On the Problem of Space-Time Symmetries in the Theory of Supergravity (Part II)

by

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Abstract

The definition of a space-time symmetry, developed in a previous paper in the framework of Simple (N=1) Supergravity, is extended to the N=2 theory. As an application, the properties of the N=2 plane wave are studied.

The mathematically related question of defining the Lie derivative of a spinor is also considered.

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1) Introduction

In a previous paper [1], henceforth referred to as "Part I", we presented a definition of the notion of a space-time symmetry in Simple Supergravity. This definition was based upon the analogous one in Gauge Theories [2,3]. More precisely, it was proposed that a solution (ϑ^a , ψ) of N = 1 Supergravity be considered as symmetric under the space-time motion generated by a vector field ξ if and only if there exist a skew-symmetric matrix Λ_{ab} and a Majorana spinor S such that[1]

$$\pounds_\xi \vartheta^a = \Lambda^a{}_b \vartheta^b + i \bar{S} \gamma^a \psi \qquad (1.1)$$

$$\pounds_\xi \psi = D S. \qquad (1.2)$$

However, it was shown that two major difficulties arose because of the presence of the Lie derivative $\pounds_\xi$ acting on the spinor-valued one-form ψ.

Firstly, the notion of a Lie derivative $\pounds_\xi$ seemed to be well defined only when the differentiating vector field ξ was a conformal Killing vector. Therefore, we were led to adjoining to (1.1), (1.2) the ad hoc restriction $\pounds_\xi g = \phi g$, where g is the metric.

Secondly, it was proven that the two usual definitions

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[1] Here and in the sequel, the notation is the same as in Part I and will therefore not be further specified. We simply recall that Greek indices refer to an arbitrary frame, whereas Latin indices refer to an orthonormal one.

[1] for the Lie derivative of a spinor, namely

$$\pounds^{(1)}_{\xi} S \equiv \xi(S) - \tfrac{1}{2} (\xi_{a;b} - \Gamma_{abc}\xi^{c}) \sigma^{ab} S \qquad (1.3)$$

$$\pounds^{(2)}_{\xi} S \equiv \xi(S) - \tfrac{1}{2} L_{ab} \sigma^{ab} S \qquad (1.4)$$

$$\pounds_{\xi} \vartheta^{a} \equiv L^{a}{}_{b} \vartheta^{b} \qquad (1.5)$$

which are equivalent in the case of a torsion-free Riemannian space, become distinct in the presence of torsion. Explicitly, one finds, for an arbitrary S :

$$\pounds^{(1)}_{\xi} S - \pounds^{(2)}_{\xi} S = \tfrac{1}{2} \xi^{c} T_{[ab]c} \sigma^{ab} S . \qquad (1.6)$$

Both difficulties will be overcome in the present paper. We shall define a generalised Lie derivative (GLD) $\pounds_{\xi}$ and we shall show that this new definition is compatible with ordinary tensor calculus for an arbitrary ξ. Moreover, we shall discuss to what extent it is unique. This will be done by drawing, in § 2, a parallel with the covariant derivative of a spinor. It will then be seen that all the results which were obtained in Part I remain valid with the GLD.

After having shown, in §2, the mathematical consistency of our framework, we shall in § 3 present the generalisation to N = 2 Supergravity of the definition of a space-time symmetry, developed in Part I for Simple Supergravity. The comparison with the Einstein-Maxwell theory will also be made.

Finally, in § 4, we shall apply the N = 2 definition of a symmetry to the N = 2 plane wave of Supergravity. The result will be that, in general, this solution does not admit the same group of motions as the plane wave of the Einstein-Maxwell theory. Furthermore, we shall show that if we restrict attention to a first-order calculation in the gravitinos $\psi^{(i)}$, $i = 1, 2$, the (approximate) symmetry group is precisely the one of the Einstein-Maxwell plane wave.

Both results are exactly analogous to those obtained in the simpler case studied in Part I. Therefore, due to the similarity of the methods used in the proofs and due to the length of the calculations for $N = 2$, we shall only present here explicitly the first-order calculation. The exact case will simply be sketched.

2) The Generalised Lie Derivative of a Spinor

The problem of defining the notion of the Lie derivative of a spinor received already much attention. (See e.g. [4-8].) Several of these approaches have in common that the authors begin by deriving an expression for $\pounds_\xi \psi$ under the assumption that ξ is a Killing vector. Then, after observing that the obtained formula is covariant, they adopt it as a definition in general, i.e. for an arbitrary vector field ξ. (See in particular [7].)

In the hypothesis in which ξ is a Killing vector, it is possible [8] to adapt to the Lie derivative, the method used by Weyl for defining the covariant derivative of a spinor [9]. This method shows clearly the origin of the difficulty which arises when trying to give a meaning to the Lie derivative of a spinor in general, and consequently, we shall begin by reviewing it here briefly. Then, we shall discuss the compatibility of the obtained equations with tensor calculus. This will also give us an indication on the uniqueness of our definition.

It will be particularly convenient to consider simultaneously the cases of the covariant derivative and the Lie derivative. Therefore, in this section, the words "derivative" and "transport" will be understood as referring to both cases: covariant and Lie derivative, covariant and Lie transport, respectively. Only when explicitly stated, shall we distinguish between covariant derivative and Lie derivative. We now describe Weyl's method for transporting spinors.

Consider a point P with coordinates x^μ. At P, one has some spinor, the components of which are $\psi(x^\mu)$ in the local orthonormal frame $e_a(P)$. To transport ψ from P to a neighbouring point Q with coordinates $x^\mu + \varepsilon\, \xi^\mu$, where ξ is an arbitrary vector field and ε is an infinitesimal parameter, one proceeds as follows [8,9]:

1) Transport $e_a(P)$ from P to Q along ξ. This is well defined since e_a, a = 0, 1, 2, 3 is a vector. Thus its transport is given by the usual laws of tensor calculus. Let $e^t_a(Q)$ be the transported frame at Q.

2) Provided the transport respects the scalar product, $e^t_a(Q)$ is also an orthonormal frame and consequently, it is related to the local orthonormal frame at Q, $e_a(Q)$, by a Lorentz transformation which can be calculated in terms of the parameters of the transport.

3) One then gives a meaning to the notion of the "transport of ψ from P to Q along ξ" by postulating that the components ψ^t of the transported ψ at Q, expressed in $e^t_a(Q)$, are the same as the components of ψ in $e_a(P)$, namely $\psi(x^\mu)$. The components of ψ^t in the local frame $e_a(Q)$ can easily be determined from the knowledge of the Lorentz transformation obtained in 2).

4) Finally, the "derivative of ψ along ξ" is defined as

$$\delta_\xi\, \psi \equiv \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\, (\psi\, (x^\mu + \varepsilon\, \xi^\mu) - \psi^t\, (x^\mu + \varepsilon\, \xi^\mu)), \qquad (2.1)$$

where δ_ξ denotes the (covariant or Lie) derivative, and both spinors (ψ and ψ^t) are expressed in $e_a(x^\mu + \epsilon\, \xi^\mu)$.

Obviously, the above procedure can be applied only if the transport respects the scalar product. Otherwise, the orthonormal frame $e_a(P)$ fails to remain orthonormal during the transport from P to Q.

In the case of the parallel transport, this requirement is satisfied if the theory is metric-compatible, i.e. if $\nabla g = 0$, where ∇ denotes the covariant derivative. Weyl's method then yields the definition of the covariant derivative of a spinor used in Part I:

$$\nabla_\xi \psi \equiv \xi(\psi) + \tfrac{1}{2} \Gamma_{abc}\, \xi^c \sigma^{ab} \psi. \tag{2.2}$$

For the Lie transport, this requirement is not fulfilled (in general), unless the transport takes place along a Killing vector field ξ. (Only then does the Lie transport respect the orthonormality of the frame.) In this particular case, a "Weyl-like" treatment [8] gives (1.4). Definition (1.3) is obtained from (1.4) by expressing L_{ab} in terms of the connection [8]. (This is similar to rewriting, in a torsion-free Riemannian space, the Killing equation $g_{\mu\nu,\alpha}\, \xi^\alpha + g_{\mu\alpha}\, \xi^\alpha_{\ ,\nu} + g_{\nu\alpha}\, \xi^\alpha_{\ ,\mu} = 0$, using the connection, as $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$.)

It is now clear that the problem which arises in Weyl's framework when one tries to define the Lie derivative of a spinor with respect to an arbitrary vector field ξ is analogous to the one of defining a covariant derivative in a non metric-compatible theory.

Therefore, we shall treat these two problems simultaneously.

It should be noted that the compatibility of any such definition with the rules of tensor calculus must be explicitly established. This is a consequence of the fact that, from two spinors ψ and χ , one can construct a vector as:

$$v^i = \bar{\psi} \gamma^i \chi , \qquad (2.3)$$

where γ^i is the ith Dirac matrix and the bar denotes the Dirac conjugate. If one generalises to spinors, an operator δ_ξ such as the covariant derivative or the Lie derivative, for which one assumes a Leibnitz rule, (2.3) implies that

$$\delta_\xi v^i = (\delta_\xi \bar{\psi}) \gamma^i \chi + \bar{\psi} (\delta_\xi \gamma^i) \chi + \bar{\psi} \gamma^i (\delta_\xi \chi). \qquad (2.4)$$

The left-hand side, being the action of the derivative on a vector, is defined. On the right-hand side, the action of the derivative on the spinors is assumed. Thus, (2.4) determines $\delta_\xi \gamma^i$. This, in turn, must be consistent with the defining property of the Dirac matrices:

$$\{\gamma^i, \gamma^j\} = 2 \eta^{ij}, \qquad (2.5)$$

in such a way that the following must hold:

$$(\delta_\xi \gamma^i) \gamma^j + \gamma^i (\delta_\xi \gamma^j) + (\delta_\xi \gamma^j) \gamma^i + \gamma^j (\delta_\xi \gamma^i) = 2 \delta_\xi \eta^{ij}, \qquad (2.6)$$

where on the right-hand side, $\delta_\xi \eta^{ij}$ is again a known quantity since η^{ij} is a tensor.

If one applies this procedure to the definition (2.2) of the covariant derivative and one assumes, for the reasons explained above, that the theory is not necessarily metric-compatible, (2.4) implies:

$$\nabla_a v^i = [e_a(\bar\psi) - \tfrac{1}{2} \Gamma_{mna} \bar\psi \sigma^{mn}] \gamma^i \chi + \bar\psi (\nabla_a \gamma^i) \chi + \bar\psi \gamma^i [e_a(\chi) + \tfrac{1}{2} \Gamma_{mna} \sigma^{mn} \chi]$$

$$= e_a(\bar\psi \gamma^i \chi) + \bar\psi \{\nabla_a \gamma^i - \tfrac{1}{2} \Gamma_{mna} [\sigma^{mn}, \gamma^i]\} \chi$$

$$= e_a(v^i) + \Gamma^i{}_{ma} v^m + \bar\psi \Phi^i{}_a \chi , \qquad (2.7)$$

$$\Phi^i{}_a \equiv \nabla_a \gamma^i + \tfrac{1}{2} H^i{}_{ja} \gamma^j, \qquad (2.8)$$

in which use has been made of the relations [8]:

$$\nabla_a \bar\psi \equiv e_a(\bar\psi) - \tfrac{1}{2} \Gamma_{mna} \bar\psi \sigma^{mn} \qquad (2.9)$$

$$[\sigma^{mn}, \gamma^i] = \eta^{in} \gamma^m - \eta^{im} \gamma^n \qquad (2.10)$$

$$\Gamma_{ijk} = - \Gamma_{jik} - H_{jik} , \qquad (2.11)$$

and ∇_a denotes the covariant derivative in the direction of the base vector e_a. Equation (2.11) is a consequence of the definition (given in the appendix) of the non-metricity $H_{ijk} \equiv g_{ij;k}$. (It should be noted that, due to the linearity of the covariant derivative, it is sufficient to discuss $\nabla_a \psi$. The covariant derivative in the direction of an arbitrary vector field ξ is obtained as $\nabla_\xi \psi = \xi^a \nabla_a \psi$.)

The two first terms on the right-hand side of (2.7) are recognised as the expression assigned to $\nabla_a v^i$ by the laws of tensor calculus. Thus, (2.7) implies that the definition (2.2), (2.9) of the covariant derivative of a spinor will be compatible with tensor calculus if and only if $\Phi^i{}_a$ vanishes, i.e. if, by virtue of (2.8), the covariant derivative of a Dirac matrix is assumed to satisfy:

$$\nabla_a \gamma^i + \tfrac{1}{2} H^i{}_{ja} \gamma^j = 0. \qquad (2.12)$$

This can be conveniently rewritten as:

$$\nabla_a \gamma^i + \tfrac{1}{2} (\nabla_a g)^i{}_j \gamma^j = 0. \qquad (2.13)$$

It must now be shown that this formula for $\nabla_a \gamma^i$ respects (2.5), i.e. that (2.6) holds for δ_ξ replaced by ∇_a . After an expansion of $\nabla_a \eta^{ij}$ in terms of the connection, and a substitution of $\nabla_a \gamma^i$ by its value from (2.12), one gets:

$$- H^{ij}{}_a = \Gamma^{ij}{}_a + \Gamma^{ji}{}_a ,$$

which is automatically satisfied, as a consequence of (2.11).

Thus, we have proven that, to ensure compatibility of the covariant derivative (2.2), (2.9) with tensor calculus, the covariant derivative of the Dirac matrices must be assumed to satisfy (2.12), which in turn is consistent with the definition (2.5) of the Dirac matrices. These results contain as the special case $H_{ijk} = 0$, the usual equations (valid in a metric-compatible theory [10]):

$$\nabla_a \gamma^i = 0 \text{ : special case of (2.12)}$$

$$\Gamma_{ijk} + \Gamma_{jik} = 0 \text{ : special case of (2.11).}$$

A similar calculation must be made for the Lie derivative. To treat simultaneously (1.3) and (1.4), we shall consider a "generalised Lie derivative" (GLD). We shall write:

$$\pounds_\xi \psi \equiv \xi(\psi) - \frac{1}{2} (L_{ab} + M_{ab}) \sigma^{ab} \psi \tag{2.14}$$

$$\pounds_\xi \bar{\psi} \equiv \xi(\bar{\psi}) + \frac{1}{2} (L_{ab} + M_{ab}) \bar{\psi} \sigma^{ab}, \tag{2.15}$$

where L_{ab} is defined by (1.5) and M_{ab} is still arbitrary. Obviously, M_{ab} vanishes for $\pounds_\xi^{(2)}$ whereas, for $\pounds_\xi^{(1)}$, one has:

$$M_{ab} = \xi_{a;b} - \Gamma_{abc} \xi^c - L_{ab} . \tag{2.16}$$

Proceeding as for the covariant derivative, it follows that

$$\pounds_\xi v^i = [\xi(\bar{\psi}) + \tfrac{1}{2} (L_{ab} + M_{ab}) \bar{\psi} \sigma^{ab}] \gamma^i \chi + \bar{\psi} (\pounds_\xi \gamma^i) \xi$$
$$+ \bar{\psi} \gamma^i [\xi(\chi) - \tfrac{1}{2} (L_{ab} + M_{ab}) \sigma^{ab} \chi]$$

$$= \xi(\bar{\psi} \gamma^i \chi) + \bar{\psi} \{\pounds_\xi \gamma^i + \tfrac{1}{2} (L_{ab} + M_{ab})[\sigma^{ab}, \gamma^i]\} \chi$$

$$= \xi(v^i) - L^i{}_j v^j + \bar{\psi} \Phi^i \chi \tag{2.17}$$

$$\Phi^i \equiv \pounds_\xi \gamma^i + \eta^{ik} \{ L_{(jk)} + M_{[jk]} \} \gamma^j$$

$$= \pounds_\xi \gamma^i + \{\tfrac{1}{2} (\pounds_\xi g)^i{}_j - \eta^{ik} M_{[kj]}\} \gamma^j, \tag{2.18}$$

in which use has again been made of (2.10) and the formula mentioned in Part I:

$$\pounds_\xi\, g = (L_{ab} + L_{ba})\, \vartheta^a \otimes \vartheta^b. \tag{2.19}$$

The two first terms on the right-hand side of (2.17) are the expression assigned to $\pounds_\xi\, v^i$ by the laws of tensor calculus (using the notation (1.5)) and thus, compatibility with tensor calculus is achieved if and only if the Lie derivative of a Dirac matrix is assumed to satisfy, by (2.17), (2.18):

$$\pounds_\xi\, \gamma^i + \{\tfrac{1}{2}\, (\pounds_\xi\, g)^i{}_j - \eta^{ik}\, M_{[kj]}\}\, \gamma^j = 0. \tag{2.20}$$

For the GLD, M_{ab} is arbitrary and therefore, so is $M_{[ab]}$. However, for $\pounds_\xi^{(2)}$, $M_{[ab]} = 0$ whereas, for $\pounds_\xi^{(1)}$, one obtains:

$$\begin{aligned} 2\, M_{[ij]} &\equiv M_{ij} - M_{ji} \\ &= \xi_{i;j} - \xi_{j;i} + (\Gamma_{jik} - \Gamma_{ijk})\, \xi^k + L_{ji} - L_{ij} \\ &= e_j(\xi_i) - e_i(\xi_j) + (\Gamma_{kji} - \Gamma_{kij} + \Gamma_{jik} - \Gamma_{ijk})\xi^k \\ &\qquad + L_{ji} - L_{ij}\, . \end{aligned} \tag{2.21}$$

It is a simple matter to transform (2.21) into the form:

$$M_{[ij]} = (T_{[ji]k} + H_{k[ij]})\, \xi^k. \tag{2.22}$$

One uses (3.10) of Part I and the three following identities, valid for an arbitrary (not necessarily metric-compatible) connection:

$$\Gamma_{kji} - \Gamma_{kij} = T_{kji} + D_{kij} \tag{2.23}$$

$$\Gamma_{jik} - \Gamma_{ijk} = 2\,\Gamma_{jik} + H_{ijk} \tag{2.24}$$

$$2\Gamma_{jik} + T_{kji} + D_{kij} + H_{ijk} = D_{ijk} - D_{jik} + T_{ikj} - T_{jki} + H_{ikj} - H_{kji}\,, \tag{2.25}$$

in which D_{ijk} denotes, as in Part I, the commutation coefficients of the basis. The position of the indices in (2.23)-(2.25) depends on the conventions which are used for the connection and which can be found in the appendix.

After obtaining the formula (2.20) for $\pounds_\xi\, \gamma^i$, it must still be verified that it is consistent with (2.6). An elementary calculation shows that this is indeed the case for an arbitrary M_{ab} .

The conclusion is then that the GLD (2.14), (2.15) is compatible with tensor calculus and with the defining relation of the Dirac matrices if and only if one imposes to the Lie derivative of the latter to be given by (2.20). Consequently, (2.14)-(2.15) represent, for an arbitrary M_{ab}, a class of operators (acting on spinors) which are consistent with the rules of tensor calculus.

To make a choice between all the members of this class, and in particular between (1.3) and (1.4), one returns to (2.20) and observes that, among all the possible choices for M_{ab}, the simplest expression for $\pounds_\xi \gamma^i$ is obtained for $M_{ab} = 0$. Moreover, this choice leads to an expression for $\pounds_\xi\, \gamma^i$ which is similar to the one for $\nabla_\xi\, \gamma^i$:

$$\nabla_\xi \gamma^i + \tfrac{1}{2} (\nabla_\xi g)^i{}_j \gamma^j = 0 \qquad (2.26)$$

$$\pounds_\xi \gamma^i + \tfrac{1}{2} (\pounds_\xi g)^i{}_j \gamma^j = 0, \qquad (2.27)$$

where (2.26) is the contraction of (2.13) with ξ^a.

Another reason, beside simplicity, to select the possibility $M_{ab} = 0$ is that, as mentioned in Part I, the quantity L_{ab} appearing in (2.14), (2.15) is independent of the connection. Thus the definition (2.14), (2.15) does not involve the connection for $M_{ab} = 0$. If one insists in having a connection-independent definition, one does not have at one's disposal a natural tensor with which one could identify M_{ab}. On the other hand, if one accepts a connection-dependent definition, one can then make an assumption such as (2.16) for M_{ab} . However, the notion of a Lie derivative should be independent of the connection and consequently, a possiblility such as (2.16) seems artificial.

It is worth mentioning, in the spirit of Weyl's method, that both terms $(\nabla_\xi\ g)$ and $(\pounds_\xi\ g)$ in (2.26), (2.27) express the variation of the scalar product of the base vectors when they are transported along ξ:

$$\begin{aligned}\delta_\xi (e_a \cdot e_b) &= \delta_\xi (g(e_a , e_b)) \\ &= (\delta_\xi g)(e_a , e_b) + g(\delta_\xi e_a , e_b) + g(e_a , \delta_\xi e_b) \\ &= (\delta_\xi g)_{ab} ,\end{aligned}$$

in which we assumed, for the last step, that $\delta_\xi e_a = 0$, and δ_ξ stands for $\pounds_\xi$ and ∇_ξ . Thus, in (2.26)-(2.27), the variation $\nabla_\xi\ \gamma^i$ or $\pounds_\xi\ \gamma^i$ of the Dirac matrices is due to the variation of the scalar product of the vectors of the frame e_a during the transport along ξ.

Accepting the above argument and taking M_{ab} to vanish, we have in fact made the choice to accept the definition (1.4) as the relevant one for the Lie derivative of a spinor. We shall continue to call it the "generalised Lie derivative", although we have selected the special case $M_{ab} = 0$, since we shall apply it without restriction on the differentiating vector field ξ, in contrast with what we did in Part I, where it was assumed that ξ was a (conformal) Killing vector. This definition is one of the two that we considered in Part I. As, in the latter, we checked explicitly that the results were valid for both definitions (1.3) and (1.4), it is clear that none of our results must be revised.

The proper notion of the Lie derivative is now at our disposal, and that will enable us to generalise, in the following section, the definition of a space-time symmetry, given in Part I for the N = 1 theory.

Remark

We always discussed in this section, the notion of the Lie derivative of a spinor. The Leibnitz rule makes straightforward the extension of this notion to a spinor-valued one-form. (See (3.4) in Part I.)

3) Definition of a Space-Time Symmetry in N = 2 Supergravity

In Part I, we based our definition of a space-time symmetry in Supergravity upon the analogous one in Gauge Theories [2,3]. The crucial role was played by the transformations of Supersymmetry. In the N = 2 theory, the independent fields are the electromagnetic one-form (A), the orthonormal frame (ϑ^a) and two spinor-valued one-forms (the gravitinos $\psi^{(i)}$, i = 1, 2). The Supersymmetry transformation which these fields undergo are [11]:

$$\delta \vartheta^a = i \sum_{m=1}^{2} \bar{S}^{(m)} \gamma^a \psi^{(m)} \qquad (3.1)$$

$$\delta \psi^{(m)} = 2 \nabla S^{(m)} + \frac{1}{\sqrt{2}} (\hat{F}_{ab} \sigma^{ab}) \sum_{n=1}^{2} \varepsilon^{mn} \gamma S^{(n)} \qquad (3.2)$$

$$\delta A = i \sqrt{2} \sum_{m,n=1}^{2} \varepsilon^{mn} \bar{S}^{(m)} \psi^{(n)}, \qquad (3.3)$$

where $S^{(m)}$, m = 1, 2, is an arbitrary Majorana spinor, γ denotes $\gamma_c \vartheta^c$ and $\hat{F}$ is the modified electromagnetic field:

$$\hat{F} = F - \frac{i}{2\sqrt{2}} \sum_{m,n=1}^{2} \varepsilon^{mn} \bar{\psi}^{(m)} \wedge \psi^{(n)}. \qquad (3.4)$$

For simplicity we shall not, in the sequel, indicate explicitly the summations over the "internal indices" m and n, nor shall we put the latter in brackets.

If one imitates, for the N = 2 theory, the definition of a space-time symmetry which we developed in Part I for

Simple Supergravity, one is led to consider a configuration (ϑ^a, ψ^a, A) as being symmetric under the space-time motion generated by a vector field ξ if and only if there exist a matrix Λ_{ab} and two Majorana spinors S^m such that

$$\pounds_\xi \vartheta^a = \Lambda^a{}_b \vartheta^b + i \bar{S}^m \gamma^a \psi^m , \Lambda_{ab} = - \Lambda_{ba} \qquad (3.5)$$

$$\pounds_\xi \psi^m = 2 \nabla S^m + \frac{1}{\sqrt{2}}(\hat{F}\sigma) \epsilon^{mn} \gamma S^n \qquad (3.6)$$

$$\pounds_\xi A = i \sqrt{2} \epsilon^{mn} \bar{S}^m \psi^n , \qquad (3.7)$$

in which, compared to (3.2), the notation has been simplified in an obvious way.

As in Part I, it is convenient to rewrite (3.5) in the form:

$$(\pounds_\xi g)_{ab} = i \bar{S}^m (\gamma_a \psi^m_b + \gamma_b \psi^m_a). \qquad (3.8)$$

Given the fact that we have explicitly shown that the GLD appearing in (3.6) is consistent in general, we do not impose to ξ to be a Killing vector, in such a way that (3.8) does not put an algebraic constraint upon S^m and ψ^m, as it was the case in (3.5), (3.6) of Part I.

The set of equations (3.6)-(3.8) must, however, be slightly modified since it does not exhibit the proper limiting behaviour when $\psi^m \to 0$ (which we call the "Einstein-Maxwell" limit). In this limit, the symmetry definition becomes:

$$\pounds_\xi g = 0 \qquad (3.9)$$

$$\pounds_\xi A = 0 , \qquad (3.10)$$

and S^n can be assumed to vanish. In the same circumstances, the field equations of N = 2 Supergravity [11] reduce to the Einstein-Maxwell equations. Therefore, (3.9), (3.10) should yield expressions which are recognisable, either from the gravitational, or from the electromagnetic point of view.

Obviously, (3.9) is the Killing equation used in General Relativity, whereas (3.10) is almost identical with the definition of a symmetry in Gauge Theories (in the Abelian case). However, (3.10) is more restrictive than the latter. One should rather have [2,3]:

$$\pounds_\xi A = d\Phi , \tag{3.11}$$

which expresses the fact that A must be invariant under Lie transport, up to a gauge transformation generated by the function Φ. The minimal modification that we must make to (3.7) to imitate as closely as possible Gauge Theories consists in adding a term $d\Phi$ (as in (3.11)). Consequently, our final symmetry criterion in N = 2 Supergravity takes the form:

"The configuration (ϑ^a,ψ^m,A) is symmetric under the space-time motion generated by ξ if and only if there exist two Majorana spinors S^m and a function Φ satisfying

$$(\pounds_\xi g)_{ab} = i\, \bar{S}^m (\gamma_a \psi^m_b + \gamma_b \psi^m_a) \tag{3.12}$$

$$\pounds_\xi \psi^m = 2 \nabla S^m + \frac{1}{\sqrt{2}} (\hat{F}\sigma)\, \varepsilon^{mn} \gamma S^n \tag{3.13}$$

$$\pounds_{\xi} A = d\Phi + i \sqrt{2}\, \varepsilon^{mn}\, \bar{S}^{m}\, \psi^{n}.\text{"} \qquad (3.14)$$

This clearly contains (3.6)-(3.8) of Part I as a special case (if one adds the further constraint that ξ be a (conformal) Killing vector).

It should be noted that, in the Einstein-Maxwell theory, one often calls "symmetric" the solutions for which only (3.9) holds. By using the Einstein-Maxwell field equations, one proves [12] that (3.11) is not necessarily verified. Explicit examples are known [13] in which (3.11) is actually violated, exhibiting an incompatibility between the symmetry of the metric and the one of the electromagnetic field. Such a conflict has been excluded, by construction, from our framework since we constructed (3.14) by requiring (3.11) to be satisfied in the Einstein-Maxwell limit. This is in the spirit of our procedure, since we try to draw the closest possible parallel with the definition of a symmetry in Gauge Theories. Consequently, we were led to postulate the limiting behaviour (3.11).

Moreover, to establish the possibility of violating (3.11) whilst satisfying (3.9), the Einstein-Maxwell equations must be used explicitly. In other words, this possibility arises from an interplay between the geometrical equation (3.9) and the dynamical equations (i.e. the Einstein-Maxwell equations). Our approach, imitating Gauge Theories, keeps the field equations separated from the symmetry principle. It is known [12] that if, in the Einstein-Maxwell theory, one insists in keeping such a separation, the obtained definition of a symmetry (3.9), (3.11) is somewhat more restrictive than otherwise. The question of the interplay between the geometrical- and the dynamical equations was already encountered in Part I (Note added in Proof), and a more detailed study is most conveniently left for a subsequent work.

Finally, it is useful to adopt in the N = 2 theory, the same terminology as in the N = 1 theory of Part I. We shall refer to the first-order approximation in ψ^m, to (3.12)-(3.14) as the "Rarita-Schwinger" limit. Its explicit form is:

$$\pounds_\xi \, g = 0 \tag{3.15}$$

$$\pounds_\xi \, \psi^m = 2 \, \nabla \, S^m + \frac{1}{\sqrt{2}} \, (F \, \sigma) \, \varepsilon^{mn} \, \gamma \, S^n \tag{3.16}$$

$$\pounds_\xi \, A = d\Phi . \tag{3.17}$$

We are now ready to apply our definition of a symmetry to the problem of the plane wave of N = 2 Supergravity. As explained in the introduction, we shall present mainly the calculations in the Rarita-Schwinger limit. The exact case will only be sketched. The results will be the same as for the N = 1 theory, namely that the plane wave of Supergravity does not, in general, admit the same symmetry as its general-relativistic counterpart, although it does admit the same symmetry if one restricts attention to the Rarita-Schwinger limit.

4) Symmetry of the Plane Wave of N = 2 Supergravity

The plane wave of N = 2 Supergravity derives from the following line-element [11]:

$$g = \eta_{ab}\, \vartheta^a \otimes \vartheta^b \tag{4.1}$$

$$\sqrt{2}\, \vartheta^0 \equiv [1-H(u,x,y)]\, du - dv \tag{4.2}$$

$$\sqrt{2}\, \vartheta^3 \equiv (1+H)\, du + dv \tag{4.3}$$

$$\vartheta^1 \equiv dx \;,\; \vartheta^2 \equiv dy, \tag{4.4}$$

where the coordinates are $x^0 \equiv u$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv v$ and H is, so far, an arbitrary function. The gravitinos take the form:

$$\psi^m = \psi_1^m\, dx + \psi_2^m\, dy \tag{4.5}$$

$$\psi_1^m = \tfrac{1}{2}\, G^m(u) \begin{bmatrix} l^{m*} + l^m \\ l^{m*} + l^m \\ i\, l^{m*} - i\, l^m \\ -\, i l^{m*} + i l^m \end{bmatrix} \tag{4.6}$$

$$\psi_2^m = \tfrac{1}{2}\, G^m(u) \begin{bmatrix} i l^{m*} - i l^m \\ i l^{m*} - i l^m \\ -l^{m*} - l^m \\ l^{m*} + l^m \end{bmatrix} \quad , \tag{4.7}$$

in which G^m, m = 1, 2 is an arbitrary function and l^m, m = 1, 2 is an arbitrary (anticommuting) constant. A useful quantity, derived from G^m and l^m is:

$$K(u,x,y) \equiv k^1 + k^2 \tag{4.8}$$

$$k^a = \tfrac{1}{2}\, i\, (l^{a*}\, l^a - l^a\, l^{a*})\, (G^a)^2 \;, \quad a = 1,\, 2\;. \tag{4.9}$$

(This function is, as in Part I, related to the torsion of space-time.)

The electromagnetic potential A is given by :

$$A = h(u,x,y)\, du, \tag{4.10}$$

where h is an arbitrary function to which Maxwell's equations impose to be harmonic:

$$0 = \Delta\, h \equiv h_{,xx} + h_{,yy}\;. \tag{4.11}$$

(Here and in the sequel, partial derivatives are indicated by a comma.) In the basis (4.2)-(4.4), the electromagnetic field F takes the form:

$$\sqrt{2}\; F = h_{,x}\, (\vartheta^1\wedge\, \vartheta^0 + \vartheta^1\wedge\, \vartheta^3) + h_{,y}\, (\vartheta^2\wedge\, \vartheta^0 + \vartheta^2\wedge\, \vartheta^3)\,. \tag{4.12}$$

The only field equation which is not automatically satisfied is Einstein's equation:

$$\Delta\, H = K^2 + h^2_{,x} + h^2_{,y}\;. \tag{4.13}$$

Our aim is to compare the plane wave of N = 2

Supergravity with the plane wave of the Einstein-Maxwell theory. Therefore, we start by assuming that the curvature tensor depends on u only. (See Part I and ref. [14].) The curvature is the same as for the N = 1 theory [11], with the same consequence, namely that H must be purely quadratic in x and y:

$$H = \varkappa(u)\ x^2 + \lambda(u)\ xy + \mu(u)\ y^2\ , \tag{4.14}$$

in which $\varkappa$, λ and μ are arbitrary functions.

If this expression for H is substituted into Einstein's equation (4.13) it follows:

$$h^2_{,x} + h^2_{,y} = 2\varkappa + 2\mu - K^2. \tag{4.15}$$

This cannot be satisfied by an arbitrary function h since $\varkappa$, μ and K depend only on u. The additional constraint that h must fulfill is:

$$\frac{\partial}{\partial x}\ (h^2_{,x} + h^2_{,y}) = \frac{\partial}{\partial y}\ (h^2_{,x} + h^2_{,y}) = 0. \tag{4.16}$$

A simple calculation using (4.11) shows then that either h must depend on u only (which leads to the degenerate case F = 0), or all the second derivatives of h must vanish. This, in turn, forces h to be linear in x and y:

$$h(u,x,y) = P(u)\ x + Q(u)\ y + Z(u), \tag{4.17}$$

where P, Q and Z are arbitrary. By (4.10), Z can be assumed to vanish without loss of generality since it does not influence F. (A further consequence of (4.17) is that F depends on u only.)

Substituting (4.17) into (4.15), one obtains:

$$\mu = \tfrac{1}{2}(K^2 + P^2 + Q^2) - \varkappa, \tag{4.18}$$

which enables one to eliminate μ from H in (4.14). A more symmetric expression is, however, obtained by renaming $\varkappa$ and λ as:

$$\varkappa \equiv \frac{1}{4}\,(\alpha + K^2 + P^2 + Q^2) \quad , \quad \lambda \equiv \tfrac{1}{2}\,\beta\ , \tag{4.19}$$

where α and β are arbitrary. The final form for H is:

$$4\,H = \alpha\,(x^2 - y^2) + 2\beta\, xy + (K^2 + P^2 + Q^2)\,(x^2 + y^2), \tag{4.20}$$

which is obviously a generalisation of (4.5) in Part I. Moreover, (4.20) together with (4.1)-(4.4), has exactly the form of the metric of the Baldwin-Jeffery [15] plane wave of the Einstein-Maxwell theory.

The conclusion of this calculation is that, with the requirement that the curvature be dependent on u only, the plane wave of N = 2 Supergravity is characterised by a H function of type (4.20), an electromagnetic potential of type (4.17) and gravitinos of the type (4.5)-(4.9). The electromagnetic field F is then also a function of u only. In all these formulae, α, β, K, P and Q are arbitrary functions, and the field equations are automatically satisfied.

We are now in the position to apply the symmetry definition of N = 2 Supergravity. We shall prove that, in the Rarita-Schwinger limit, the generators ξ of the symmetry are precisely the Killing vectors of the plane wave of the Einstein-Maxwell theory. However, beyond this limit, the symmetry of the plane wave of N = 2 Supergravity is, in general, different from its relativistic counterpart.

For a metric of the form (4.20), (4.1)-(4.4), it is known that the Killing vectors are [15]:

$$\xi = q(u) \frac{\partial}{\partial x} + r(u) \frac{\partial}{\partial y} + [m-(q'x + r'y)] \frac{\partial}{\partial v} , \qquad (4.21)$$

in which m is a constant and q, r are arbitrary solutions of the system

$$2\begin{bmatrix} q'' \\ r'' \end{bmatrix} = \begin{bmatrix} K^2+P^2+Q^2+\alpha & \beta \\ \beta & K^2+P^2+Q^2-\alpha \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix} . \qquad (4.22)$$

(Here and in the sequel, a prime over a function denotes a derivative with respect to u.) It is now a well defined question to investigate whether the vector fields (4.21), (4.22) satisfy the definition of a symmetry in Supergravity (3.12)-(3.14) in the case of the plane wave (4.1)-(4.10), (4.20).

In the subsequent calculation, we shall prove that the symmetry equations are satisfied in the Rarita-Schwinger limit (3.15)-(3.17). As we shall also give, at the end of this section, some indications on the exact treatment (i.e. beyond the Rarita-Schwinger limit), it is convenient not to apply immediately the approximation (3.15)-(3.17), but to consider at first the exact equations (3.12)-(3.14). The limit will be taken explicitly in (4.46) below.

As in Part I, we start by calculating successively the following quantities, all expressed in the basis (4.2)-(4.4):

$$\sqrt{2}\, L^{a}{}_{b} = \begin{bmatrix} 0 & q' & r' & 0 \\ q' & 0 & 0 & q' \\ r' & 0 & 0 & r' \\ 0 & -q' & -r' & 0 \end{bmatrix} \qquad (4.23)$$

$$\sqrt{2}\; \mathcal{L}_{\xi}\, \psi^{m} = (q'\, \psi^{m}_{1} + r'\, \psi^{m}_{2})(\vartheta^{0}+\vartheta^{3}) + (q'\mu + r'\nu)(\psi^{m}_{1}\, \vartheta^{1} + \psi^{m}_{2}\, \vartheta^{2}) \qquad (4.24)$$

$$\nabla\, S^{m} = [E_{0}(S^{m}) + \tfrac{1}{2}\,(H_{x}\,\mu + H_{y}\,\nu - K\,\tau) S^{m}]\, \vartheta^{0}$$
$$+ [E_{1}(S^{m}) - \tfrac{1}{2}\, K\, \nu\, S^{m}]\, \vartheta^{1} + [E_{2}(S^{m}) + \tfrac{1}{2}\, K\, \mu\, S^{m}]\, \vartheta^{2}$$

$$+[E_3(S^m) + \tfrac{1}{2}\,(H_x\,\mu + H_y\,\nu - K\,\tau)S^m]\,\vartheta^3 \qquad (4.25)$$

$$\hat{F}_{ab} \equiv F_{ab} - \frac{i}{\sqrt{2}}\,\varepsilon^{mn}\,\bar{\psi}^m_a\,\psi^m_b = F_{ab} \qquad (4.26)$$

$$\pounds_\xi\, A = (q\,P + r\,Q)\,\frac{1}{\sqrt{2}}\,(\vartheta^0 + \vartheta^3) \qquad (4.27)$$

$$\pounds_\xi\, g = 0, \qquad (4.28)$$

in which we have put:

$$\sqrt{2}\,E_0 \equiv \frac{\partial}{\partial u} - (1 + H)\,\frac{\partial}{\partial v} \qquad (4.29)$$

$$\sqrt{2}\,E_3 \equiv \frac{\partial}{\partial u} + (1 - H)\,\frac{\partial}{\partial v} \qquad (4.30)$$

$$E_1 \equiv \frac{\partial}{\partial x} \quad , \quad E_2 \equiv \frac{\partial}{\partial y} \qquad (4.31)$$

$$4\mu \equiv [\gamma^0\,,\,\gamma^1] - [\gamma^1\,,\,\gamma^3] \qquad (4.32)$$

$$4\nu \equiv [\gamma^0\,,\,\gamma^2] - [\gamma^2\,,\,\gamma^3] \qquad (4.33)$$

$$4\tau \equiv [\gamma^1\,,\,\gamma^2] \qquad (4.34)$$

$$H_x \equiv H_{,x} \quad , \quad H_y \equiv H_{,y}\,.$$

Moreover, it has been assumed (consistently with (4.5)-(4.7)) that

$$\psi_0^m = 0 \ , \ \psi_3^m = 0 \ , \ \psi_a^m \equiv \begin{bmatrix} A_a^m \\ A_a^m \\ B_a^m \\ -B_a^m \end{bmatrix} \ , \ A_{a,x}^m = A_{a,y}^m = 0 \tag{4.35}$$

$$B_{a,x}^m = B_{a,y}^m = 0, \ a = 1, \ 2 \quad .$$

Observing that, for ψ_a^m given by (4.35), $\mu\ \psi_a^m$ and $\nu\ \psi_a^m$ both vanish identically (as in the analogous case of Part I), the application of the the symmetry equations (3.12), (3.14) and (3.13) greatly simplifies and yields:

$$0 = \bar{S}^m\ (\gamma_a\ \psi_b^m + \gamma_b\ \psi_a^m\ ) \tag{4.36}$$

$$(q\ P + r\ Q)\ \frac{1}{\sqrt{2}}\ (\vartheta^0 + \vartheta^3) = i\ \sqrt{2}\ \varepsilon^{mn}\ \bar{S}^m\ \psi^n + d\Phi \tag{4.37}$$

$$q'\ \psi_1^m + r'\ \psi_2^m = 2\ \sqrt{2}\ [E_0(S^m) + \tfrac{1}{2}\ (H_x\ \mu + H_y\ \nu - K\ \tau)S^m]$$

$$- \sqrt{2}\ (P\ \mu + Q\ \nu)\ \varepsilon^{mn}\ \gamma_0\ S^n \tag{4.38}$$

$$q'\ \psi_1^m + r'\ \psi_2^m = 2\ \sqrt{2}\ [E_3(S^m) + \tfrac{1}{2}\ (H_x\ \mu + H_y\nu - K\ \tau)S^m]$$

$$- \sqrt{2}\ (P\ \mu + Q\ \nu)\ \varepsilon^{mn}\ \gamma_3\ S^n \tag{4.39}$$

$$2\ S_x^m = K\ \nu\ S^m + (P\ \mu + Q\ \nu)\ \varepsilon^{mn}\ \gamma_1\ S^n \tag{4.40}$$

$$-2\ S_y^m = K\ \mu\ S^m - (P\ \mu + Q\ \nu)\ \varepsilon^{mn}\ \gamma_2\ S^n\ . \tag{4.41}$$

It is a simple matter to prove, using the appendix of Part I, that $\mu\,\gamma_0 = \mu\,\gamma_3$, $\nu\,\gamma_0 = \nu\,\gamma_3$, $\mu\,\gamma_2 = -\,\nu\,\gamma_1$, $\nu\,\gamma_2 = \mu\,\gamma_1$, in such a way that (4.38)-(4.41) imply:

$$\frac{\partial}{\partial v} S^m = 0 \qquad (4.42)$$

$$2\,S^m_u + \sqrt{2}\,(H_x\,\mu + H_y\,\nu - K\,\tau)S^m = q'\,\psi^m_1 + r'\,\psi^m_2$$

$$+\ \sqrt{2}\,(P\,\mu + Q\,\nu)\,\varepsilon^{mn}\,\gamma_0\,S^n \qquad (4.43)$$

$$2\,S^m_x = K\,\nu\,S^m + (P\,\mu + Q\,\nu)\,\varepsilon^{mn}\,\gamma_1\,S^n \qquad (4.44)$$

$$-\,2\,S^m_y = K\,\mu\,S^m + (P\,\nu - Q\,\mu)\,\varepsilon^{mn}\,\gamma_1\,S^n\,. \qquad (4.45)$$

So far, no approximation has been made. Given the length of the calculations, we shall now restrict attention to the Rarita-Schwinger limit. As mentioned above, some indications on the exact treatment will be briefly stated at the end of this section.

In the Rarita-Schwinger, one neglects the products $\psi^m\,\psi^n$ and $S^m\,\psi^n$. Therefore, (4.36) is automatically satisfied and (4.37) becomes, by virtue of (4.2), (4.3):

$$(q\,P + r\,Q)\,du = d\Phi\ , \qquad (4.46)$$

whereas, in (4.42)-(4.45), one must neglect K which is quadratic in ψ^m [11]. Due to the fact that the left-hand side of (4.46) depends on u only, it is obvious that a solution $\Phi(u)$ can be found by integration.

To establish that (4.42)-(4.45) are compatible, it is sufficient to find a particular solution. It will be seen that such a solution exists for a spinor S^m of a form similar to (4.35):

$$S^m = \begin{bmatrix} s^m \\ s^m \\ t^m \\ -t^m \end{bmatrix} \qquad (4.47)$$

For a spinor of this type, one has: $\mu\, S^m = \nu\, S^m = \mu\, \gamma_1\, S^m = \nu\, \gamma_1\, S^m = 0$ and consequently, (4.43)-(4.45) reduce to

$$2\, S^m_u = q'\, \psi^m_1 + r'\, \psi^m_2 + \sqrt{2}\, (P\, \mu + Q\, \nu)\, \varepsilon^{mn}\, \gamma_0\, S^n \qquad (4.48)$$

$$S^m_x = S^m_y = 0. \qquad (4.49)$$

By (4.42) and (4.49), S^m depends on u only. Moreover, putting

$$q'\, \psi^m_1 + r'\, \psi^m_2 \equiv \begin{bmatrix} a^m \\ a^m \\ b^m \\ -b^m \end{bmatrix}, \qquad (4.50)$$

which is consistent with (4.35), and splitting (4.48) in components, one obtains the following system of equations:

$$2 \frac{d}{du} V = \sqrt{2}\, M V + W \,, \tag{4.51}$$

$$V \equiv \begin{bmatrix} s^1 \\ t^1 \\ s^2 \\ t^2 \end{bmatrix}, \quad W \equiv \begin{bmatrix} a^1 \\ b^1 \\ a^2 \\ b^2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & -P & -Q \\ 0 & 0 & -Q & P \\ P & Q & 0 & 0 \\ Q & -P & 0 & 0 \end{bmatrix} . \tag{4.52}$$

Thus, finding a solution to (4.47),(4.48) is equivalent to solving (4.51),(4.52). This is a linear system for V, with a non-vanishing determinant and therefore it does admit some solution. This proves that the symmetry equations are satisfied, in the Rarita-Schwinger limit, by the Killing vectors ξ (4.21),(4.22) of the Einstein-Maxwell plane wave.

If one wants to investigate the question beyond the Rarita-Schwinger limit, one must return to the exact equations (4.36), (4.37), (4.42)-(4.45). It is possible but tedious to imitate the treatment of Part I. We shall not present this calculation explicitly. The various steps are similar to those of Part I. For instance, (4.30), (4.31), (4.38) and (4.39) of Part I generalise as:

$$P^m = x\,[K\,\varepsilon^m + \varepsilon^{mn}\,(Q\,\varepsilon^n - P\,\delta^n)] - y\,[K\,\delta^m + \varepsilon^{mn}\,(P\,\varepsilon^n + Q\,\delta^n)] + L^m \qquad (4.53)$$

$$Q^m = x\,[K\,\delta^m - \varepsilon^{mn}\,(P\,\varepsilon^n + Q\,\delta^n)] + y\,[K\,\varepsilon^m + \varepsilon^{mn}\,(P\,\delta^n - Q\,\varepsilon^n)] + M^m \qquad (4.54)$$

$$\delta'^m - \frac{\sqrt{2}}{4}\,K\,\varepsilon^m = 0 \qquad (4.55)$$

$$\varepsilon'^m + \frac{\sqrt{2}}{4}\,K\,\delta^m = 0\ , \qquad (4.56)$$

where the notation is in close parallel with the one of Part I.

It is clear that the N = 2 plane wave cannot satisfy the symmetry equations in general (for the symmetry generators used above, namely the Killing vectors of the Einstein-Maxwell solution) since it contains the N = 1 plane wave as a special case, and it has been shown in Part I that the latter does not have the same symmetry as the relativistic solution (in general). For this reason, it is not justified to go into the details of the exact derivation of the exact N = 2 case. It is, however, a non-trivial result that the symmetry property generalises, in the Rarita-Schwinger limit, from the particular N = 1 case to the more general N = 2 case.

It should be noted that, as in Part I, the obstruction to satisfying the exact symmetry equations is the algebraic constraints (4.36). Such constraints arise, in our new framework, only because of the fact that we preassigned the values of the symmetry generators ξ , in such a way that $\pounds_\xi$ g is then a given quantity. They would not be present if we used (3.12)-(3.14) to determine the vector fields ξ , knowing the fields $(\vartheta^a , \psi^a , A)$.

This is very different from the situation encountered in Part I, where these constraints were unavoidable, being a consequence of the restrictions put on the differentiating vector field ξ of the Lie derivative $\pounds_\xi$ when acting on spinors. This point was already mentioned briefly in § 3.

5) Conclusion

In this paper we extended to N = 2 Supergravity, the definition of a space-time symmetry developed, in an earlier work, in the framework of Simple Supergravity. We considered a configuration of fields (ϑ^a , ψ^a , A) as symmetric under the space-time motion generated by a vector field ξ if and only if there exist a scalar function Φ and a Majorana spinor S^m , m = 1, 2, such that

$$(\pounds_\xi \, g)_{ab} = i \, \bar{S}^m \, (\gamma_a \, \psi^m_b + \gamma_b \, \psi^m_a) \tag{5.1}$$

$$\pounds_\xi \, \psi^m = 2 \, \nabla \, S^m + \frac{1}{\sqrt{2}} \, (\hat{F} \, \sigma) \, \varepsilon^{mn} \, \gamma \, S^n \tag{5.2}$$

$$\pounds_\xi \, A = d\Phi + i \sqrt{2} \, \varepsilon^{mn} \, \bar{S}^m \, \psi^n \, , \tag{5.3}$$

in which the metric g and the modified electromagnetic field $\hat{F}$ are derived from ϑ^a , ψ^m and A.

The problem of defining the notion of the Lie derivative of a spinor has also been investigated. This was necessary in order to give a definite meaning to (5.2). We considered the expression

$$\pounds_\xi \, \psi \equiv \xi(\psi) - \tfrac{1}{2} \, (L_{ab} + M_{ab}) \, \sigma^{ab} \, \psi \, , \tag{5.4}$$

in which L_{ab} is defined by (1.5), and M_{ab} is arbitrary. We proved that all the operators of this class are compatible with the rules of tensor calculus. Moreover, we gave an argument favouring the choice $M_{ab} = 0$ in (5.4).

Finally, we applied this definition to the problem of the plane wave of N = 2 Supergravity. It was shown that this wave does not, in general, admit the same symmetry group as the plane wave of the Einstein-Maxwell theory but that, at the first order in ψ^m, the (approximate) symmetry is the same as the one of the Einstein-Maxwell plane wave.

6) Appendix: Conventions for Non Metric-Compatible Connections

In an arbitrary basis (not necessary orthonormal), the conventions which we used in Section 2 are:

Torsion: $T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]$ (6.1)

Non-Metricity: $H(X,Y,Z) \equiv (\nabla_Z g)(X,Y)$ (6.2)

Components of torsion: $T \equiv T^{\gamma}{}_{\alpha\beta}\, \vartheta^{\beta} \otimes \vartheta^{\alpha} \otimes e_{\gamma}$ (6.3)

Components of non-metricity:

$$H \equiv H_{\alpha\beta\gamma}\, \vartheta^{\alpha} \otimes \vartheta^{\beta} \otimes \vartheta^{\gamma} \tag{6.4}$$

Components of connection:

$$\nabla_{\alpha}\, e_{\beta} \equiv \Gamma^{\gamma}{}_{\beta\alpha}\, e_{\gamma} \tag{6.5}$$

Commutation coefficients: $[e_{\alpha}\, ,\, e_{\beta}] \equiv D^{\gamma}{}_{\alpha\beta}\, e_{\gamma}$ (6.6)

From the above, it follows that

$$\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma} = T_{\alpha\gamma\beta} + D_{\alpha\beta\gamma} \tag{6.7}$$

$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = e_{\gamma}(g_{\alpha\beta}) - H_{\alpha\beta\gamma}\ . \tag{6.8}$$

Therefore, the explicit expression of the connection components $\Gamma_{\alpha\beta\gamma}$ in terms of $e_{\alpha}(g_{\beta\gamma})$, $D_{\alpha\beta\gamma}$, $T_{\alpha\beta\gamma}$, $H_{\alpha\beta\gamma}$ is:

$$\Gamma_{\alpha\beta\gamma} = [\alpha\beta\gamma] + C_{\alpha\beta\gamma} + Q_{\alpha\beta\gamma} - K_{\alpha\beta\gamma}$$

$$2\ [\alpha\beta\gamma] \equiv e_{\beta}(g_{\alpha\gamma}) + e_{\gamma}(g_{\alpha\beta}) - e_{\alpha}(g_{\beta\gamma}) = 2\ [\alpha\gamma\beta]$$

$$2\ C_{\alpha\beta\gamma} \equiv D_{\gamma\alpha\beta} + D_{\beta\alpha\gamma} - D_{\alpha\beta\gamma} = -\ 2\ C_{\beta\alpha\gamma}$$

$$2\ Q_{\alpha\beta\gamma} \equiv T_{\gamma\beta\alpha} + T_{\beta\gamma\alpha} - T_{\alpha\gamma\beta} = -\ 2\ Q_{\beta\alpha\gamma}$$

$$2\ K_{\alpha\beta\gamma} \equiv H_{\gamma\alpha\beta} + H_{\beta\alpha\gamma} - H_{\beta\gamma\alpha} = 2\ K_{\alpha\gamma\beta}$$

In the special case of an orthonormal basis (indicated by the use Latin indices), the above formulae simplify and one proves easily the statements (2.23)-(2.25) of § 2.

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