86-19

FM85/A005

MANY FERMION GREEN FUNCTIONS AND DYNAMICAL ALGEBRA

Allan I Solomon
Faculty of Mathematics
Open University
Milton Keynes
UK

and

Joseph L Birman
Physics Department
City College, CUNY
New York 10031
USA

86-19

Talk presented at the XIV International Colloquim on Group Theoretical Methods in Physics, Seoul, August 1985

MANY FERMION GREEN FUNCTIONS AND DYNAMICAL ALGEBRA

Joseph L. Birman

Dept. of Physics, City College, C.U.N.Y., New York, NY 10031,USA

and

Allan I. Solomon

Faculty of Mathematics, The Open University, Milton Keynes, U.K.

ABSTRACT

The mean-field Hamiltonian for a many-fermion system is an element in a Dynamical Algebra - a classical Lie Algebra. The thermal Matsubara and T=0 Green Functions are sums of products of factors determined by certain structure constants of the algebra. These describe the automorphisms of the Algebra.

In the mean-field approximation the Hamiltonian of an interacting many-fermion system is a sum of bilinear products of fermion creation/destruction operators with complex coefficients. This applies for systems such as a singlet BCS superconductor; and a coexisting charge-density-wave superconductor; and coexisting charge - and spin-density-wave superconductor. The Hamiltonian can be taken as

$$H = \sum_k H_k$$

where (k) is a wave-vector index. We define a basis (A_j) by relabelling the standard Fermi operators as $(A_1, A_2, \ldots) = (a_{k\uparrow}, a^{\dagger}_{-k\downarrow}, \ldots)$. Then $H_k = \sum_{ij} \lambda_{ij} X_{ij}$ with $X_{ij} = A_i^{\dagger} A_j$. The matrix λ with $(\lambda)_{ij} = \lambda_{ij}$ represents the complex coupling coefficients. The set of all pairs X_{ij} closes under Lie Bracket: $[X_{ij}, X_{\ell n}] = X_{im}\, \delta_{\ell j} - X_{\ell j}\, \delta_{im}$. If (i,j) = 1...N the algebra is an SO(4N), (or D series) algebra. The set of all singles (A_i) and $(A_i^{\dagger})$ <u>plus</u> all pairs (X_{ij}) generates an SO(4N+1) (or

B series) algebra under Lie Bracket. Evidently the Hamiltonian H_k in the mean-field approximation is an element in the SO(4N) algebra. If we pass to a Cartan-Weyl realization of the basis, then H_k can be re-written as

$$H_k = \sum_j \mu_j \, h_j + \sum_{\alpha\pm} \mu_\alpha \, e_\alpha \tag{1}$$

where the (h_j, e_α) are the generators in C-W form and (μ_j, μ_α) are some combinations of λ_{ij}.

The T=0 Green Function can be written, (suppressing spin) as

$$G(k,t) = -i < \phi' | T_t \; a_k(t) \; a_k^\dagger \; | \phi' > \tag{2}$$

here $|\phi'>$ is the (mean-field) ground state of H_k, and

$$a_k(t) = (\exp i \, H_k \, t) \; a_k \; (\exp - i \, H_k \, t) \tag{3}$$

Using the (A_i) notation we find

$$a_k(t) = A_r(t) = \sum_j (e^{-it\lambda})_{rj} \; A_j \tag{4}$$

so,

$$a_k(t) \; a_k^\dagger \equiv A_r(t) \; A_r^\dagger = \sum_j (e^{-it\lambda})_{rj} \; X_{rj}^\dagger \tag{5}$$

Examining the above we can identify the matrix elements $(e^{-it\lambda})_{rj}$ as "structure constants" of an "Heisenberg" automorphism ϕ: $A_r \overset{\phi}{\rightarrow} \{A_r\}$ of single fermion operators onto themselves in the B algebra.

The H_k can be rotated to "diagonal" form: $U \, H_k \, U^{-1} = \sum_j \gamma_j \, h_j \equiv H_k'$, by means of a unitary transformation $U = \exp i \, (\sum_{\alpha\pm} \theta_\alpha \, e_\alpha^j)$, with coefficients θ_α. We can identify the coefficients γ_j as a <u>second</u> set of "structure constants" of an automorphism ϕ' of pairs of fermion operators onto themselves: $(h_j, e_\alpha) \overset{\phi'}{\rightarrow} (h_j)$ produced by the mapping U.

After rotation H_k is a sum of generators (h_j) of the Cartan sub-algebra of D. It is natural to label the kets $|\{\lambda\}>$ of H_k' by eigen-values of the h_j: $h_j \; |\lambda'\}> = \lambda_j' \; |\{\lambda'\}>$ where (λ') are the lowest eigen-values. In terms of the eigenfunction $|\{\lambda'\}>$ the state $|\phi'>$ is $|\phi'> = U^{-1} \; |\{\lambda'\}>$.

The Green Function G(k,t) can be written for t>0 as

$$-i \sum_j (e^{-it\lambda})_{rj} \quad < \{\lambda'\}| \, U X^{\dagger}_{rj} \, U^{-1} \, |\{\lambda'\} > \tag{6}$$

But $U X^{\dagger}_{rj} U^{-1} \Rightarrow U(h_\ell, e_\beta) U^{-1}$ which is again an automorphism in D $\{h_\ell, e_\beta\} \; \Phi' \; \{h_\ell, e_\beta\}$ with structure constants (γ)

$$U(h_\ell, e_\beta) \, U^{-1} = \sum_j \gamma_j h_j + \sum_{\alpha\pm} \gamma_\alpha e_\alpha \tag{7}$$

Taking the diagonal matrix element of (7) only the terms $\sum \gamma_j h_j$ contribute. Then for $t>0$

$$G(k,t) = (-i) \sum_j (e^{-it\lambda})_{rj} \sum_\ell \gamma_\ell \lambda'_\ell \tag{8}$$

with a similar expression for $t<0$. This result gives the factorization of $G(k,t)$ into a sum of products of factors determined by the Dynamical Algebras (D and B), namely:

(1) Structure constants $(e^{-it\lambda})$ of ϕ;

(2) Structure constants γ_ℓ of ϕ';

(3) Eigenvalues $\{\lambda'\}$ of the operators h_ℓ of the Cartan sub-algebra.

A similar factorization applies for the T=0 Gor'kov (anomalous) function $F(k,t) = < \Phi'| \, T_t \, a_k(t) \, a_{-k} \, |\Phi'>$

The Thermal Matsubara Green Function ($T\neq 0$) is defined as

$$G(k,\tau) \equiv - < T_\tau \, a_k(\tau) \, a^{\dagger}_k > \tag{9}$$

with $(\tau = it)$ and in this case the thermal average is defined as

$$< \hat{O} > \equiv \mathrm{Tr}\, (e^{-\beta H}\hat{O})/\mathrm{Tr} \;\; e^{-\beta H} \tag{10}$$

A similar analysis as in the T=0 case gives $(\tau>0)$.

$$G(k,\tau) = \sum_j (e^{-\tau\lambda})_{rj} \; \Big(\sum_{\{\lambda_p\}} \Big(\sum_p e^{-\beta E_p \lambda_p} \gamma_p \lambda_p\Big)\Big) / \sum_{p\{\lambda_p\}} e^{-\beta E_p \lambda_p} \tag{11}$$

with a similar expression for $\tau<0$. Again, $G(k,\tau)$ is constructed as a sum of terms each of which is composed of factors which are symmetry determined from structure constants of automorphisms of the Dynamical Algebra and eigenvalues of the generators (h_ℓ). *Mutatis mutandis* the thermal Gor'kov anomalous function has a similar structure.

Details of the analysis and illustrations for simple BCS-like theory of superconductivity are given elsewhere[1), 2)].

This work was supported in part by grants from the PSC-BHE Faculty Research Award Program, and The Open University.

REFERENCES

1) J.L. Birman and A.I. Solomon, Prog. Theor. Phys. Supp. 80, 62-75 (1984); Phys. Rev. Lett. 49, 230 (1982).

2) See also A.I. Solomon and J.L. Birman, Phys. Lett. 88A, 413 (1982); 104A, 235 (1984); and *ibid* 1985 in press.