DIAS-STP-87-46

To appear in the Proceedings of the Second Hungarian Workshop on General Relativity ("Recent Developments in Relativity", Budapest I-5 September 1987.)

On the Time Evolution of some Robinson-Trautman Line Elements

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Abstract

We present a review of some aspects of the linearised Robinson-Trautman metrics, mainly the study of the initial-value problem and a long-term expansion of a special case of these line elements. The problem of wire-singularities is also considered.

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INTRODUCTION

The purpose of this lecture is to expose some aspects of the dynamics of the linearised* Robinson-Trautman (henceforth RT) line elements. This question received much consideration (e.g. in [1-5]) and therefore we shall have to restrict attention and to concentrate here on a few topics only.

In the first section, we shall give a physical motivation for the study of the RT metrics. Their interpretation as the gravitational field produced by an accelerated point-mass will then become clear. The necessity of considering the question of "wire-singularities" in the RT framework will also be obvious.

In the second section, we shall present an algorithm for studying the initial-value problem of the RT metrics. This will enable us to find a large class of line elements exhibiting no wire-singularities, at least for a small period of evolution from the initial conditions.

Finally, in the third section, we shall investigate the long-term behaviour of the RT metrics and compare their time evolution with the one of an ordinary heat wave in one spatial dimension.

I) PHYSICAL MOTIVATION OF THE *RT* LINE ELEMENTS

To motivate the RT metrics, it is convenient to start by considering briefly the Minkowski space and by introducing Synge's "retarded construction" [6]. This will enable us to define a vector field $\overrightarrow{k}$ which will play a fundamental role when making the transition to General Relativity.

Consider the Minkowski space M expressed in Cartesian co-ordinates x^μ. Let C be a time-like curve on which τ is the proper time. The equation of C is: $x^\mu = Z^\mu(\tau)$, Z^μ being known. (C will be interpreted later as the world line of a

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* The reader is referred to the lecture of Prof. D. Kramer in this volume for non-linear aspects .

point-particle generating the RT gravitational field.) For any point p of M, we can draw the past null cone of p, which intersects C at a unique corresponding point q. (Alternatively we can draw the future null cone of q, which in turn contains p.) We define $\vec{\xi}$ as the vector $\overrightarrow{qp}$ and $\vec{r}$ as the vector passing through p and perpendicular to the unit tangent vector $\vec{V}$ (velocity vector) to C at q. In this way, we can attach to p the vectors $\vec{\xi}$ and $\vec{r}$, and also the scalar τ defined as the value of the proper time of q. (See Fig. 1.) By construction, the scalar field τ is constant on each future null cone emanating from C.

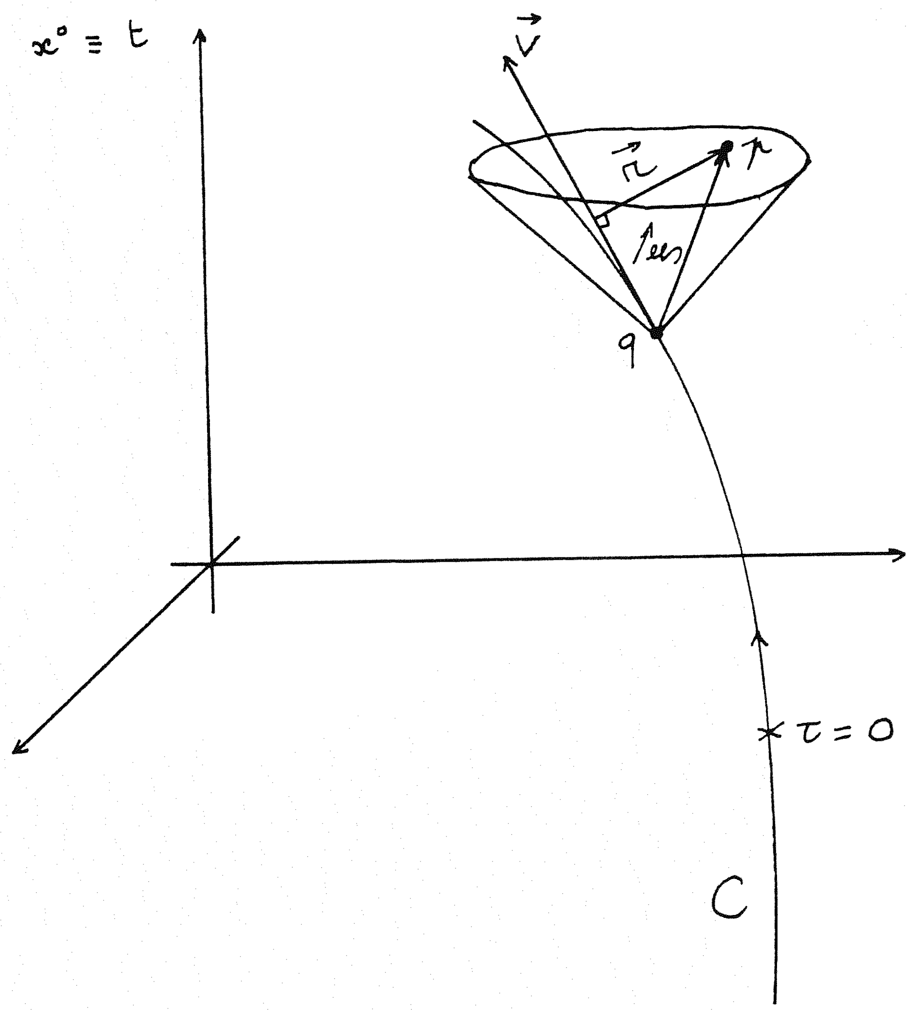

Fig. 1 Future null cone based on a time-like curve C.

Putting: $\vec{k} = \frac{\vec{\xi}}{r}$, $r \equiv -||\vec{r}||$, one shows easily that $\vec{k}$ is related to τ as:

$$k^{\mu} = \eta^{\mu\nu} \frac{\partial \tau}{\partial x^{\nu}} \equiv \eta^{\mu\nu} \tau_{,\,\nu} \tag{1.1}$$

Moreover, the vector field $\vec{k}$ is:

$$\text{null: } k^{\mu}\, k_{\mu} = 0 \tag{1.2}$$

$$\text{geodesic: } k^{\mu}_{,\,\nu}\, k^{\nu} = 0 \tag{1.3}$$

$$\text{shear-free: } \sigma^2 \equiv \frac{1}{2} k_{(\mu,\,\nu)}\, k^{\mu,\,\nu} - (\frac{1}{2} k^{\mu}_{,\,\mu})^2 = 0 \tag{1.4}$$

$$\text{twist-free: } \omega^2 \equiv \frac{1}{2} k_{[\mu,\,\nu]}\, k^{\mu,\,\nu} = 0 \tag{1.5}$$

$$\text{diverging: } \Theta \equiv \frac{1}{2} k^{\mu}_{,\,\mu} = \frac{1}{r} \quad . \tag{1.6}$$

Finally, we can decide to use r as the parameter along the integral curves of $\vec{k}$, in such a way that:

$$k^{\mu} = \frac{dx^{\mu}}{dr} \quad . \tag{1.7}$$

We can paraphrase this construction by saying that, in the Minkowski space, there exists a family of null hypersurfaces $\tau =$ constant (the null cones), such that the vector field defined by (1.1), with the choice of the parameter (1.7), has the properties (1.2)-(1.6).

To obtain the RT line elements, we now envisage the following problem: Consider a Riemannian space and a family of null hypersurfaces $\sigma =$ constant. From this family, construct a vector field $\vec{k}$ in an analogous way to (1.1), namely:

$$k^{\mu} = g^{\mu\nu} \sigma_{,\,\nu} \quad . \tag{1.8}$$

One can ask the question whether there exist solutions of Einstein's equations (in vacuo) such that the vector field $\vec{k}$ defined by (1.8), with the parameter choice (1.7) has the same properties as in the Minkowski space (i.e.: (1.2)-(1.6), with covariant derivatives, rather than partial ones). It turns out that the answer to this question is affirmative and that the solutions which have the required properties are the RT metrics:

$$ds^2 = h(\sigma,\varsigma,\bar{\varsigma})\ d\sigma^2 + 2\ drd\sigma - 2r^2P^{-2}(\sigma,\varsigma,\bar{\varsigma})\ d\varsigma d\bar{\varsigma}\ , \tag{1.9}$$

in which the co-ordinates are $\sigma, r, \varsigma, \bar{\varsigma}$ and h is given in terms of P as:

$$h \equiv K - 2Hr - 2\frac{m}{r} \tag{1.10}$$

$$H \equiv \frac{\partial}{\partial\sigma}\ln P \tag{1.11}$$

$$K \equiv \Delta \ln P \tag{1.12}$$

$$\Delta \equiv 2P^2\frac{\partial^2}{\partial\varsigma\partial\bar{\varsigma}}\ , \tag{1.13}$$

whereas P is a solution of:

$$\Delta K + 12mH = 0\ , \tag{1.14}$$

m being a constant.

Having obtained the form of the RT line elements, one can get some further insight in them by returning briefly to the Minkowski space M. From Fig. 1 and the definition of $\overrightarrow{k}$, it follows that:

$$x^\mu = Z^\mu(\tau) + r\ k^\mu\ . \tag{1.15}$$

Applying a procedure attributed to Robinson [1], one considers (1.15) as a co-ordinate transformation in M, x^μ being the Cartesian co-ordinates and two of the new co-ordinates being τ and r. Using the properties (1.2)-(1.6) of $\overrightarrow{k}$, one proves [1] that it is possible to find a complex co-ordinate ς such that:

$$\overrightarrow{k} = (1 + \frac{1}{2}\varsigma\bar{\varsigma}\ ,\ \frac{1}{\sqrt{2}}(\varsigma + \bar{\varsigma})\ ,\ -\frac{i}{\sqrt{2}}(\varsigma - \bar{\varsigma})\ ,\ 1 - \frac{1}{2}\varsigma\bar{\varsigma})P_0^{-1} \tag{1.16}$$

$$P_0 \equiv (1 + \frac{1}{2}\varsigma\bar{\varsigma})V^0 - (1 - \frac{1}{2}\varsigma\bar{\varsigma})V^3 - (V^1 - iV^2)\frac{\varsigma}{\sqrt{2}} - (V^1 + iV^2)\frac{\bar{\varsigma}}{\sqrt{2}}\ . \tag{1.17}$$

One can interpret ς as the co-ordinate labelling the generator of the null cone to which the point p belongs. (See Fig. 1.) Alternatively, one could use the two ordinary (real) spherical co-ordinates θ and ϕ defined as: $\varsigma = \tan\frac{\theta}{2}\, e^{i\phi}$.
It is then a simple matter to calculate the expression of the Minkowski metric in these co-ordinates:

$$ds^2 = h_0\, d\tau^2 + 2drd\tau - 2r^2 P_0^{-2} d\varsigma d\bar{\varsigma} \ , \tag{1.18}$$

in which:

$$h_0 \equiv K_0 - 2H_0\, r \tag{1.19}$$

$$H_0 \equiv \frac{\partial}{\partial\sigma}\ln P_0 \tag{1.20}$$

$$K_0 \equiv \Delta_0 \ln P_0 = 1 \tag{1.21}$$

$$\Delta_0 \equiv 2P_0^2 \frac{\partial^2}{\partial\varsigma\partial\bar{\varsigma}} \ . \tag{1.22}$$

The analogy between (1.18)-(1.22) and (1.9)-(1.13) is striking. The main difference between the two settings (apart from the term involving m in (1.10)) is that, in the RT case, P is a solution of the field equation (1.14), rather than being a known function, totally determined by the curve C, as in the Minkowski case (1.17).

The above analysis provides us with a convenient framework to investigate the gravitational field produced by an accelerated point-mass. Assuming that the field is weak, the solution will be approximately the Minkowski metric and the following procedure is justified [1]:
1)Impose the motion of the source in the Minkowskian background (i.e.: take for the curve C, the (preassigned) world line of the source) and calculate the corresponding P_0 by (1.17).
2)Express that the total P of (1.9) is a perturbation of this P_0:

$$P = P_0 \times (1 + \epsilon(\sigma, \varsigma, \bar{\varsigma})) \ , \ |\epsilon| \ll 1 \ . \tag{1.23}$$

3)Determine ϵ in (1.23) by the requirement that the total P should satisfy (1.14) to the first order in ϵ.

This shows the close analogy which exists between the RT metrics and the Liénard-Wiechert potentials in Electromagnetism: An accelerated point-charge generates an electromagnetic field, whereas an accelerated point-mass generates a gravitational field. Both fields can be calculated given the (preassigned) motion of the source. As an illustration of the above-described procedure, we consider briefly the example of a point-mass which oscillates harmonically along the z-axis.

According to step 1), we choose for the curve C, the world line:

$$x^0 = \alpha \ , \quad x^3 = \lambda \cos \omega \alpha$$

$$x^1 = x^2 = 0$$

$$|\lambda \omega| < 1 \ ,$$

in which λ is the amplitude of the oscillation and α is a parameter along C, related to the proper time σ as:

$$\sigma = \frac{1}{\omega} E(\omega\alpha, \lambda\omega) \ ,$$

where E is the elliptic function of the second kind. This yields the expression for P_0 by (1.17).

Following step 2), we express that the total P is a perturbation of P_0. Due to the fact that the motion of the source is rectilinear in the Minkowskian background, it is reasonable to assume that the solution is axially symmetric. Therefore it must be independent of the azimuthal angle ϕ. This will be achieved by imposing that P depends on ς and $\bar{\varsigma}$ only by the combination $\varsigma\bar{\varsigma}$.

Finally, we solve the field equation as explained is step 3), with the result:

$$P(\sigma \ , \ \xi) = 2\frac{g(\alpha)}{1-\xi} \left(1 - m\, f(\alpha)\, \xi \, \ln|1-\xi^2|\right) + o(2) \ , \tag{1.24}$$

in which the notation is:

$$g(\alpha) \equiv \sqrt{\frac{1+\lambda\omega \sin\omega\alpha}{1-\lambda\omega\sin\omega\alpha}} \tag{1.25}$$

$$\xi \equiv \frac{\frac{1}{2}\varsigma\bar{\varsigma} - g^2}{\frac{1}{2}\varsigma\bar{\varsigma} + g^2} \tag{1.26}$$

$$f(\alpha) \equiv -\frac{\lambda\omega^2 \cos\omega\alpha}{(1-\lambda^2\omega^2\sin^2\omega\alpha)^{3/2}} \tag{1.27}$$

To determine the arbitrary functions which appear in the course of the integration leading to (1.24)-(1.27), we calculated the Riemann tensor and imposed that the only singularities which may possibly occur are "point-like" singularities, i.e. poles in the curvature. It turned out that we did not have at our disposal enough functions of integration to remove all the other singularities. In particular, the component ψ_4 of the curvature in the Newman-Penrose formalism has a "wire-singularity", i.e. a line of infinite curvature for $\xi = \pm 1$:

$$\psi_4 = -\frac{2m}{r}\,(\frac{\bar{\varsigma}}{P_0})^2\,(3f^2 + \frac{df}{d\sigma}\,\frac{\xi(3-\xi^2)}{(1-\xi^2)^2})\ .$$

The values $\xi = \pm 1$ are equivalent, by (1.26), to $\varsigma\bar{\varsigma} = \infty$ or 0, i.e. to the polar angle θ being either π or 0. This corresponds to the z-axis, the axis of oscillation of m.

This simple example shows that, in general, it is not possible to preassign to the source an arbitrary path in the Minkowskian background. Such an arbitrary path would require that energy from infinity be supplied to the source (through the wire-singularities) in order to maintain the assumed motion. Some very general results have been obtained [3] on the type of motion for which no such singularities occur. In the sequel, we shall present an alternative formalism to investigate the same problem. We shall prove that there exists a large class of approximate RT line elements which do not exhibit wire-singularities. In Section II), we shall study the initial-value problem for the RT metrics and in Section III), a long-term expansion of a special class of solutions.

II) INITIAL-VALUE PROBLEM FOR THE *RT* SOLUTIONS

II.1) Introduction

If one restricts the RT line element (1.9) to a fixed value of σ and r, and performs a rescaling, one obtains a family of two-dimensional metrics indexed by the parameter σ:

$$ds^2(\sigma = \text{constant}) = 2P^{-2}(\sigma = \text{constant}\ ,\ \varsigma\ ,\ \bar{\varsigma})\ d\varsigma\ d\bar{\varsigma}\ . \tag{2.1}$$

This family is interpreted as representing the geometry of the wave fronts. Moreover, K in (1.12) and Δ in (1.13) are then the Gaussian curvature and the invariant Laplacian on (2.1). Therefore, if the wave fronts (2.1) are known at a given value of σ, say $\sigma = 0$, i.e. if P is known at $\sigma = 0$, one can calculate ΔK by (1.12) and (1.13) in such a way that the field equation (1.14) determines $(\frac{\partial P}{\partial \sigma})(\sigma = 0)$:

$$12m\ \frac{\dot{P}(\sigma = 0)}{P(\sigma = 0)} = -(\Delta K)(\sigma = 0)\ \ ,\ \ \cdot \equiv \frac{\partial}{\partial \sigma}\ . \tag{2.2}$$

Thus, by a Taylor expansion, P can be obtained (approximately) for $\sigma > 0$.

In Section II.2, we shall systematise these ideas and show how to construct an algorithm from which one can calculate the successive derivatives of P with respect to σ at $\sigma = 0$ in terms of the known function $P(\sigma = 0\ ,\ \varsigma\ ,\ \bar{\varsigma})$. This will enable us to study the initial-value problem for the RT line elements. We shall insist here mainly on the idea of this method, the reader being referred to [7] for the details.

II.2) Construction of the Power Expansion of the Solution

Our aim is to form the Taylor polynomial:

$$P(\sigma) = P(0)\ +\ \dot{P}(0)\ \sigma\ +\ \ddot{P}\ \frac{\sigma^2}{2}\ +\ \dots\ . \tag{2.3}$$

However, it will be simpler to expand H, for it is H which appears directly in the field equation (1.14). It is straightforward to transform the expansion for H in the required expansion for P using (1.11).

The values of the derivatives of H at $\sigma = 0$ can be calculated by the field equation (1.14):

$$\partial^n\ H = -\frac{1}{12m}\partial^n \Delta K\ \ ,\ \ \partial^n \equiv \frac{\partial^n}{\partial \sigma^n}\ . \tag{2.4}$$

For the rest of this section, all the equations will be understood as restricted to the value $\sigma = 0$. The argument $\sigma = 0$ will be omitted everywhere to simplify the notation.

Moreover, by (1.11) and (1.13):

$$\partial \Delta K = 2H \; \Delta K + \Delta \dot{K} \; . \qquad (2.5)$$

Iterating the derivation:

$$\partial^n \; \Delta K = 2 \sum_{j=0}^{n-1} C^j_{n-1} \; (\partial^j \; \Delta K) \; (\partial^{n-1-j} \; H) \; + \; \partial^{n-1} \; \Delta \dot{K} \; \; , \; \; n \geq 1 \; , \qquad (2.6)$$

where C^j_n is the binomial coefficient. This expression becomes, with (2.4):

$$\partial^n \; H = 2 \sum_{j=0}^{n-1} C^j_{n-1} \; (\partial^j \; H) \; (\partial^{n-1-j} \; H) \; - \; \frac{1}{12m} \partial^{n-1} \; \Delta \dot{K} \; \; , \; n \geq 1 \; . \qquad (2.7)$$

If the second term on the right-hand side were absent, (2.7) would be the solution of the initial-value problem since it would enable us to calculate the nth-order derivative of H in terms of derivatives up to order $n-1$ at most. So, using repeatedly (2.7), it would be possible to express $\partial^n \; H$ (at $\sigma = 0$) as a function of $H(0)$, which is known by (2.4) with $n = 0$.

To evaluate the term involving K in (2.7), we apply a system similar to (2.5) and (2.6), with the result:

$$\dot{K} = 2HK + \Delta H \qquad (2.8)$$

$$\partial^n \; \Delta(\partial^m \; K) = 2 \sum_{j=0}^{n-1} C^j_{n-1} (\partial^j \; \Delta(\partial^m \; K))(\partial^{n-1-j} \; H) + \partial^{n-1} \; \Delta(\partial^{m+1} \; K), \; n \geq 1 \; . \qquad (2.9)$$

These formulae make it possible to compute terms of the type $\partial^n \; \Delta(\partial^m \; K)$, appearing in (2.7), as combinations of expressions such as $\partial^{n-1} \; \Delta(\partial^m \; K)$, $\partial^{n-1} \; H$ and $\partial^{n-1} \; \Delta(\partial^{m+1} \; K)$, in such a way that a repeated application of (2.7), (2.8) and (2.9) yields $\partial^n \; H$ in terms of $\partial^j \; H$, $0 \leq j \leq n-1$ and $\Delta(\partial^j \; K)$, $1 \leq j \leq n$. Consequently, what remains to be done is to express $\Delta(\partial^j \; K)$, $1 \leq j \leq n$ as a function of $\partial^j H$, $0 \le j \le n-1$. This is similar to the above treatment. We get [7]:

$$\partial^n K = 2\sum_{j=0}^{n-1} C_{n-1}^j \, (\partial^j K)\, (\partial^{n-1-j} H) \;+\; \partial^{n-1} \Delta H \;\;,\;\; n \ge 1 \qquad (2.10)$$

$$\partial^n \Delta(\partial^m H) = 2\sum_{j=0}^{n-1} C_{n-1}^j (\partial^j \Delta(\partial^m H))(\partial^{n-1-j} H) + \partial^{n-1} \Delta(\partial^{m+1} H),\; n \ge 1\,. \qquad (2.11)$$

To show how this algorithm is used in practice, we shall apply it to the case in which $P(0)$ describes wave fronts which are axially symmetric perturbations of a sphere. Moreover, we shall perform the expansion in σ only up to order 1. At order 0 in σ, the calculation is trivial: Δ and K are known by (1.13) and (1.12), and $H(0)$ is calculated by (2.4) with $n = 0$. At order 1, we get from (2.7) and (2.8):

$$\dot{H} = 2H^2 - \frac{1}{12m}\Delta \dot{K} \qquad (2.12)$$

$$\dot{K} = 2HK + \Delta H \;. \qquad (2.13)$$

The complete framework (2.7)-(2.11) comes in operation only from order 2 on [7]. We shall now particularise (2.12), (2.13) to the case where $P(0)$ describes an axially symmetric perturbation of a sphere.

II.3) Axially Symmetric Perturbations of Spheres

Let Σ be the surface obtained by rotating about the z-axis, the curve Γ of the xz-plane. Let the equation of Γ be given in polar co-ordinates as: $x = \chi(\theta)\sin\theta$, $y = \chi(\theta)\cos\theta$, where θ is the angle between the z-axis and the radius-vector of an arbitrary point of Γ, and χ is the magnitude of this radius-vector. The line element of Σ is then:

$$dl^2 = (\chi'^2 + \chi^2)d\theta^2 + \chi^2 \sin^2\theta \;\; d\phi^2 \;\;, \qquad (2.14)$$

in which ϕ denotes the azimuthal angle. Obviously, the expression $\chi(\theta) = R$, (where R is a constant) is the equation of a sphere of radius R and therefore an axially symmetric perturbation of this sphere will be characterised by: $\chi(\theta) = R(1+\epsilon(\theta))$, $|\epsilon| \ll 1$. The metric (2.14) then takes the form:

$$dl^2 = 2R^2 P^{-2} d\varsigma d\bar{\varsigma} \ , \tag{2.15}$$

with:

$$P \equiv \sqrt{2}\frac{1-\epsilon}{1+\xi} \ , \ \xi \equiv \cos\theta \ , \ \varsigma \equiv \tan\frac{\theta}{2}\, e^{i\phi} \ . \tag{2.16}$$

Thus, we have expressed the line element of Σ in a manner which is appropriate to identify it with the metric (2.1) of the wave fronts of the RT solutions at the initial value of σ. Moreover, (2.16) gives us the explicit equation of the function P in terms of the (known) perturbation ϵ. Consequently, it is a simple routine to apply the equations of the initial-value problem (2.12), (2.13) to obtain the first-order expansion in σ of H. (See [7].) The results are:

$$12mH(0) = 4(3\xi^2-1)\epsilon^{II} + 8\xi(\xi^2-1)\epsilon^{III} + (1-\xi^2)^2\epsilon^{IV} \tag{2.17}$$

$$\begin{aligned} 144m^2\dot{H}(0) =& 96(1-3\xi^2)\epsilon^{II} + 64\xi(21-33\xi^2)\epsilon^{III} + 24(-19+134\xi^2-131\xi^4)\epsilon^{IV} \\ &+ 96\xi(-9+26\xi^2-17\xi^4)\epsilon^{V} + 32(2-15\xi^2+24\xi^4-11\xi^6)\epsilon^{VI} \\ &+ 32\xi(1-\xi^2)^3\epsilon^{VII} - (1-\xi^2)^4\epsilon^{VIII} \ . \end{aligned} \tag{2.18}$$

Here, Roman numerals denote derivatives with respect to ξ. It should be noted that $H(0)$ and $\dot{H}(0)$ are both polynomials in ξ with coefficients being the derivatives of the perturbation ϵ with respect to ξ. A straightforward calculation of the curvature in the basis $\omega^1 \equiv rP^{-1}d\varsigma$, $\omega^2 \equiv rP^{-1}d\bar{\varsigma}$, $\omega^3 \equiv d\sigma$, $\omega^4 \equiv dr + \frac{h}{2}d\sigma$ shows that the curvature components (in the Newman-Penrose notation) ψ_1, ψ_2, ψ_3, ψ_4 are also polynomials in ξ with derivatives of ϵ as coefficients. Therefore, these quantities are regular for any value of ξ provided the coefficients are finite for all ξ. (In fact, $-1 \leq \xi \leq 1$, by (2.16).) Thus, we have proved that all the RT solutions for which the wave fronts are smooth perturbations of spheres (i.e. all the derivatives of ϵ are finite) evolve in time without creating wire-singularities.

However the above approach, giving only a power expansion of the metric, is valid only for small values of σ. It would therefore be desirable to investigate the behaviour of the solutions for arbitrarily large values of σ, at least for a restricted class of line elements. Several methods have been devised for this purpose, e.g. the Liapunov formalism [8], applicable to wave fronts which are topologically spheres. As a complement to these studies, we shall develop here some other formalism which will enable us to investigate (approximately) perturbed cylinders. This is complementary to [8], since the cylinder is the simplest surface which is not topologically a sphere. We shall explain only the fundamentals here, referring the reader to [9] for more details.

III) LONG-TERM EXPANSION OF THE RT METRICS

III.1) Introduction and Form of the Metric

We shall mainly be interested in axially symmetric perturbations of cylinders. When a RT line element is axially symmetric, it is sometimes convenient to introduce the Robinson co-ordinates ξ and η [9]. It is sufficient to know that the metric is:

$$ds^2 = hd\sigma^2 + 2dr\, d\sigma - r^2 \left[f^{-1}(d\xi + af d\sigma)^2 + f d\eta^2 \right] \ , \tag{3.1}$$

where a and f depend on σ and ξ, and are solutions of:

$$\frac{\partial}{\partial \xi}[f(\frac{\partial^3 f}{\partial \xi^3} - 12ma)] = 0 \tag{3.2}$$

$$\frac{\partial f}{\partial \sigma} + f^2 \frac{\partial a}{\partial \xi} = 0 \ . \tag{3.3}$$

The field equations (3.2), (3.3) are simplified by extracting a from (3.2), substituting the value in (3.3) and removing, via a simple change of variables, the arbitrary function generated by the integration of (3.2). The final result is:

$$12m \frac{\partial f}{\partial \sigma} + f^2 \frac{\partial^4 f}{\partial \xi^4} = 0 \ . \tag{3.4}$$

We are now going to solve approximately (3.4) in the case where the wave fronts are axially symmetric perturbations of cylinders.

III.2) Application to Perturbed Cylinders

By (3.1), the metric of the wave fronts $\sigma = \sigma_0$ (constant), $r = r_0$ (constant) is:

$$ds^2 \, (\sigma_0 \, , r_0) = -r_0^2 \, [f^{-1}(\sigma_0 \, , \xi) \, d\xi^2 \, + f(\sigma_0 \, , \xi) \, d\eta^2 \,] \; . \tag{3.5}$$

It would be possible to develop a method to express an arbitrary surface of revolution in this form and to find the corresponding function f. (See [9].) For our purposes, it will be sufficient to note that if $f = 1$, (3.5) is flat. Moreover it has a built-in axial symmetry and is free of singularities. Therefore, $f = 1$ must represent a cylinder. (A complete proof is available in [9].) Thus, we shall define a "perturbed cylinder" as a surface on which the line element is (3.5), for f given by:

$$f(\sigma \, , \, \xi) = 1 + \epsilon(\sigma \, , \, \xi) \; . \tag{3.6}$$

To obtain the equation for ϵ, one substitutes (3.6) in (3.4) and keeps only the terms which are linear in ϵ, getting:

$$\frac{\partial \epsilon}{\partial \sigma} + \frac{\partial^4 \, \epsilon}{\partial \xi^4} \; , \tag{3.7}$$

in which the constant $12m$ has been absorbed in the definition of σ. This equation is very similar to the one satisfied by the temperature of a heat-conducting bar: $\frac{\partial T}{\partial \sigma} - \frac{\partial^2 T}{\partial \xi^2} = 0$ and, for this reason, one expects that the behaviour of the solutions ϵ and T will also be similar. By a standard technique of Fourier integrals, the time evolution of ϵ is found to be:

$$\epsilon(\sigma, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; \epsilon(0, x) \; \Phi(\sigma, \xi - x) \tag{3.8}$$

$$\Phi(\sigma, \xi) \equiv \frac{1}{|t|} \, \chi(\frac{\xi}{t}) \; , \; t \equiv \sqrt[4]{\sigma} \tag{3.9}$$

$$\chi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\ e^{-ixy}\ e^{-y^4} \quad . \tag{3.10}$$

This is analogous to the case of the heat conduction:

$$T(\sigma,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ T(0,x)\ \Phi(\sigma,\xi - x) \tag{3.11}$$

$$\Phi(\sigma,\xi) \equiv \frac{1}{|t|}\ \chi(\frac{\xi}{t})\ ,\ t \equiv \sqrt[2]{\sigma} \tag{3.12}$$

$$\chi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ e^{-ixy}\ e^{-y^2} = \frac{1}{\sqrt{2}}\ e^{-x^2/4} \quad . \tag{3.13}$$

We shall not pursue any further the comparison with the heat equation. The point which is still worth mentioning here is that one of the conserved quantities which can be constructed from ϵ is related to the total heat Q in a heat diffusion. To $Q \equiv \int_{-\infty}^{\infty} d\xi\ T(\sigma,\xi)$ corresponds $D \equiv \pi R^2 \int_{-\infty}^{\infty} d\xi\ \epsilon(\sigma,\xi)$, the latter representing the limiting value of the excess in area of a slice $-z \leq \xi \leq z$ of the (perturbed) wave front, compared with the unperturbed one (a cylinder or radius R), when z tends to infinity.

To conclude the study of the wire-singularities, one can find a useful upper bound on the derivatives of ϵ:

$$|\frac{\partial^n \epsilon}{\partial \xi^n}| \leq \frac{1}{4\pi}\, \Gamma(\frac{n+1}{4})\ \frac{1}{|t|^{n+1}} \int_{-\infty}^{\infty} dx\ |\epsilon(0,x)| \quad , \tag{3.14}$$

in which the notation is the same as in (3.9), and Γ is the Γ-function. This equation shows that, if the perturbation of the wave front is absolutely integrable at the initial time, the solution and all its derivatives tend to 0 when the time of evolution from the initial conditions tends to infinity. Moreover, the Gaussian curvature of the wave fronts tends to 0 and there are no wire-singularities. All these metrics tend therefore asymptotically to the (flat) unperturbed cylinder. Thus, we have found a class of (approximate) RT metrics which are free of wire-singularities for any time of evolution from the initial conditions. Moreover, the comparison with the heat equation gives a clear intuitive picture of this evolution: at the initial time, the total excess D of area between the actual wave front and the unperturbed one is given. When σ increases, this excess diffuses similarly to a heat wave, with the condition that this total excess remains constant, as the total quantity of heat Q remains also constant in a heat diffusion.

IV) CONCLUSION

In this lecture, we started by motivating physically the RT metrics. We showed how it is possible to consider the linearised RT solutions as the gravitational analogues of the Liénard-Wiechert electromagnetic potentials. We then investigated the question of the "wire-singularities" appearing in the RT line elements. Using two different formalisms, we proved that there exist (approximate) RT metrics which exhibit no such singularities. In some special case, the RT solutions evolve in time in a similar way to a heat wave.

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