DIAS-STP-86-32

The U(1)-Anomaly, Phase shifts and the η-invariant

Andreas W. Wipf*

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

The violation of classical conservation laws, so-called anomalies, arise from the interaction of spinor fields with external fields. Their true significance was first appreciated in the work of Adler and of Bell and Jackiw [1]. Although the anomalies have already been studied since then from several different perspectives (e.g. simple models, lattice formulations and index theorems) there are still questions in need of answers, e.g. why is the anomaly reflected both as a high energy and as a low energy phenomena, what is the role of regulators and what is the physical interpretation of index theorems and the fractional part of certain anomalies?

In this talk I want to report on some recent results [2,3] concerning the low energy aspects of the U(1)-anomaly on open spaces. They provide an extension of Levinson's theorem and link the fractional part of the anomaly with the (measurable) low energy phase shifts of $-\not{D}^2$ and thus clearly emphasize the long distance aspects of the problem. In addition they generalize the index theorem to open spaces and allow a physical interpretation of both, the index of the Dirac operator

$$i\not{D} = i\gamma^\mu(\partial_\mu + iA_\mu) = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix} \tag{1}$$

on an even-dimensional space and the fractional part of the U(1)-anomaly, and thus clarify the last of the above mentioned problems.

In studying anomalies one conveniently starts from the effective fermion action in the presence of an external gauge field

$$\Gamma = \log\det i\not{D} = tr\,\log i\not{D} \tag{2}$$

which on open spaces is both IR and UV divergent. The IR divergence comes from the fact that the *operator* $\log i\not{D}$ does not exist when the spectrum of $i\not{D}$ is not

---

* Address after 1.1.'87: Theoretical Division, Los Alamos National Laboratory
Talk presented at the Siofok-Conference, Sept. 1986

bounded away from zero (as it typically happens on open spaces) whereas the UV divergence comes from the fact that the *trace* of $\log i\not{D}$ does not exist. To cure the IR divergence one adds a small imaginary mass term to $i\not{D}$. However, a mass is not chirally invariant ($m \rightarrow m\exp(2i\alpha\gamma_5)$) and it is actually more convenient to replace m by a small chiral doublett $M = m + in\gamma_5$ which rotates under chiral transformations. The UV divergence is removed by using one of the conventional UV regularization schemes. Having removed the divergences in this way one may write

$$\Gamma = tr_u \log i(\not{D} + M), \tag{3}$$

where u denotes the UV-regularization, and it makes sense to talk of chiral variations of Γ. It is then easy to see that the chiral variation of Γ naturally splits into two parts corresponding to the variation of $\not{D}$ and M

$$\delta\Gamma = \delta\Gamma_D + \delta\Gamma_M \tag{4}$$

where

$$\delta\Gamma_D = i\, tr_u\Big(\frac{1}{\not{D} + M}\gamma^\mu\gamma_5\alpha_{,\mu}\Big) \tag{5}$$

is IR-convegent even for $M = 0$, only mildly UV-divergent and proportional to $\alpha_{,\mu}$ and

$$\delta\Gamma_M = 2i\rho^2\, tr\,\Big(\frac{1}{\rho^2 + H_+} - \frac{1}{\rho^2 + H_-}\Big)\,\alpha, \quad \rho^2 = m^2 + n^2, \quad H_\pm = Q^\mp Q^\pm \tag{6}$$

is UV-convergent and proportional to α. From now on we shall consider only the global part $\delta\Gamma_M$ of the anomaly which contains all the IR information.

On *compact spaces* the spectrum of $i\not{D}$ is discrete and one sees at once from (6) that for a constant α

$$\lim_{\rho\rightarrow 0}\frac{\partial\Gamma_M}{\partial\alpha} = 2i(n_+ - n_-), \tag{7}$$

where $n_\pm$ are the multiplicities of the the right and left handed zero modes of $-\not{D}^2$ or equivalently the number of zero modes of $Q^\pm$. Note that the index formula (7) also holds in the more general case where the zero eigenvalues of $i\not{D}$ are isolated from the rest of the (possibly continuous) spectrum.

On *open spaces*, however, the continuous part of the spectrum of $-\not{D}^2$ generally stretches down to zero and in this more general case

$$\lim_{\rho\rightarrow 0}\frac{\partial\Gamma_M}{\partial\alpha} = 2i\lim_{\rho\rightarrow 0}\rho^2\int_0^\infty \frac{d\delta(E)}{E + \rho^2} = \frac{2i}{\pi}\big(\delta_+(0) - \delta_-(0)\big). \tag{8}$$

Here $\delta(E)/\pi$ denotes the trace of the difference $P(E) = P_+(E) - P_-(E)$ of the spectral measures of H_+ and H_-, i.e. tr $\int f(E)dP(E) = 1/\pi\int f(E)d\delta(E)$ for sufficiently fast decaying functions f, and $\delta(0)$ means the limit of $\delta(E)$ as E tends to zero from the $+$ direction. The notation in (8) in by no means accidential. In fact, what I now wish to sketch, and this is our main result, is that $\delta_{\pm}(E)$ are just the phase shifts of $H_{\pm}$. Indeed, with the representation of the spectral measures

$$P_{\pm}(E) = \frac{1}{2\pi i} \lim_{\epsilon \searrow 0} \log \frac{E - H_{\pm} - i\epsilon}{E - H_{\pm} + i\epsilon},$$

and the identity

$$\frac{E - H_{\pm} - i\epsilon}{E - H_{\pm} + i\epsilon} = \frac{1}{E + \Delta + i\epsilon}\, \Sigma_{\pm}(\epsilon, E)\, (E + \Delta - i\epsilon),$$

where $\Sigma_{\pm}(\epsilon, E)$ becomes the S-matrix $S_{\pm}(E) = \exp(2i\delta_{\pm}(E))$ corresponding to $H_{\pm}$ after the limit $\epsilon \searrow 0$ has been taken, one obains [2]

$$\lim_{\rho \to 0} \frac{\partial \Gamma_M}{\partial \alpha} = 2i(n_+ - n_-) + \frac{2i}{\pi} \sum_l \left(\delta^l_+(0) - \delta^l_-(0)\right). \tag{9}$$

In deriving (9) we used the fact that the integer part of $\delta^l_{\pm}(0)/\pi$ counts the number of normalizable right- resp. left-handed zero modes and therefore $\delta^l_{\pm}(0) \in (0, \pi]$. The eq. (9) is the desired generalization of (7) to the non-compact case. One sees that the low energy scattering states (actually the zero energy resonance states) produce the fractional part of the anomaly.

It is generally accepted that in $d = 2n$ dimensions the U(1) chiral variation $\delta\Gamma$ is given by the formula

$$\delta\Gamma = 2i\kappa \int \alpha(x)\Phi(x) d^d x, \quad \kappa = \frac{1}{n!(4\pi)^n}, \tag{10}$$

where $\Phi(x)$ is a pseudo-scalar which is a divergence of a local function of the gauge potential, i.e.

$$\Phi(x) = \epsilon^{\alpha_1 \ldots \alpha_{2n}} F_{\alpha_1 \alpha_2} \cdots F_{\alpha_{2n-1}\alpha_{2n}} = \partial_\mu \Phi_\mu(x) \tag{11}$$

For a constant α we can eliminate the anomaly from (10) and (9) and obtain a generalization to non-compact spaces of the Atiyah-Singer index theorem, namely

$$\kappa \oint \Phi_\mu(x) dS^\mu = (n_+ - n_-) + \frac{1}{\pi} \sum_l \left(\delta^l_+(0) - \delta^l_-(0)\right). \tag{12}$$

This formula is independent of the anomaly, and we will sketch how one derives it directly, i.e. by using only ordinary quantum mechanical scattering theory, for the simplest case of a ***two-dimensional space*** (d=2). In this case the phase shifts $\delta^l_{\pm}(0)$

reduce to the (supersymmetric) Bohm-Aharanov phase shifts. Let us chose the gauge $A_\delta = -\epsilon_{\delta\alpha} x_\alpha \Phi(x)/r^2$ and the representation $\gamma_0 = \sigma_2$, $\gamma_1 = \sigma_1$, such that

$$i\not{D} = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\theta}\big(\partial_r - (L+\Phi)/r\big) \\ e^{-i\theta}\big(-\partial_r - (L+\Phi)/r\big) & 0 \end{pmatrix}, \quad (13)$$

where the orbital angular momentum $L = \frac{1}{i}\partial_\theta$ has the eigenvalues $l \in \{0, \pm1, \pm2, ...\}$.

Zero modes: It is well known that the zero modes of the supersymmetric hamiltonian $-\not{D}^2$ fulfil the first order differential equation

$$\begin{aligned} Q^+\phi_+ &= 0 \quad \text{and} \quad \phi_- = 0 \quad \text{or} \\ Q^-\phi_- &= 0 \quad \text{and} \quad \phi_+ = 0\,, \end{aligned} \quad (14)$$

where $\phi_\pm$ denotes the $\gamma_5 = \pm1$ component of the Dirac spinor. We assume $\Phi = \Phi(r)$ for simplicity, in which case L commutes with $-\not{D}^2$. By integrating the equations (14) over a circle of radius r one finds the following flux-equations ($\epsilon = \pm$)

$$\begin{aligned} 2\pi\Phi(r) &= -\oint \frac{\partial_\theta \phi_\epsilon}{\phi_\epsilon} d\theta \;-\; \epsilon \oint \frac{\partial_r \phi_\epsilon}{\phi_\epsilon} r\, d\theta \\ &= -2\pi l - \epsilon \oint \frac{\partial_r \phi_\epsilon}{\phi_\epsilon} r\, d\theta. \end{aligned} \quad (15)$$

From this equation one can already find the number of bound states, since they must be smaller than $1/r$ both as $r \to 0$ and as $r \to \infty$. Hence they are just those for which

$$l\epsilon < 1 \quad \text{and} \quad (\Phi + l)\epsilon > 1, \quad (16)$$

where $\Phi = \Phi(\infty)$ is the total flux and we have used the fact that $\Phi(0) = 0$. It follows from (16) that

$$\Phi\epsilon > 0 \quad \text{and} \quad 0 \le |l| \le |\Phi| - 1. \quad (17)$$

For example, for positive Φ there are only right handed ($\epsilon = +$) bound states with angular momenta $l = 0, -1, .., [1 - \Phi]$. In any case, one sees at once that the integer part of the flux is equals to the index,

$$[\Phi] = n_+ - n_-, \quad (18)$$

which proves part of (12).

Scattering states: For finding the scattering states we, contrary to the zero modes, must solve the second order differential equations

$$-\not{D}^2\psi = E\psi \quad \Rightarrow \quad (-D^2 + \frac{\epsilon}{2}B)\psi_\epsilon = E\psi_\epsilon, \qquad E > 0, \quad \epsilon = \pm. \quad (19)$$

In the outside region $r \ge a$, where B=0, we can solve these second order equations explicitly, whereas in the interior region we can approximate them by the zero-energy

solutions and have good control on the error because the relevant perturbation parameter $a \cdot k$ tends to zero.
outside: Here B vanishes and in a fixed angular momentum sector the equation (19) reduces to the Bessel equation and with $W = |\Phi + l|$ and $E = k^2$ one obtains

$$\psi_\epsilon^l = \alpha J_W(kr) + \beta J_{-W}(kr), \tag{20}$$

which yields (we supress ϵ and l for the moment)

$$\tan\delta = \frac{\beta - \alpha}{\beta + \alpha} \tan\frac{\pi W}{2}. \tag{21}$$

In particular $\tan\delta = \pm\tan\pi W/2$ for $\alpha = 0$ and $\beta = 0$ respectively.
inside: We only need the logarithmic radial derivatives for matching the inside to the outside solutions. With $\psi_\epsilon = \phi_\epsilon + \delta\psi_\epsilon$, where ϕ_ϵ is the zero-energy solution, one easily finds from (19) and (15)

$$\frac{r\partial_r\psi_\epsilon}{\psi_\epsilon}|_{r=a} \sim -\epsilon(\Phi + l) - (ka)^2\Delta^2, \tag{22}$$

with a strictly positive and k-independent Δ^2.
matching: To match the logarithmic derivatives of the outside and inside solutions at $r = a$ we use the fact that the Bessel functions in (20) may be approximated for $ka \to 0$ by their values in the neighbourhood of the origin. In this way one ends up with

$$\frac{\beta}{\alpha} = \text{const}\,(ka)^{2W}\,\frac{W + \epsilon(\Phi + l) + (ka\Delta)^2}{W - \epsilon(\Phi + l) - (ka\Delta)^2} \tag{23}$$

as the equation to determine β/α. From (23) one sees that $\beta/\alpha \to 0$ in the low energy limit and hence $\tan\delta = -\tan\pi W/2$ in all cases except when $\epsilon(\Phi + l) = W < 1$, in which case $\alpha/\beta \to 0$ as $k \to 0$ and $\tan\delta = \tan\pi W/2$. Since furthermore $l\epsilon < 1$ from eq. (16) there is only one special angular momentum l_s and only one chirality for which the sign of the phase shift is the reverse of the normal phase shift $\pi\langle\Phi\rangle/2$. Here $\langle\Phi\rangle$ denotes the fractional part of the flux, i.e. $\Phi = [\Phi] + \langle\Phi\rangle$. We conclude that

$$\delta^l(0) = \delta_+^l(0) - \delta_-^l(0) = \pi\langle\Phi\rangle\delta_{l,l_s} \tag{24}$$

which, together with (18), establishes the generalized index theorem (12) for the two-dimensional case.

The reason for the fractional anomaly is the long range gauge field which forbids a compactification of the underlying space. Another way to deal with such a situation is the introduction of a (spherical) boundary. The fractional part is then recovered as the remaining boundary term after the limit r(boundary) $\to \infty$ has been taken. In dealing with manifolds with boundaries one may employ the Atiyah-Patodi-Singer (APS) index theorem. Let us briefly recall this theorem in order to prepare the ground for what follows. The Dirac operator on an $d = 2n$-dimensional space X with boundary ∂X can be brought into the standart form (1). Near ∂X we assume that

$$Q^{\pm} = \mp\partial_u + B, \tag{25}$$

where B is a selfadjoint operator on the boundary. We parametrize X near ∂X by (u, y) such that $(u = 0, y)$ are the coordinates on the boundary. Then, by using the eigenfunctions e_l of B, $Be_l = \omega_l e_l$, we may expand any function near ∂X as

$$\psi(u, y) = \sum_l \begin{pmatrix} f_l(u) \\ g_l(u) \end{pmatrix} e_l(y) \tag{26}$$

in terms of which the boundary conditions (BC) for the Dirac operator read

$$\begin{aligned} f_l(0) &= 0 \qquad \text{for} \quad w_l \geq 0 \\ g_l(0) &= 0 \qquad \text{for} \quad w_l < 0. \end{aligned} \tag{27}$$

With these domains for Q^+ and Q^- it is possible to demonstrate that Q^+ and Q^- are adjoints of each other. The η-function (for simplicity we assume that B has no zero modes)

$$\eta(s) = \sum_l \mathrm{sgn}(\omega_l)|\omega_l|^{-s} \tag{28}$$

can be continued to $s = 0$ and $\eta(0)$ is the celebrated η – invariant. With these definitions the APS index theorem reads

$$\kappa \int_X \Phi(x) = n_+ - n_- - \frac{1}{2}\eta(0). \tag{29}$$

In comparing (29) with (12) one expects that if the boundary conditions (27) simulate those on L_2, $\eta(0)$ ought to be proportional to the difference $\delta_+(0) - \delta_-(0)$. To compare the η-invariant with the phase shifts one conveniently uses a different characterization of the first one [4]. For that purpose one takes the high temperature expansion of the 'difference' of the partition functions of H_+ and H_- on the *outside* region, namely

$$K(\beta) = tr\left(e^{-\beta H^+} - e^{-\beta H^-}\right) \sim \sum_k a_k \beta^k \qquad \text{as } \beta \to 0 \tag{30}$$

and observes that

$$\eta(0) = -2a_0. \tag{31}$$

Note, that the domains (27) for $Q^{\pm}$ imply the following domains for $H^{\pm} = Q^{\mp}Q^{\pm}$:

$$\begin{aligned} f_l(0) &= 0 \qquad & Q^- g_l(0) &= 0 \qquad \text{for} \quad \omega_l \geq 0 \\ g_l(0) &= 0 \qquad & Q^+ f_l(0) &= 0 \qquad \text{for} \quad \omega_l < 0. \end{aligned} \tag{32}$$

The Laplace transform of $K(\beta)$,

$$G(\rho^2) = \int_0^\infty e^{-\beta\rho^2} K(\beta) d\beta = tr\left(\frac{1}{\rho^2 + H_+} - \frac{1}{\rho^2 + H_-}\right)$$

we have already encountered in the anomaly (6) (now for the Dirac operator on the outside region with boundary conditions (27)) and therefore

$$\frac{\partial \Gamma_M}{\partial \alpha} = 2i\, \rho^2 G(\rho^2). \tag{33}$$

As an example, assume $2iG(\rho^2) = 1/\rho^2 \lim_{\rho\to 0} \partial\Gamma_M/\partial\alpha$. Then

$$K(\beta) = \frac{1}{2i} \lim_{\rho\to 0} \frac{\partial \Gamma_M}{\partial \alpha} = a_0 = -\frac{1}{2}\eta(0) \tag{34}$$

is temperature independent and the η-invariant is equals (up to a factor i) to the global part of the anomaly. However, to deal with more general cases one may employ a more direct relation between the phase shifts and the η-invariant, namly the characterization

$$K(\beta) = \sum_l K_l(\beta) = \sum_l \int e^{-\beta E} \frac{d\delta_l}{dk} dk \tag{35}$$

of the partition function, which can be proven by using similar arguments than those given in the derivation of eq.(9).

As a first example let us consider the simple case when the Dirac operator is defined on the *cylinder* $\{u, \theta\} \in R^+ \times S^1$ with $A_u = 0$ and $A_\theta = \Phi$ is constant. Then the operators $Q^\pm$ are of the form (25), wherein

$$B = -\frac{1}{i}\partial_\theta - \Phi \tag{36}$$

has eigenvalues $w_l = -(l + \Phi)$. The scattering solutions of $H_l^\pm = -\partial_u^2 + w_l^2$ which fulfil the BC (32) are

$$\begin{aligned} f_l &= \sin ku \qquad & g_l &= \frac{1}{k} Q_l^+ f_l = \sin(ku + \delta_-^l) \qquad & &\text{for } \omega_l \geq 0 \\ g_l &= \sin ku \qquad & f_l &= \frac{1}{k} Q_l^- g_l = \sin(ku + \delta_+^l) \qquad & &\text{for } \omega_l < 0, \end{aligned} \tag{37}$$

where the phase shifts solve the equations $\tan \delta_\pm^l = \mp k/\omega_l = \mp\sqrt{E - w_l^2}/w_l$. By applying (35) we finally end up with

$$K_l(\beta) = -e^{-\beta\omega_l^2} \int e^{-\beta k^2} \frac{|\omega_l|}{\omega_l^2 + k^2} = -\frac{1}{2} \mathrm{sgn}(\omega_l)\, \mathrm{erfc}\left(\sqrt{\omega_l^2 \beta}\right), \tag{38}$$

which yields the correct APS η-invariant $\eta = \langle\Phi\rangle - 1/2$. It may be worth mentioning, that the anomaly vanishes in this case since the continuum of the Dirac operator is separated from zero. Indeed, by using the identity (8) or by applying Green-functions techniques, one easily can show that

$$i\frac{\partial\Gamma_M}{\partial\alpha} = \sum_l \Big(\frac{\omega_l}{\sqrt{\rho^2+\omega_l^2}} - \mathrm{sgn}(\omega_l)\Big) \tag{39}$$

which, by the way, exactly reproduces the L_2 result obtained in [3,5], and as expected tends to zero (like ρ^2) for small ρ. However, for large values of ρ^2 the Φ-derivative of (39),

$$i\frac{\partial}{\partial\Phi}\frac{\partial\Gamma_M}{\partial\alpha} = -\sum \frac{1}{\rho}\Big(1+\frac{(l+\Phi)^2}{\rho^2}\Big)^{-3/2},$$

approaches the integral $-\int dx(1+x^2)^{-3/2} = -2$, and since $\partial\Gamma_M/\partial\alpha$ is periodic in Φ with period 1 and furthermore vanishes at $\Phi = 1/2$ one sees at once that

$$\frac{\partial\Gamma_M}{\partial\alpha} \longrightarrow i(2\langle\Phi\rangle - 1) = i\eta(0) \qquad \text{as} \quad \rho\to\infty. \tag{40}$$

Thus we find that on the cylinder, where the threshold energy of $-\not{D}^2$ is strictly positive, the η-invariant coincides with the high energy limit of $\partial\Gamma_M/\partial\alpha$ which in turn can be identified as the Pauli-Villars regulator.

As second example we reconsider the *supersymmetric Bohm-Aharonov effect.* In this case the operators $Q^\pm$ in (13) are not of the standart APS-form (25) and it is not clear a priori which part of $Q^\pm$ should be identified with the boundary operator B. To proceed we observe that on the outside region the total angular momentum $J = L - \gamma_5/2$ commutes with $\not{D}$ and on a fixed J-sector ($j \in \{\pm 1/2, \pm 3/2, ..\}$)

$$i\not{D}_j = \begin{pmatrix} 0 & \partial_r + 1/2r - \omega_j/r \\ -\partial_r - 1/2r - \omega_j/r & 0 \end{pmatrix}, \qquad \omega_j = j + \Phi.$$

Since $\partial_r + 1/2r$ is the adjoint of $-\partial_r - 1/2r$ on $L_2(R^+, rdr)$ one is tempted to take $B_J = (J+\Phi)/a$ with eigenvalues ω_j/a as boundary operator. However, with this choice of B and the appropriate BC (32), the right and left handed scattering states (we take $\omega_j > 0$ and suppress the index j)

$$\begin{aligned} f &= \alpha J_{\omega+1/2}(kr) + \beta J_{-(\omega+1/2)}(kr) \\ g = \frac{1}{k}Q^+ f &= \alpha J_{\omega-1/2}(kr) - \beta J_{-(\omega-1/2)}(kr), \end{aligned} \tag{41}$$

have, after properly normalized with the free solutions, exactly the same phase shifts. From (8) and (35) we immediately find the discouraging result that $\partial\Gamma_M^j/\partial\alpha$ and $K_j(\beta)$ both are $\frac{1}{2}\pi$ (the free part) for $\omega > 0$ and $-\frac{1}{2}\pi$ for $\omega < 0$. Thus the sum $\sum K_j(\beta)$ is ill-defined.

On the other hand, when one considers the second order operator

$$\not{D}^2 = \begin{pmatrix} -\partial_r^2 - 1/r\,\partial_r + (L+\Phi)^2/r^2 & 0 \\ 0 & -\partial_r^2 - 1/r\,\partial_r + (L+\Phi)^2/r^2 \end{pmatrix}, \tag{42}$$

one sees that one is forced to take $B_L = L+\Phi$ as boundary operator in order to obtain a convergent sum in (35). However, with this choice and the accompanying boundary conditions (27) Q^+ is not anymore the adjoint of Q^-, due to the factors $\exp(\pm i\theta)$ in (13). Adjointness demands that the BC (27) go with the eigenvalues of B_J and not with those of B_L. It turns out that when one takes $\delta(k) = \sum_l (\delta_+^l(k) - \delta_-^l(k))$, where $\delta_\pm^l$ are the phase shifts in the *same B_L-sector*, but computed with the BC (27) wherein one takes the eigenvalues of B_J, then

$$\frac{\partial \Gamma_M}{\partial \alpha} = 2i\langle\Phi\rangle \tag{43}$$

is ρ-independent and, according to (34), equals to $i\eta(0)$. In addition, both the low and high energy limit (Pauli-Villars regulator) of the quantity $\partial\Gamma_M/\partial\alpha$ coincide and reproduce the correct L_2-result. In the low energy region the anomaly $\lim_{\rho\to 0}\partial\Gamma_M/\partial\alpha$ comes, as we have seen, from the zero energy phase shifts. What happens is that the resonance state in the special angular momentum sector l_s causes a jump at zero energy of the corresponding phase shift.

We conclude by remarking that the methods presented in the first part of this talk are not restricted to a particular dimension and that our 2-dimensional computations, which have been presented in the second part, can be done in higher dimensions as well which in turn may help to expose the different high- and low-energy aspects of anomalies.

ACKNOWLEDGEMENTS: I am grateful to my collaborators P. Forgacs, P. Horvathy, R. Musto and L .O'Raifeartaigh. This work was supported in part by the Deutsche Forschungsgemeinschaft.

REFERENCES

[1] S.L. Adler, Phys. Rev. 177 (1969) 2426
J.S. Bell and R. Jackiw, Nuovo Cimento A 60 (1969) 47

[2] R. Musto, L. O'Raifeartaigh and A. Wipf, Phys. Lett. B 175 (1986) 433

[3] K. Kiskis, Phys. Rev. D 15 (1977) 2329
D. Blankenbecler and D. Boyanovsky, Phys. Rev. D 31 (1985) 3234
T. Jaroszewicz, Harvard preprint HUTP-86/B02
M. Stone, Ann. Phys. (N.Y.) 155 (1984) 56

[4] M. Atiyah, V. Patodi and I. Singer, Math. Proc. Cambridge Philos. Soc. 77 (1975) 42; 78 (1975) 405; 79 (1976) 71
M. Ninomiya and C.I. Tan, Nucl. Phys. B 257 (1985) 199

[5] A. Wipf, Nucl. Phys. B 269 (1986) 24