_3 x \tau_\downarrow x \tau_0\} \sim u(2)$;

two independent u(2) spin models, for the k and $\bar{k}$ systems respectively.

We show the descent from SU(8) to the subgroups corresponding to models (i) to (iii) in Table 2.

TABLE 2: Subgroup Descent from SU(8)



Notation: SSC = Singlet Superconductor
TSC = Triplet Superconductor
CDW = Charge Density Waves
SDW = Spin Density Waves

8. General Hamiltonian and Self-Consistency

We may now write down the most general Hamiltonian for the coexistence of superconductivity (singlet and triplet) and density waves (charge and spin) within the context of our su(8) algebra. This will generalize the expression (6.1), thus:

$$H = \sum_k H(k)$$

$$\begin{aligned} H(k) = 1/2\ (\varepsilon + \varepsilon')\ N + (\varepsilon - \varepsilon')\ S + p\ P\ + & \\ + \Delta_\mu^{(\rho)}\ D_\mu^{(\rho)} & \quad \text{(superconducting terms)} \\ + \gamma_\mu^{(\rho)}\ \Gamma_\mu^{(\rho)} & \quad \text{(density waves)} \\ + \alpha_\mu^{(\rho)}\ A_\mu^{(\rho)} & \quad \text{(anomalous terms)} \\ + \underline{H}_{ext}^{(\rho)} \cdot \underline{\sigma}^{(\rho)} & \quad \text{(magnetic field terms)} \end{aligned} \tag{8.1}$$

We sum over repeated indices in (8.1); $\mu = 0,1,2,3$ and $\rho = 1,2,3,4$. We have written $\underline{\sigma}^{(4)} \equiv \underline{\sigma}$ for conciseness; and have included a momentum term p P, where $\hat{P} = 2\ S_3 x \tau_0$, in order to attain the full complement of 63 operators. The magnetic field terms in (8.1) enable calculation of susceptibilities, as has been carried out for the SDW sub-algebra of su(8) [7].

The expression (8.1) has the virtue of explicitness; however, a more concise, if less transparent, form of the mean-field Hamiltonian H is given by

$$H = \sum_k m_{ij}\ (k)\ X_{ij}\ (k) \tag{8.2}$$

(summation over repeated indices i,j) in terms of the operators

$X_{ij}(k) \equiv B_i^{\dagger}(k)\ B_j(k)$ introduced in Section 2. These satisfy the commutation relations

$$[X_{ij}(k),\ X_{rs}(k')] = \delta_{kk'}\ (\delta_{jr}\ X_{is}(k) - \delta_{is}\ X_{rj}(k)) \tag{8.3}$$

We may consider the mean-field Hamiltonian (8.2) to have arisen from a pairing Hamiltonian H^{red} in the following way. We require that H^{red} conserve number N, momentum P..., in fact, all seven Cartan operators which are broken in the passage to the lower symmetry, mean-field system. These operators have the form $\sum_k \lambda_i(k)\ X_{ii}(k)$ (i = 1,2...., 8) (adding in the identity) and it is straightforward to verify that the Hamiltonian

$$H^{red} = 1/2 \sum_{i,j,k,k'} g_{ij}(k,k')\ X_{ij}(k)\ X_{ij}(k')^{\dagger} \tag{8.4}$$

conserves these quantities. Thus (8.4) is a suitable choice of pairing Hamiltonian. If we choose

$$g_{ij}(k,k) = 2\varepsilon_i(k)\,\delta_{ij} \tag{8.5}$$

and note that $[X_{ii}(k)]^2 = X_{ii}(k)$, then the kinetic energy terms are also included in (8.4). With this choice the coupling constants satisfy

$$g_{ij}(k,k') = g_{ji}(k',k). \tag{8.6}$$

In addition, from the hermiticity of H^{red} we have

$$g_{ij}^{*}(k,k') = g_{ij}(k',k) \quad . \tag{8.7}$$

Now define

$$m_{ij}(k) = \langle\langle \sum_{k'} g_{ij}(k,k') X_{ij}^{\dagger}(k') \rangle\rangle \quad (i \neq j) \tag{8.8}$$

(no summation over ij), where << >> refers to a thermal average with respect to the pairing Hamiltonian (8.4),

$$\langle\langle Q \rangle\rangle \equiv \text{trace}\,\{\exp(-\beta H^{red})\, Q\}/\text{trace}\,\exp(-\beta H^{red}) . \tag{8.9}$$

We now apply a Hartree-Fock linearisation to H^{red}, and obtain as an approximation the mean-field form (8.2), using relations (8.6) and (8.7). We now introduce the thermal Green's functions [15], and for simplicity take << >> $\sim$ < > below:

$$G_{ij}(k,\tau) = -\langle T_\tau (B_i(k,\tau)\, B_j^{\dagger}(k,0)) \rangle \tag{8.10}$$

where, at the level of mean-field theory, the thermal average < > is with respect to the mean-field Hamiltonian H of (8.2), as is the Heisenberg τ-evolution

$$B_i(k,\tau) = \exp(H\tau)\, B_i(k)\, \exp(-H\tau) .$$

Here T_τ is the τ-ordering operator, so that

$$G_{ij}(k,0^-) = \langle X_{ij}^{\dagger}(k) \rangle . \tag{8.11}$$

Writing the conventional ω - transform of the Green's function, and replacing $\sum_{k'}$ in (8.8) by the integral, we obtain the self-consistent equation

$$m_{ij}(k) = \frac{1}{\beta}\sum_n \int \frac{d^3k'}{(2\pi)^3}\, g_{ij}(k,k')\, G_{ij}(k',\omega_n) \quad . \tag{8.12}$$

In mean-field approximation the Green's function is explicitly known [16]; in matrix form

$$G(k,\omega_n) = (i\,\omega_n\, I - \hat{H}(k))^{-1} \tag{8.13}$$

where $\hat{H}(k)$ is the 8x8 representation of (8.2), $\hat{H}(k)_{ij} \equiv m_{ij}(k)$. Thus equation (8.12) becomes

$$m_{ij}(k) = \frac{1}{\beta}\sum_n \int \frac{d^3k'}{(2\pi)^3}\, g_{ij}(k,k')\, tr[e_{ji}\; G(k,\omega_n)] \tag{8.14}$$

where e_{ij} is the same matrix as was introduced in Section 2. A slightly more conventional form of (8.14) is obtained by using the Hamiltonian (8.1) in the triple-Nambu representation (2.9), thus - for simplicity - taking $g_{ij}(k,k') = -g$, $(i\neq j)$ independent of k,k'

$$m_{abc} = -\frac{g}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, tr[(\tau_a \times \tau_b \times \tau_c)\; G(k,\omega_n)] \tag{8.15}$$

where we have written $\hat{H} = \sum m_{abc}\, \tau_a \times \tau_b \times \tau_c$. Thus, for example, using (2.8) and (3.2) to determine the coefficient $\Delta_0^{(1)}$ of $D_0^{(1)}$ in (8.1), and taking $\Delta = \Delta_0 = \Delta_0'$ real in (6.2) we have

$$m_{301} = - \Delta = \frac{-g}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \; tr[(\tau_3 \times \tau_0 \times \tau_1) \; G \; (k, \omega_n)],$$

a self-consistent equation for the singlet superconducting gap in this formalism.

9. Conclusions

Starting with the simple model Hamiltonian (2.1) of Section 2, we are led by algebraic closure of the operators therein, to the general Hamiltonian of (8.1). This new system includes many new phenomena not present in the original system, involving as it does 63 parameters against the original 14. Two questions concerning the algebraically generated operators arise naturally:

(i) Is it really necessary to include them in the theory?

(ii) Do they give rise to physically observable phenomena?

The answer to (i) is that even if the new operators are not present in the original Hamiltonian, they will be generated by the time evolution of the dynamics acting on the operators already present; and so they must be included for completeness. The physical detection of the corresponding order parameters will depend on their not vanishing in the ground state of the system; this requires diagonalization of the Hamiltonian. This calculation has been carried out for a simplified so(4) ⊗ so(4) version [3] of the complete su(8) model , where it was found that a new operator (triplet Q=0 superconductor) not present in the original Hamiltonian had non-zero expectation in the ground state of the original Hamiltonian. These questions may also be tackled by conventional self-consistent methods; and we sketched this approach in Section 8.

References

[1] R. Sooryakumar and M.V. Klein, Phys. Rev. Lett. 45, 660 (1980), and Phys. Rev. B23, 3213 (1981).

[2] C.Y. Huang, Crit. Rev. Sol. St. Mat. Science 12, 78 (1984); M. Ishikawa, Contemporary Physics 23, 442 (1982).

[3] A.I. Solomon and J.L. Birman "Mechanism for Triplet Superconductivity" (to be published)

[4] J.L. Birman and A.I. Solomon, Phys. Rev. Lett. 49, 230 (1982).

[5] J.L. Birman and A.I. Solomon, Proc. of XIIIth Int'l Colloquium on Group Theoretical Methods in Physics, ed. W.W. Zachary, (World Scientific, 1985, p.445).

[6] For a review of some of these systems, see J.L. Birman and A.I. Solomon, "Spectrum Generating Algebras in Condensed Matter Physics" in "Spectrum Generating Algebras in Physics" ed. A. Bohm and Y. Ne'eman (World Scientific Publ. Co., Singapore, 1986).

[7] A.I. Solomon and J.L. Birman, Phys. Lett. 104A, 235 (1984).

[8] A.I. Solomon, "Differential Geometric Methods in Mathematical Physics" ed. S. Sternberg, Reidel, Dordrecht, 1984, p.279).

[9] L. O'Raifeartaigh, Rep. Prog. Phys. 42, 159 (1979).

[10] E.W. Fenton and G.C. Psaltakis, Sol. St. Comm. 45, 5 (1982) see also G.C. Psaltakis and E.W. Fenton, J. Phys. C16, 3913 (1983); and E.W. Fenton, Prog. Theor. Phys. Suppl. 80, 94 (1984)

[11] A.I. Solomon, Ann. N.Y. Acad. Sci. 337, 161 (1980).

[12] A.I. Solomon, J. Phys. A14, 2177 (1981).

[13] Y. Hasegawa, T. Usugawa and F. Iwamoto, Prog. Theor. Phys. 62, 1458 (1979).

[14] A.I. Solomon and J.L. Birman, Phys. Lett. 88A, 413 (1982).

[15] T. Matsubara, Prog. Theor. Phys. 14, 351 (1955); and A.L. Fetter and J.D. Walecka, Quantum Theory of Many Particle Systems" (McGraw-Hill, New York, 1971).

[16] A.I. Solomon and J.L. Birman, Phys. Lett. 111A, 423 (1985); J.L. Birman and A.I. Solomon, Prog. Theor. Phys. (Kyoto), Suppl.80, 62 (1984).

Acknowledgements

The authors would like to thank Prof. Mario Rasetti and the Institute for Scientific Interchange at Torino for the hospitality extended to them while this work was completed, and Prof. M. Berger and Prof. L. Michel for the hospitality of the Institut des Hautes Etudes Scientifiques, Bures, where some of the earlier work was done. Partial support at City College by the PSC-BHE Faculty Research Award Program, and the NSF-USA is acknowledged. We also acknowledge partial support of this work by NATO grant (RG.86/0223).