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LARGE DEVIATION PRINCIPLE FOR PRODUCT MEASURES

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Abstract.

Assuming that the Large Deviation Principle (LDP) is satisfied in each component of the product space an elementary proof of the LDP for the product measure is given.

1. Introduction.

The Large Deviation Principle (LDP) is the main hypothesis in the statement of Varadhan's theorem which provides an effective method for computing the asymptotic behavior of some integrals. In equilibrium statistical mechanics the limiting specific free-energy can be treated by this method. In the case of quantum spin systems interacting with a second quantum system, LDP was used to obtain upper and lower bounds for the free-energy [1] .
In [1] the LDP for product measures has been derived from Varadhan's theorem [2] .
In this note we present a direct, elementary proof.

2. Large Deviation Principle.

Let X be a complete separable metric space. A sequence $\{K_n , n \geq 1\}$ of Borel probability measures defined on X is said to satisfy the LDP with rate-function I and constants $\{c_n , n \geq 1\}$ if the following conditions are fulfilled [2] :

(LD0) The sequence $\{c_n, n \geq 1\}$ of positive real numbers is such that $c_n \longrightarrow \infty$ as $n \longrightarrow \infty$

(LD1) $I : X \longrightarrow [0,\infty]$ is lower semicontinuous

(LD2) $\{ x \in X ; I(x) \leq \lambda \}$ is a compact set for each $\lambda \geq 0$

(LD3) For every closed set $C \subset X$, $\limsup_{n \to \infty} \frac{1}{c_n} \log K_n(C) \leq - \inf_{x \in C} I(x)$

(LD4) For every open set $G \subset X$, $\liminf_{n \to \infty} \frac{1}{c_n} \log K_n(G) \geq - \inf_{x \in G} I(x)$

We will formulate and prove the theorem for the product of two spaces. A similar result for the product of n spaces (where $n \in N$) follows directly from the case $n = 2$.

<u>Theorem.</u>

Let X and Y be complete separable metric spaces. Suppose that K_n and L_n are sequences of Borel probability measures on X and Y satisfying the LDP with rate-functions $I(x)$ and $H(y)$ respectively, with the same constants c_n. Then the sequence of the product measures $M_n = K_n \times L_n$ defined on $X \times Y$ satisfies the LDP with the rate-function $J(x,y) = I(x) + H(y)$ and the constants c_n.

Proof:

(LD1) is obvious since $J(x,y)$ is the sum of non-negative and lower semicontinuous functions.

(LD2) Clearly the following inclusion holds:

$$\{(x,y) \in X \times Y \; ; \; J(x,y) \leq \lambda\} \subset \{x \in X \; ; \; I(x) \leq \lambda\} \times \{y \in Y \; ; \; H(y) \leq \lambda\}.$$

Since the left hand side is a closed subset of a compact set, it is compact.

In what follows, by an open (closed) rectangle in $X \times Y$ we shall mean a set of the form $A \times B$ where the sets A and B are open (closed) in X and Y respectively. It is easy to see that (LD3) and (LD4) are true for rectangles. Indeed, for a rectangle $A \times B$ we have $\log M_n(A \times B) = \log K_n(A) + \log L_n(B)$ and both properties follow immediately from the assumptions.

To prove (LD4) in general, let us fix an open set $G \subset X \times Y$ and a point $z = (x,y) \in G$. There exists an open rectangle $R_z \subset G$ which contains z such that

$$\liminf_{n \to \infty} \frac{1}{t_n} \log M_n(G) \geq \liminf_{n \to \infty} \frac{1}{t_n} \log M_n(R_z) \geq -\inf J(R_z) \geq -J(x,y)$$

Since $z \in G$ was arbitrary, (LD4) follows by taking the supremum with respect to $z \in G$.

For the proof of (LD3), we need the following lemmata describing the approximation of closed set by finite union of rectangles.

<u>Lemma 1.</u>

If c is a constant and $S_1, \ldots S_N$ are sets such that for $j = 1, \ldots, N$

$$\limsup_{n \to \infty} \frac{1}{t_n} \log M_n(S_j) \leq c \,, \quad \text{then} \quad \limsup_{n \to \infty} \frac{1}{t_n} \log M_n\Big(\bigcup_{j=1}^{N} S_j\Big) \leq c$$

<u>Lemma 1a.</u>

If $S_1, \ldots S_N$ are rectangles then

$$\limsup_{n \to \infty} \frac{1}{t_n} \log M_n\Big(\bigcup_{j=1}^{N} S_j\Big) \leq -\inf J\Big(\bigcup_{j=1}^{N} S_j\Big)$$

Proof of lemma 1 and 1a:

Indeed, $\frac{1}{t_n} \log M_n\Big(\bigcup_{j=1}^{N} S_j\Big) \leq \frac{1}{t_n} \log N + \frac{1}{t_n} \log (M_n(S_{jn}))$ where

$S_{jn} \in \{S_1, \ldots S_N\}$ is chosen so that $M_n(S_{jn}) = \max_{j=1,\ldots N} M_n(S_j)$

Observe now that $\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(S_{jn}) = \max_{j=1,..N} \limsup_{n\to\infty} \frac{1}{c_n}\log M_n(S_j) \leq c$

For rectangular sets it is clear that

$$\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(S_{jn}) \leq \max_{j=1,..N} \{-\inf J(S_j)\} \leq -\inf J(\bigcup_{j=1}^{N} S_j)$$

■

If C is a compact set then (LD3) follows from the above lemmata and semicontinuity of $J(x,y)$.

Suppose now that C is an arbitrary closed set in $X \times Y$.

<u>Lemma 2.</u>

For any $\varepsilon > 0$ and $a \in X \times Y$ there exists an open rectangle $R_a \subset X \times Y$ such that $a \in R_a$ and $\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(R_a \cap C) \leq -\inf J(C) + \varepsilon$

Proof: Let $\delta = -\inf J(C)$

Fix $a \in X \times Y$. If $a \notin C$ the statement is obvious. Assume that $a \in C$. In view of lower semicontinuity of $J(x,y)$, there exists an open rectangle R_a containg the point a such that $\inf J(\bar{R}_a) \geq J(a) - \varepsilon$. Thus by (LD3) for the closed rectangle we have $\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(R_a \cap C) \leq \limsup_{n\to\infty} \frac{1}{c_n}\log M_n(\bar{R}_a) \leq$

$$\leq -\inf J(\bar{R}_a) \leq -J(a) + \varepsilon \leq \delta + \varepsilon$$

■

Now we are in position to prove (LD3) for an arbitrary closed set C. Take $m \in N$ such that $-m < \delta$. The set $F := \{(x,y) \in X \times Y \; ; \; J(x,y) \leq m\}$ is compact so it can be covered by a finite number of rectangles $R_{a_1}, \dots, R_{a_N}$ chosen as in lemma 2. Then by lemma 1 and 2 we have

$$\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(C \cap \bigcup_{j=1}^{N} R_{a_j}) \leq \delta + \varepsilon \tag{1}$$

Notice that if $A \times B$ is an open rectangle in $X \times Y$ then $X \times Y \setminus A \times B = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$ and hence the complement of $A \times B$ is the union of two closed rectangles. Thus De Morgan's Laws combined with the Distributive Laws for $\cap$ and $\cup$ imply that $X \times Y \setminus \bigcup_{j=1}^{N} R_{a_j}$

is a finite union of closed rectangles. Therefore it follows from lemma 1a that

$$\limsup_{n\to\infty} \frac{1}{c_n}\log M_n(C \setminus \bigcup_{j=1}^{N} R_{a_j}) \leq \limsup_{n\to\infty} \frac{1}{c_n}\log M_n(X \times Y \setminus \bigcup_{j=1}^{N} R_{a_j}) =$$

(2)

$$= \limsup_{n\to\infty} \frac{1}{c_n}\log M_n(\bigcup_{j=1}^{M} Q_j) \leq -\inf J(\bigcup_{j=1}^{M} Q_j) \leq -m < \delta < \delta + \varepsilon$$

where $\bigcup_{j=1}^{M} Q_j$ is a finite union of closed rectangles contained in the complement of F. From (1), (2) and lemma 1 we have

$$\limsup_{n\to\infty} \frac{1}{c_n} \log M_n(C) \leq \mathcal{J} + \varepsilon = -\inf J(C) + \varepsilon$$

which completes the proof of the theorem.

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