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On the stability of monopoles

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Abstract

A monopole with non-Abelian charge Q admits $2\sum|2\alpha(Q)|-1$ negative modes where α is a root of the residual group. These modes can be constructed by techniques of geometric quantization. Each topological sector admits a unique stable monopole.

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As first pointed out by Brandt and Neri [1] and by Coleman [2], most non-Abelian monopoles are unstable under small perturbations. In fact [3] a monopole with non-Abelian charge Q (see Refs 2,4) is stable if and only if, for each root α of the residual symmetry group H,

$$|2\alpha(Q)| \leq 1 \qquad (1)$$

The clue to further investigations is the observation [2,5] that at large distances the monopole problem reduces to a pure Yang-Mills (YM) theory with gauge group H. Yang-Mills over S^2 is but a special case of that over a Riemann surface [6], for which the index (the number of negative modes) can be expressed in topological terms. Applied to S^2 this becomes [7]

$$\nu = 2 \sum_{\alpha(Q) \neq 0} |2\alpha(Q)| - 1 \qquad (2)$$

so that $\nu = 0$ implies (1).

The point is that the case of S^2 is so simple that the negative modes can be found explicitly, namely by the technique previously introduced in geometric quantization [8]. The problem is in fact to find the holomorphic sections of line bundles with Chern class $n_\alpha = |2\alpha(Q)| - 2$.

It has been claimed [2] that each topological sector contains exactly one stable monopole. Coleman illustrates this for $H = SU(N)/\mathbb{Z}_N$ [2], and Goddard and Olive [3] prove for the case when H has a 1-dimensional center. Our proof (valid for an arbitrary symmetry breaking pattern) uses the trick of reducing the problem to AdH, the adjoint group, which is semisimple. The statement follows then from the structure theory of Lie algebras [9]. Remarkably, $SU(N)/\mathbb{Z}_N$ is just the adjoint group of $U(N)$.

Consider a pure YM theory on a principal H-bundle P over S^2 given by

(3) $$E(A)=\int_{S^2} tr(F\wedge *F) \quad ,$$

where H is assumed to be compact and connected, F is the field strength tensor F=DA and * is the Hodge operator on S^2. The solutions of the associated field equation D*F=0 are characterized by an (up to conjugation unique) vector Q in h, the Lie algebra of H. Q is quantized, exp $4\pi Q=1$ [2,4].

The stability properties of a solution are determined by the Hessian

(4) $$\frac{1}{2}\,\delta^2 E\,(\eta,\eta) = \int_{S^2} tr \quad (D^* D\,\eta + *[*F,\eta])\,\eta$$

where the variation η is an adP-valued 1-form on S^2.

Ω , the space of 1-forms on S^2, is decomposed according to the eigenvalues $\mp i$ of *, $\Omega = \Omega^{(1,0)} + \Omega^{(0,1)}$ (This decomposition is analogous to that in 4-dimensional YM theory to self-dual and anti-self-dual forms). The eigenvalues of $\hat{F}$: $\eta \longrightarrow *[*F, \eta]$ on $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are $-q_\alpha$ and q_α respectively, where $q_\alpha = \alpha(Q)/i$, α being a root of h. There is no loss of generality in assuming $q_\alpha \geq 0$ for all positive root α, since this can always be achieved by a gauge rotation. D*D is a positive operator on $\Omega^{(1,0)}$, and Atiyah and Bott [6] show that its first non-zero eigenvalue on the subspace where $\hat{F}=-q_\alpha$ is at least $2q_\alpha$, so to be a negative mode η must satisfy $D*\eta=0$, $D\eta=0$. Splitting η and D as $\eta=\eta'+\eta''$, $D=D'+D''$ according to the eigenvalues of *, this condition reduces to

(5) $$D''\eta' =0.$$

Now, according to a theorem of Koszul and Malgrange, a complex vector bundle with connection over S^2 has a unique holomorphic structure whose holomorphic sections are exactly the solutions of (5). We conclude that the negative modes are just the holomorphic sections of $\mathrm{AdP} \otimes \Omega^{(1,0)}$.

If Q is a monopole, $\chi(e^{it}) = \exp 4\pi Qt$ is a homomorphism of $U(1)$ into H and we can form the associated bundle $Y\times_\chi H$, where Y is the principal $U(1)$ bundle of Hopf over S^2, whose Chern class is one, c(Y)=1. In fact, this is isomorphic to P ,the monopole bundle.

The decomposition $h^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum \mathfrak{g}_\alpha$ of the complexified Lie algebra (where $\mathfrak{t}$ is a Cartan algebra and the $\mathfrak{g}_\alpha$'s are the root spaces) implies the analogous decomposition

(6) $$\mathrm{adP}^{\mathbb{C}} = \mathcal{P}_0 + \sum_\alpha \mathcal{P}_\alpha ,$$

where

$$\mathcal{P}_0 = Y\times_\chi \mathfrak{t} \quad \text{and} \quad \mathcal{P}_\alpha = Y\times_\chi \mathfrak{g}_\alpha .$$

Both $\mathcal{P}_0$ and $\mathcal{P}_\alpha$ are holomorphic line bundles, $\mathcal{P}_0$ is trivial, $c(\mathcal{P}\ )=0$ and $\mathcal{P}_\alpha$ has Chern class $c(\mathcal{P}_\alpha) = 2q_\alpha$. By (6) the sections of $\mathrm{adP}^{\mathbb{C}} \otimes \Omega^{(1,0)}$ are obtained from those of

(7) $$(\mathcal{P}_0 \otimes \Omega^{(1,0)}) \oplus \sum_\alpha (\mathcal{P}_\alpha \otimes \Omega^{(1,0)}).$$

The Chern class of a tensor product is the sum of the Chern classes, and $\Omega^{(1,0)}$ has Chern class(-2). Hence $c(\mathcal{P}_0 \otimes \Omega^{(1,0)})=-2$ has no holomorphic sections. On the other hand, $c(\mathcal{P}_\alpha \otimes \Omega^{(1,0)}) = n_\alpha = 2q_\alpha - 2$. Geometric quantization [8] tells us then that for $n_\alpha \geq 0$ a line bundle with Chern class n_α admits $n_\alpha + 1 = |2\alpha(Q)|-1$ holomorphic sections.

To construct the negative modes explicitly remember [8] that $(\mathbb{C}^2)^*/\mathbb{Z}_n$ is a line bundle L_n of order n over $\mathbb{C}P_1 = S^2$. S^2 admits the complex coordinates $z = z_1/z_2$ and $w = z_2/z_1$ defined for $U_1 : z_2 \neq 0$ and $U_2 : z_1 \neq 0$ respectively. Furthermore,

$$(8) \qquad s_1(\ell) = (c z_2)^n \quad , \quad s_2(\ell) = -(c z_1)^n$$

(where $\ell = c(z_1, z_2)$, $c \in \mathbb{C}$) is a smooth, non-vanishing section of L_n with transition function $(-z^n)$. The holomorphic sections of L_n - the wave functions of the anti-holomorphic polarization $\partial_{\bar{z}}$ of S^2 [8]- are hence of the form

$$(9) \qquad f_1(z)\, z_2^{\,n} \qquad\qquad -f_2(w)\, z_1^{\,n} \, ,$$

where $f_1(z)$ and $f_2(w) = - z^n f_1(1/z)$ are both holomorphic. Consequently they are both polynomials of order at most n.

Returning to our problem, the negative modes are hence linear combinations of

$$(10) \qquad \eta^{\alpha}_{k} = z_1^{\,k}\, z_2^{\,n_\alpha - k}\, dz \, , \qquad k = 0, \dots n_\alpha$$

if $n_\alpha \geq 0$, i.e. if $|\alpha(2Q)| \geq 2$. This proves the index formula (2), and hence also the stability condition(1).

The "Brandt-Neri" condition (1) means clearly that adQ may only have 0 or $\pm i/2$ for eigenvalues. This has been analyzed by Goddard and Olive [3] who have found it to be equivalent to the requirement that the semisimple part Q' of Q must be a sum of minimal weights. More precisely, a minimal weight $\hat{\xi}_k$ of a simple Lie algebra $\mathfrak{h}'$ is a vector in $\mathfrak{h}'$, dual to one of the simple roots $(\alpha_j(\hat{\xi}_k) = i\,\delta_{jk})$ α_k such that α_k appears with coefficient 1 in the expansion of the highest root. They can be read off the Dynkin diagram [9].

Now any compact $\mathfrak{h}$ can be decomposed into $\mathfrak{h} = \mathfrak{z} + \mathfrak{h}'$, where $\mathfrak{z}$ is the centre and $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ is semisimple. Accordingly, $Q = z(Q) + Q'$. $\mathfrak{h}'$ is further decomposed into simple factors, $\mathfrak{h}' = \mathfrak{h}_1 + \ldots + \mathfrak{h}_\ell$, and the result of [3] tells that to be stable Q' must be

$$(11) \qquad 2Q_0' = \sum_j \hat{\xi}^{(j)} ,$$

where $\hat{\xi}^{(j)}$ is either 0 or a minimal weight of the simple factor $\mathfrak{h}_j$.

What we want to prove now is that each topological sector contains exactly one monopole whose semisimple part is of the form (11). This will follow from an algebraic description of π_1 of a compact group.

First, if α is a root of a semisimple $\mathfrak{h}'$, define η_α by $\alpha(\cdot) = \mathrm{tr}\,(\eta_\alpha, \cdot\,)$ If $\alpha_1, \ldots, \alpha_r$ the simple roots, when set

$$(12) \qquad \eta_j = 2\eta_{\alpha_j} / \alpha_j(\eta_{\alpha_j}) \qquad \text{and} \qquad \alpha_j(\xi_k) = i\,\delta_{jk} .$$

These vectors generate the two lattices

$$(13) \quad \Gamma_\eta' = \Big\{ \sum_j n_j \eta_j \Big\} \quad \text{and} \quad \Gamma_\xi' = \Big\{ \sum m_j \xi_j \Big\} , \quad n_j, m_j \in \mathbb{Z} .$$

If H is any compact group with algebra $\mathfrak{h}$, its roots are those of $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$. The η-lattice Γ_η is still defined by (13), and the unit lattice Γ is all $\chi \in \mathfrak{h}$ such that $\exp 2\pi\chi = 1$. For example, $2Q$, twice the non-Abelian charge of a monopole, is in Γ . If z is the projection $z: \mathfrak{h} \to \mathfrak{z}$ onto the center, $z(\Gamma)$ is a lattice in $\mathfrak{z}$, and we define

$$(14) \quad \Gamma_\xi = z(\Gamma) + \Gamma_\xi' = \Big\{ \zeta + \xi \;\Big|\; \zeta \in z(\Gamma),\ \xi \in \Gamma_\xi' \Big\}$$

These lattices are ordered according to $\Gamma_\eta \subset \Gamma \subset \Gamma_\xi$.

If H' is compact and semisimple, $\pi_1(H')$ is known to be Γ'/Γ'_γ. Now we extend this description to any compact H.

Remember first that if H' is compact and semisimple, then it has a simply connected covering group $\tilde{H}'$ with projection $\sigma' : \tilde{H}' \to H'$, and thus
$\pi_1(H') \simeq \operatorname{Ker} \sigma$.

For a general compact H denote by H' the subgroup whose Lie algebra is $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$, and define $\sigma : \mathfrak{z} \times \tilde{H}' \to H$ by

$$\sigma(\zeta, \tilde{h}) = (\exp 2\pi\zeta)\, \sigma'(\tilde{h}). \tag{15}$$

Then $\pi_1(H) \simeq \operatorname{Ker} \sigma$. If $\mathfrak{t}$ is a Cartan algebra of $\mathfrak{h}$, let $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{h}'$ and denote T' and $\tilde{T}'$ the generated maximal tori. The restriction of σ yields $\sigma : \mathfrak{z} \times \tilde{T}' \to T$ and $\pi_1(H) \simeq \operatorname{Ker} \sigma$ is still valid. $\mathfrak{t}$ is a simply connected covering space for both T and $\mathfrak{z} \times \tilde{T}'$ with covering maps $\mathfrak{t} \ni \chi \to \exp 2\pi\chi \in T$ and $\tau(\chi) = (z(\chi), \widetilde{\exp}\, 2\pi\chi')$, where $\widetilde{\exp}$ is the exponential in $\tilde{H}'$ and $\chi' = \chi - z(\chi)$.

$$(\sigma \circ \tau)(\chi) = (\exp 2\pi z(\chi)) . \sigma'(\widetilde{\exp}\, 2\pi\chi') =$$

$$= \exp 2\pi z(\chi) . \exp 2\pi\chi' = \exp 2\pi\chi$$

shows that

$$\begin{tikzcd} & \mathfrak{t} \arrow[dl, "\tau"'] \arrow[dr, "\exp 2\pi(\cdot)"] & \\ \mathfrak{z} \times \tilde{T}' \arrow[rr, "\sigma"] & & T \end{tikzcd}$$

commutes.

It follows that Ker $\sigma \simeq$ Ker exp $2\pi(\cdot)/\mathrm{Ker}\,\tau$. But Ker τ is $\tilde{\Gamma}' \simeq \Gamma'_\eta \simeq \Gamma_\eta$, Ker exp $2\pi(.) = \Gamma$, and thus

(16) $$\pi_1(H) = \Gamma/\Gamma_\eta$$

for any compact H.

The topological sectors of monopole theory are those classes in $\pi_1(H)$ which belong to the image of the connecting homomorphism $\delta: \pi_2(G/H) \to \pi_1(H)$, where G is the original gauge group. A sector is represented by $2Q+\eta$, where 2Q is in Γ and η belongs to Γ_η. Consequently

(17) $$\rho([2Q]) = z(2Q)$$

is a well-defined map from π_1 onto $z(\Gamma)$. ρ is linear and its kernel is $\Gamma \cap \mathfrak{z}'/\Gamma_\eta \simeq \pi_1(H')$. But $z(\Gamma)$ is a free Abelian group, $\mathbb{Z}^p$ where p is the dimension of $\mathfrak{z}$, and $\pi_1(H')$ is finite. Thus

(18) $$\pi_1(H) \simeq z(\Gamma) + \pi_1(H') \simeq \mathbb{Z}^p + \pi_1(H').$$

(direct sum). In particular, ρ (introduced previouslly by topological means [10]) is an isomorphism between the free part of $\pi_1(H)$ and $z(\Gamma)$. Choosing a $\mathbb{Z}$-basis $\zeta_1, \ldots, \zeta_p$ for $z(\Gamma)$, we get "quantum" numbers $m_1, \ldots, m_p$ by $\rho(2Q) = \sum m_j \zeta_j$.

$\pi_1(H') \simeq \mathrm{Ker}\,\sigma'$ is a subgroup of $Z(\tilde{H}')$, the centre of $\tilde{H}'$, so two extreme cases may arise. First, H' may be simply connected. This happens for example if H is the little group of a vector in the adjoint representation.

The other extreme case is that of Ker $\sigma' = Z(\tilde{H}')$. Then $\Gamma' = \Gamma'_\xi$, the vectors in Γ'_ξ generate loops. For any compact H, AdH, the adjoint group, does have this property. AdH $\simeq \tilde{H}'/Z(\tilde{H}')$, and has h' for its lie algebra. Decomposing to simple factors

(19) $$\pi_1(AdH) \simeq \left(\Gamma_\xi^{(1)}/\Gamma_\eta^{(1)}\right) + \ldots + \left(\Gamma_\xi^{(\ell)}/\Gamma_\eta^{(\ell)}\right) .$$

The crucial fact for our proof is that the points of the (discrete) centre of $\tilde{H}'_j$, (the covering group of the j^{th} factor) is uniquely exp $2\pi\, \hat{\xi}^{(j)}$ where $\hat{\xi}^{(j)}$ is either 0 or a minimal weight, see [9]. In other words, π_1 (AdH) is generated by those loops

(20) $$(\exp 2\pi\, \hat{\xi}^{(1)} t) \ldots\ldots (\exp 2\pi\, \hat{\xi}^{(\ell)} t)$$

For a monopole theory with AdH as residual group, the existence and uniqueness of a stable monopole is hence established, since each sector contains exactly one Q'_0 of the form (11).

For a general compact H we can proceed as follows: a loop in H is also a loop in AdH, and if $2Q_1$, $2Q_2$ generate homotopic loops in H, then $2(Q_1-Q_2)$ is in Γ_η and generates thus a contractible loop also in AdH. This yields a well-defined map

(21) $$\mu : \pi_1(H) \longrightarrow \pi_1(Ad\, H)$$

If Q' is stable as in (11), $\tilde{h} = \widetilde{\exp}\, 4\pi Q'$ is in $Z(\tilde{H}')$, and thus $h = \sigma'(\tilde{h})$ is in $Z(H') = Z(H)_0 \cap H'$ (the subscript $\circ$ means here connected component).

However, as proved in Ref. 10, exp $2\pi(\cdot)$ maps $z(\Gamma)$ onto $z(H)_0 \cap H'$ with kernel $\Gamma_z = \mathfrak{z} \cap \Gamma$. Hence $h = \exp 2\pi\zeta$ for some ζ in $z(\Gamma)$, and ζ is unique up to a vector χ in Γ_z. But for such a χ exp $2\pi\chi$ is never contractible since $\Gamma_z \cap \Gamma_\gamma$ is empty.

We get therefore the following algorithm for constructing the unique stable monopole Q_0 of a given topological sector: first, choose the unique stable Q_0' from $\mu([Q_0])$ in $\pi_1(\mathrm{Ad}H)$ as in (11). Second, the equation

(22) $$\exp 2\pi\zeta = \exp(-4\pi Q_0')$$

admits, as we have just proved, a unique solution ζ in our sector.

(23) $$2Q_0 = \zeta + 2Q_0' = \zeta + \sum_j^{\ell} \hat{\xi}^{(j)}$$

is then the unique stable monopole in our sector.

Furthermore, any other monopole of the sector has charge

(24) $$2Q = 2Q_0 + \gamma,$$

where γ is in Γ_γ.

Referneces

[1] Brandt R.A. and Neri F. Nucl. Phys. B161, 253 (1979)

[2] Colemann S. in The Unity of Fundamental Interactions, A. Zichich, ed. Plenum, New York (1983)

[3] Goddard P. and Olive D. Nucl. Phys. B191, 528 (1981)

[4] Goddard P. and Olive D. Rep. Progr. Phys. 41, 1357 (1978)

[5] Horváthy P.A. O'Raifeartaigh L. and Rawnsley J. (to be published)

[6] Atiyah M.F. and Bott R. Phil. Trans. Roy Soc. London A308, 523 (1982)

[7] Friedrich Th. and Habermann L. Commun. Math. Phys. 100, 231 (1985)

[8] Kostant B. Quantization and Unitary Representations, in Lecture Notes in Math. 170, Berlin, Springer (1970); Souriau J-M. Structure des systemes dynamiques, Paris, Dunod (1970); Gawedzki Fourier-like kernels in Geometric Quantization, Diss. Math. 128, Warszawa (1976)

[9] Helgason S. Differential Geometry, Lie Groups and Symmetric Spaces, New York, Academic (1978)

[10] Horváthy P.A. and Rawnsley J.H. Commun. Math. Phys. 99, 517 (1985)