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ON THE CONSTRUCTION OF HIGGS SECTORS*

J. Burzlaff** and L. O'Raifeartaigh

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

Abstract

The problem of constructing Higgs sectors, that is, finding Higgs representations and potentials that will produce a given spontaneous breakdown $G \rightarrow H$ (or hierarchy $G \rightarrow H \rightarrow K$...) is considered from a general point of view. The concepts of ordering of little groups, of little spaces and of symmetric algebras are shown to be of use, and it is also shown that in many cases a sum-of-squares potential can be very effective. A number of applications are presented and the pseudo-Goldstone problem is discussed.

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** Permanent address: School of Math. Sciences, NIHE Dublin 9, Ireland.

1. INTRODUCTION

In theory the problem of spontaneous breakdown of Higgs potentials is the following: given a Higgs (scalar) field Φ belonging to a given representation of a group G (not necessarily irreducible) and a given (renormalizable) potential $V(\Phi)$, find the vector $\overset{o}{\Phi}$ (more precisely the orbit of vectors $\overset{o}{\Phi}$) which minimizes $V(\Phi)$. Then the little group H of that orbit gives the breakdown i.e., $G \to H$. In practice, however, because the Higgs sector is not known experimentally, the pattern $G \to H$ (or hierarchy $G \to H \to K$...) is deduced from the fermionic (and possibly gauge) sector, so the problem is rather the reverse: given a desired pattern $G \to H$ (or $G \to H \to K$...), find a corresponding Higgs sector i.e. find a representation of the Higgs field Φ, and a potential $V(\Phi)$ that will produce it. This problem is not easy to solve because, not only are Φ and $V(\Phi)$ not necessarily unique, but they are strongly constrained by the following conditions. The representation R of Φ should be as irreducible as possible and the irreducible components should be as small as possible (in order to minimize the number of Higgs parameters) and the potential $V(\Phi)$ should be quartic (in order to be non-trivial, bounded below and renormalizable). It is clear that the constraint on the potential is both the stronger and the more inflexible of the two.

As a matter of fact, if there were no minimality constraints for the parameters and no renormalizability constraint for the potential, there would be no problem at all, because there is a theorem (due to Mostow[1)]) which guarantees that every closed subgroup of a compact Lie group occurs as a little group for <u>some</u> finite-dimensional (but not necessarily irreducible) representation, and there is a theorem (due to Schwarz[2)]) which states that the invariants in any finite-dimensional representation of a Lie group separate the orbits in the representation. Hence to obtain a given subgroup H as the little group for <u>some</u> representation R and <u>some</u> potential $V(\Phi)$, all that one has to do is to choose a representation for which H is a little group, choose an H-orbit in this representation, let $I(\Phi)$ be a function of the invariants such that $I(\Phi) = 0$ on this orbit alone, and then set $V = I(\Phi)^2$. Thus the problem of finding a suitable R and V i.e. of constructing a Higgs sector is due entirely to the constraints on the parameters and the degree of V.

So far there does not seem to have been any systematic approach to this problem. Indeed in many grand-unification schemes the problem is ignored, partly because so little is known about the Higgs sector anyway and partly because the problem is not one that lends itself to a general analysis. In the present talk I wish to present some general results, which although they do not by any means solve the problem, go some way towards clarifying it, and in some cases help to solve it. The results are essentially of three kinds, namely, on the ordering of little groups, on symmetric algebras, and on the construction of a certain type of Higgs potential, namely one which consists only of squared terms. The little group and symmetric algebra results may be thought of as first steps in clarifying the problem and the sum-of-squares potential as a solution for certain special (but relevant) cases. A generalization of the sum-of-squares method has been in use independently by the Naples group[3].

2. ORDERING OF LITTLE GROUPS

It is now well-known that in a given representation R of a group G, the little groups can be partially-ordered (by inclusion, up to conjugation). Thus one has in general a pattern such as

$$\begin{tikzcd}[row sep=small]
 & & & H_6 \arrow[dl] & H_9 \arrow[l] \\
 & & H_3 \arrow[dl] & & \\
 & H_1 \arrow[dl] & & & \\
H_0 & & H_4 \arrow[ul] \arrow[dl] & H_7 \arrow[l] & H_{10} \arrow[l] \\
 & H_2 \arrow[ul] & & & \\
 & & H_5 \arrow[ul] & & \\
 & & & H_8 \arrow[ul] & H_{11} \arrow[l]
\end{tikzcd} \tag{2.1}$$

e.g. for the adjoint representation of $SU(5)$

$$\begin{tikzcd}[row sep=small]
 & & S(U(2)^2 \times U(1)) \arrow[dl] & U(4) \arrow[l] \arrow[dl] \\
U(1)^4 & S(U(2) \times U(1)^3) \arrow[l] & & \\
 & & S(U(3) \times U(1)^2) \arrow[ul] & S(U(3) \times U(2)) \arrow[l] \arrow[uul]
\end{tikzcd}$$

where $H \leftarrow K$ means that K contains a G-conjugate of H. The maximal little groups are not unique, but the minimal little group H_0 is unique and is the little group for the generic orbits in the representation (which form a dense set in the orbit space).

The interest of this ordering for symmetry breaking is that it is found empirically that for non-isotropic quartic potentials (i.e. quartic potentials which depend non-trivially on invariants other than the norm (Φ, Φ)) the smaller and more irreducible the representation R, the larger is the little group. In fact, there was a conjecture by Michel[4] that if the representations are irreducible the little group would always be maximal, and although counter-examples to this conjecture are now known (one of which will be constructed later by the sum-of-squares method) it is true in most cases of interest, and illustrates the general trend. In any case the ordering of the groups clearly plays an important role in surveying the problem of finding suitable Higgs representations R.

There are some more precise statements on the ordering that can be made. First, there is a result[5)] which relates the minimal little group H_0 to the dimension of an (irreducible) representation R and to the value of the so-called index $I(R)$ of R, defined as

$$I(R) = \frac{trX^2(R)}{trX^2(A)} = \frac{C(R)\ dimR}{C(A)\ dimA} = \frac{(h, h+\delta)\ dimR}{(a, a+\delta)\ dimA}\ , \tag{2.2}$$

where $X(R)$ denotes any generator, $C(R)$ the second-degree Casimir and h the highest weight of R, A, a the corresponding quantities for the adjoint representation and δ the sum of positive roots. The result may be expressed conveniently by the following table of equivalences

$$\begin{array}{ccccc} I(R) > 1 & \longleftrightarrow & dimR > dimA & \longleftrightarrow & H_0\ finite \\ I(R) = 1 & \longleftrightarrow & R = A & \longleftrightarrow & H_0\ abelian \\ I(R) < 1 & \longleftrightarrow & dimR < dimA & \overset{*}{\longleftrightarrow} & H_0\ non-abelian \end{array}$$

where the star in the last equivalence means that it does not hold in two exceptional cases, namely the 2-dimensional representation of $SU(2) \approx SO(3)$ and the 6-dimensional (symmetric 2-tensor) representation of $SU(3)$. Actually the set of representations for which $dimR < dimA$ is quite limited, and, apart from the fundamental (defining) representation of each of the simple compact groups (except $E(8)$) and its conjugate for $SU(n)$, the set includes only the traceless symmetric 2-tensors of $SU(n)$, $n \geq 3$, the anti-symmetric 2-tensors of $SU(n)$, $n \geq 4$ and $Sp(2n)$, $n \geq 2$, the totally anti-symmetric 3-tensors of $SU(6)$, $SU(7)$, $SU(8)$ and $Sp(6)$ and the lowest-dimensional spinorial representations of $SO(7)...SO(14)$.

A second precise statement that can be made concerns the dimensions of the 'little spaces' R_i of the H_i, which are defined as the linear spans of all the vectors $\overset{o}{\Phi}$ for which the H_i are little groups ($H_i\ \overset{o}{\Phi} = \overset{o}{\Phi}$). Let us consider any ordered chain in (2.1) i.e. a chain

$$H_0 \leftarrow H_1 \leftarrow H_2 \leftarrow ... \leftarrow H_{m-1} \leftarrow H_m \tag{2.3}$$

where H_m is maximal. Then since $H_{i+1}\ \overset{o}{\Phi} = \overset{o}{\Phi}$ implies $H_i\ \overset{o}{\Phi} = \overset{o}{\Phi}$ it is clear that the spaces R_i are ordered in the opposite way i.e.

$$R_0 \supseteq R_1 \supseteq R_2 \supseteq ... \supseteq R_{m-1} \supseteq R_m \tag{2.4}$$

and in fact, one can say more, namely, that the ordering is strict i.e.

$$R_0 \supset R_1 \supset R_2 \supset ... \supset R_{m-1} \supset R_m \quad \text{or} \quad dimR_{i-1} - dimR_i \geq 1\ . \tag{2.5}$$

To see this one notes that if R_i were equal to R_{i-1} then any vector which had H_{i-1} as little group would also have H_i as little group so H_{i-1} could not be a little group (maximal stability group) of $\overset{o}{\Phi}$ to begin with.

A corollary of this result is that if the 'little vector' $\overset{o}{\Phi}$ of a little group H is unique (up to normalization) then H must be a maximal little group i.e. $dim R_i = 1 \Rightarrow R_i = R_m$. In all cases of irreducible representations that we know, the converse is also true i.e. $dim R_m = 1$, but we know of no general proof and it might be interesting to find a counter-example.

3. SYMMETRIC ALGEBRAS

Let R be a representation of a group G and suppose that the symmetric product $(R \times R)_s$ contains the representation R at least once. For most representations of interest this can be checked using the Slansky tables[6]. Then a G-invariant symmetric algebra $(R \times R)_s \rightarrow R$ can be defined by letting

$$(a \bigtriangledown b)^\alpha = C^\alpha_{\beta\gamma} a^\beta b^\gamma, \quad a, b \in R \tag{3.1}$$

where $C^\alpha_{\beta\gamma}$ are the CG-coefficients for $(R \times R)_s \rightarrow R$. It is then obvious that one can, in fact, define an independent symmetric algebra for each occurrence of R in $(R \times R)_s$. Let us consider some examples.

1) Adjoint representation of $SU(n)$: It is well known that this representation is contained (once) in its symmetric product, and if the elements of the representation-space R are realized as traceless, hermitian $n \times n$ matrices A, B then the product is defined by

$$A \bigtriangledown B = \{A, B\} - \frac{1}{n} \mathbf{I}\, tr\, \{A, B\} \ , \tag{3.2}$$

where curly brackets denote the anti-commutator of the two matrices.

2) The symmetric tensor representation of $SO(n)$: This is a special case of the previous one when A and B are real.

3) The symmetric tensor plus anti-symmetric tensor (adjoint) representation of $SO(n)$: For $SO(n)$ the adjoint representation A does not contain A in the symmetric product. So a symmetric algebra cannot be constructed from A alone. However, the symmetric product $(A \times A)_s$ does contain the symmetric tensor representation of example 2. Hence if we adjoin these two representations to form the direct sum representation $S \oplus A$, we can define a symmetric algebra, and, in an obvious notation it reads

$$S \bigtriangledown S = S\ , \qquad S \bigtriangledown A = A\ , \qquad A \bigtriangledown A = S\ . \tag{3.3}$$

It is interesting to contrast (3.3) with the Lie algebra for $SU(n)$ which reads

$$[A, A] = A\ , \quad [A, S] = S\ , \quad [S, S] = A\ . \tag{3.4}$$

4) The 27-dimensional (or (2,2)) representation of $SU(3)$: This representation has the property that it is contained <u>twice</u> in its symmetric product. We describe the vectors for this representation by 8×8 symmetric matrices X^a_b, which satisfy the conditions

$$X^a_a = 0\ , \quad d^a_{bc} X^c_a = 0\ , \tag{3.5}$$

where the d's are the CG-coefficients for $(8 \times 8)_s \rightarrow 8$ (Gell-Mann matrices). Note that any symmetric matrix X can be modified so as to satisfy (3.5) by letting

$$X^a_b \longrightarrow S^a_b(X) = X^a_b - \frac{3}{5} d^a_{bk} d^r_{ks} X^s_r - \frac{1}{8} \delta^a_b X^k_k\ . \tag{3.6}$$

In this notation the two independent symmetric algebras[7)] may be defined as

$$(X \bigtriangledown Y)^a_b = S^a_b(F(X, Y)) \quad \text{and} \quad (X \vee Y)^a_b = S^a_b(\{X, Y\})\ , \tag{3.7}$$

where

$$F^a_b(X, Y) = f^a_{ij} f^r_{bs} X^i_r X^j_s\ ,$$

and $\{X, Y\}$ denotes the anti-commutator of X and Y.

A useful property of the symmetric algebras is that they leave the little spaces of the previous section invariant. For if X and Y are H_i-invariant, and the operation $\bigtriangledown$ is G- (and therefore H_i-) invariant, the product $X \quad Y$ must be H_i-invariant and therefore lie in R_i. Thus

$$R_i \bigtriangledown R_i \subset R_i\ . \tag{3.8}$$

(Although the mapping (3.8) is onto all of R_i, there may exist vectors which map themselves to zero i.e. vectors k for which

$$k \bigtriangledown k = 0\ , \quad k \in R_i\ , \tag{3.9}$$

and we shall see an example later.)

In particular, if R_m is one-dimensional i.e. the little vector is unique (and therefore H_m is maximal) then the little vector $\overset{o}{\Phi}$ is an idempotent of the symmetric product,

$$\overset{o}{\Phi} \bigtriangledown \overset{o}{\Phi} = \lambda\ \overset{o}{\Phi} \tag{3.10}$$

where λ is some constant. Similarly, if R_i is 2-dimensional, with base-vectors $\overset{o}{\Phi}$, $\overset{o}{\Psi}$ say, then

$$\overset{o}{\Phi} \bigtriangledown \overset{o}{\Phi} = \lambda \overset{o}{\Phi} + \mu \overset{o}{\Psi} \tag{3.11}$$

where λ and μ are constants (and similarly for $\overset{o}{\Psi}$). If there are two or more symmetric algebras, then (3.10) and (3.11) hold for each one separately. For example,

$$\overset{o}{\Phi} \vee \overset{o}{\Phi} = \lambda_1 \overset{o}{\Phi} + \mu_1 \overset{o}{\Psi} \tag{3.12}$$

$$\overset{o}{\Phi} \bigtriangledown \overset{o}{\Phi} = \lambda_2 \overset{o}{\Phi} + \mu_2 \overset{o}{\Psi} \tag{3.13}$$

In this case there exists a symmetric algebra for which $\overset{o}{\Phi}$ is an idempotent,

$$\overset{o}{\Phi} \diamond \overset{o}{\Phi} = \lambda \overset{o}{\Phi} , \quad \lambda = \mu_2 \lambda_1 - \mu_1 \lambda_2 , \tag{3.14}$$

namely the linear combination $\diamond = \mu_2 \vee - \mu_1 \bigtriangledown$, or

$$A \diamond B = \mu_2 A \vee B - \mu_1 A \bigtriangledown B . \tag{3.15}$$

4. SUM-OF-SQUARES POTENTIALS

We have already mentioned that for renormalizable (quartic) potentials, the more irreducible the Higgs representation the more likely it is that the little group will be maximal or near-maximal. Thus in many cases one is interested in maximal and near-maximal little groups (at least at the separate stages $H_i \rightarrow H_{i-1}$ of a sequence of spontaneous symmetry breakdowns). For such little groups it is often possible to turn the tables on the renormalization constraint and exploit the fact that $V(\Phi)$ is quartic, in order to obtain the required little group. This is done by writing $V(\Phi)$ as a sum of squares, and although the technique does not always work, when it does work it is very effective. A generalization of this approach has been adopted by the Naples group[3], who have noted that if R_α are those representations in the expansion $(R \times R)_s$ for which $(\overset{o}{\Phi} \vee \overset{o}{\Phi})_\alpha$ is zero, where $\overset{o}{\Phi}$ is a little vector of H, then the potential

$$V(\Phi) = \sum C_\alpha (\Phi \vee \Phi)_\alpha (\Phi \vee \Phi)_\alpha , \quad C_\alpha > 0 , \tag{4.1}$$

vanishes for $\Phi = \overset{o}{\Phi}$, and that in many cases (for a sufficiently large number of R_α) it vanishes <u>only</u> when $\Phi = \overset{o}{\Phi}$ (or a conjugate). What I shall use below is a different variation of the sum-of-squares method, based on the symmetric algebra. This approach uses the fact, that if a required little vector $\overset{o}{\Phi}$ is an idempotent of a symmetric algebra, and is the only (normalized) vector with the given eigenvalue λ (up to conjugation) then the potential

$$V = h((\Phi,\Phi)^2 - c^2)^2 + k(\Phi \bigtriangledown \Phi - \lambda\Phi)^2 \ , \quad h,k > 0 \tag{4.2}$$

where c is the norm, will vanish, if, and only if, Φ is conjugate to $\overset{o}{\Phi}$. But this is best illustrated by examples.

5. EXAMPLES OF HIGGS SECTORS

Example 1

The Higgs field Φ belongs to the adjoint representation of $SU(n)$ and the most general quartic potential is

$$V(\Phi) = h(tr\Phi^2)^2 + g\ tr\Phi^4 + f\ tr\Phi^3 - m\ tr\Phi^2 \ . \tag{5.1}$$

It is well-known that (unless $g = f = 0$) the minimum of the Higgs potential (5.1) must fall on an orbit with a maximal subgroup i.e. that the Michel conjecture mentioned earlier holds in this case. (For completeness I have inserted the latest, and to my mind the shortest and most elegant, proof[8] of this result in the appendix.) In any case, for the adjoint representation of $SU(n)$, the maximal little groups are just $S(U(p) \times U(q)), p+q = n$, and they have as little vectors the matrices

$$\overset{o}{\Phi}_{pq} = x \begin{pmatrix} p\mathbf{I}_q & 0 \\ 0 & -q\mathbf{I}_p \end{pmatrix} , \quad tr\ \overset{o}{\Phi}{}^2 = x^2 npq \ , \quad \overset{o}{\Phi} \bigtriangledown \overset{o}{\Phi} = 2x(p-q)\ \overset{o}{\Phi} \ , \tag{5.2}$$

(and their conjugates) where x is arbitrary. Note that this is one of the cases for which the little vector $\overset{o}{\Phi}$ is unique (up to conjugation) i.e. for which $dim R_m = 1$, and that for $p = q$ it is one of the cases for which there is a vector which maps itself to zero.

The questions that concern us here are the following. Are all the maximal little groups $S(U(p) \times U(q))$ realized for a potential of the form (5.1), and, for those that are, how can the parameters in $V(\Phi)$ be arranged so as to realize it? Both questions can be answered at once by simply choosing the potential to be the sum of squares

$$V(\Phi) = k(tr\Phi^2 - npq)^2 + s\ tr(\Phi \bigtriangledown \Phi - 2(p-q)\Phi)^2 \ , \quad k,s > 0 \ . \tag{5.3}$$

From (5.3), one sees that $V(\Phi) = 0$ if, and only if, Φ is conjugate to $\overset{o}{\Phi}_{pq}$. Thus all maximal little groups are realizable, and are realized for the given (p,q) by (5.3).

Example 2

Suppose we wish to break $SO(10)$ down to $U(3)\times U(2)$ (not $S(U(3)\times U(2))$). Since $U(3) \times U(2)$ is not a maximal little group (in fact, the dimension of its little space is three) this cannot be done using the symmetric tensor or adjoint representation alone, but can easily be done by the combination $S \oplus A$, which has a symmetric algebra, as follows: First, we write the generic quartic potential in the form

$$V(S,A) = \lambda_S((S,S)-\mu_S)^2 + \lambda_A((A,A)-m_A)^2$$

$$+\lambda(m_A(S,S)-\mu_S(A,A))^2 + \sigma(S\wedge A, S\wedge A) + f(F,F) + g(G,G) \;, \tag{5.4}$$

where

$$F = F(\alpha) = S \bigtriangledown S \; cos\alpha + S \; sin\alpha \;, \quad G = A \bigtriangledown A \; cos\beta + F(\gamma) sin\beta \;, \tag{5.5}$$

the symbol $\wedge$ denotes commutator, and the inner-product is defined as the trace. This becomes a sum of squares if $\lambda_S, \lambda_A, \lambda, \sigma, f$ and g are all positive and then at the absolute minimum we have

$$S \bigtriangledown S = -tan\alpha \; S \;, \quad A \bigtriangledown A = -tan\beta \frac{sin(\gamma-\alpha)}{cos\alpha} S \;, \quad S\wedge A = 0 \tag{5.6}$$

plus some normalization conditions. It is clear that by choosing $tan\alpha$ suitably we can break $SO(10)$ to $S(O(6) \times O(4))$ (just like $SU(10) \rightarrow S(U(6) \times U(4))$ in the previous example) and that A then decomposes into an $SO(6)$ A and an $SO(4)$ A separately. It is then easy to see from standard adjoint-of-$SO(n)$ analysis that, for suitable choices of the parameters, the potentials for these A's break $SO(6)$ to $U(3)$ and $SO(4)$ to $U(2)$ respectively. Thus the final symmetry group is $U(3) \times U(2)$ as required.

It is only fair to add, however, that it may not be possible to obtain all symmetry breaking patterns for $S \oplus A$ by a sum-of-squares method. For example, the above procedure does not lead to $SO(6) \times U(2)$ or $SU(3) \times O(4)$.

6. COUNTER-EXAMPLE TO MICHEL CONJECTURE

As a final example of sum-of-squares potentials, and as an example of the use of two symmetric algebras, we construct a counter-example to the Michel conjecture mentioned earlier (that for irreducible representations, and non-isotropic quartic potentials the little group must be maximal). There are actually only two known counter-examples[7,9] at present (for Lie groups) and we shall use the one constructed from the 27-dimensional or (2,2) representation of $SU(3)$. There are three maximal little groups for this representation, namely, $SO(3), U(2)$ and $W_3 \wedge (U(1) \times U(1))$ where W_3 is the Weyl group for $SU(3)$ and $\wedge$ denotes semi-direct product. Note that, although the third little group is maximal its algebra $u(1) \times u(1)$ is not, which shows that global properties of groups play an important role in these considerations. Each of the above three little groups has a unique little vector (up to normalization and conjugation) so that again we have the empirical result $dim R_m = 1$. Let us denote the three normalized little vectors by $\overset{o}{x}_\alpha, (\overset{o}{x}_\alpha, \overset{o}{x}_\alpha) = c^2, \alpha = 1, 2, 3$, respectively. Since they are unique they are idempotents of both symmetric algebras

$$\overset{o}{x}_\alpha \vee \overset{o}{x}_\alpha = \lambda(\alpha)\, \overset{o}{x}_\alpha\ , \quad \overset{o}{x}_\alpha \bigtriangledown \overset{o}{x}_\alpha = \mu(\alpha)\, \overset{o}{x}_\alpha \qquad (\alpha \textit{ not summed}) \tag{6.1}$$

where $\lambda(\alpha)$ and $\mu(\alpha)$ are numerical coefficients[7)]. Consider now the 2-space spanned by x_2 and x_3. It is clear that the little group for a generic point $\overset{o}{x}$ in the 2-space is the intersection group $W = U(2) \cap W_3 \wedge (U(1) \times U(1)) = W_2 \wedge (U(1) \times U(1))$ where W_2 is the Weyl group for $SU(2)$, and that this little group is not maximal. But from the discussion of section 3, we know that there exists a symmetric algebra $\diamondsuit$ for which $\overset{o}{x}$ is an idempotent

$$\overset{o}{x} \diamondsuit \overset{o}{x} = \sigma(\overset{o}{x})\, \overset{o}{x}\ , \quad (\overset{o}{x}, \overset{o}{x}) = c^2\ . \tag{6.2}$$

Furthermore, it is easy to check[7)] that although the $\overset{o}{x}_\alpha, \alpha = 1, 2, 3$ are also idempotents of this algebra, their eigenvalues are different

$$\overset{o}{x}_\alpha \diamondsuit \overset{o}{x}_\alpha = \sigma(\alpha)\, \overset{o}{x}_\alpha\ , \quad \sigma(\alpha) \neq \sigma(\overset{o}{x})\ , \quad (\overset{o}{x}_\alpha, \overset{o}{x}_\alpha) = c^2 \quad (\alpha \textit{ not summed})\ . \tag{6.3}$$

Hence, if we choose as quartic potential the quantity

$$V(\Phi) = h((\Phi, \Phi) - c^2)^2 + k(\Phi \diamondsuit \Phi - \sigma(\overset{o}{x})\Phi)^2\ , \quad h, k > 0 \tag{6.4}$$

it is clear that $V(\overset{o}{x}_\alpha) > 0$ and $V(\overset{o}{x}) = 0$, so the vectors $\overset{o}{x}_\alpha$ with maximal little groups cannot minimize it. Thus $V(\Phi)$ is minimized by the next-to-maximal little group $W_2 \wedge (U(1) \times U(1))$ (and possibly by some other non-maximal little groups). Finally, one should check that $SU(3)/Z_3$ really is the maximal invariance group of V. This follows from the fact that, by the construction of the symmetric products, the invariance group of V must leave the structure constants f^a_{bc} and the CG-coefficients d^a_{bc} (which are expressible in terms of one another) invariant and hence must belong to the group of automorphisms of $SU(3)/Z_3$ which are all inner.

7. THE PSEUDO-GOLDSTONE PROBLEM AND MASS-SPECTRUM

In forming the sum-of-squares, or indeed any, Higgs potential, there is the danger that the minimum point will be unstable in the sense that small variations in the parameters may change the little group. This is a particularly important problem in quantum field theory, because the parameters of the classical potential may be changed by the radiative corrections. (This is what actually happens in the case of the Coleman-Weinberg potential[10], for example.) Since, in general, $V(p,\Phi) \to V(p+\delta p,\Phi)$ implies $\overset{o}{\Phi} \to \overset{o}{\Phi} + \delta\overset{o}{\Phi}$, where p are the parameters one sees that in order to guarantee stability one needs to guarantee that the given little group H is not only the little group of $\overset{o}{\Phi}$, but also of its immediate neighbourhood.

It turns out that a practical way to check whether the combination $(H,\overset{o}{\Phi})$ is stable in this sense, is to check that there are no pseudo-Goldstone fields i.e. that there are only $dimG/H$ massless fields at $\overset{o}{\Phi}$. The reason for this is the following. If K is the little group of the neighbourhood $(K(\overset{o}{\Phi}+\delta\overset{o}{\Phi}) = \overset{o}{\Phi}+\delta\overset{o}{\Phi})$ then by continuity $K\overset{o}{\Phi}=\overset{o}{\Phi}$ which means that $K \subseteq H$. (Note that this implies that the little group increases only at the boundaries of (locally) open sets.) Now, in the neighbourhood the number of massless fields $\geq dimG/K$, which is the number of Goldstone fields for K. Hence, by continuity, the number of massless fields at $\overset{o}{\Phi} \geq dimG/K$. But $dimG/K = dimG/H + dimH/K$. Hence the number of massless but non-Goldstone (i.e. pseudo-Goldstone) fields at $\overset{o}{\Phi} \geq dimH/K$. Thus if there are no pseudo-Goldstone fields $dimH/K = 0$ and $H = K$. Thus the absence of pseudo-Goldstone fields is a good criterion for stability. Indeed, if the non-Goldstone fields at $\overset{o}{\Phi}$ are of the order unity on the general mass-scale, there should be no other little groups anywhere near $\overset{o}{\Phi}$. In that case one says that $\overset{o}{\Phi}$ is deep in the H-basin.

The sum-of-squares potentials are particularly good for applying the pseudo-Goldstone criterion and indeed for determining the mass-spectrum in general. For consider the potential

$$V(\Phi) = \frac{1}{2}\sum_{\alpha} V_\alpha(\Phi)^2\,, \quad \text{where} \quad V(\overset{o}{\Phi}) = 0\,. \tag{7.1}$$

Then the first variation of $V(\Phi)$ at $\Phi = \overset{o}{\Phi}$ vanishes, and the second variation yields

$$\delta^2 V(\overset{o}{\Phi}) = \sum_{\alpha}\left[\left(\frac{\partial V_\alpha(\Phi)}{\partial\Phi_a}\right)_{\Phi=\overset{o}{\Phi}}\delta\Phi_a\right]^2 \tag{7.2}$$

which is again a sum of squares. Thus the masses of the various fields are simply

$$M_a^2 = \frac{1}{2}\sum_{\alpha}\left(\frac{\partial V_\alpha(\Phi)}{\partial\Phi_a}\right)^2_{\Phi=\overset{o}{\Phi}} \tag{7.3}$$

which is easy to compute. In particular, the massless fields can only be those which satisfy the set of independent linear equations

$$\left(\frac{\partial V_\alpha(\Phi)}{\partial \Phi_a}\right)_{\Phi=\overset{o}{\Phi}} \delta\Phi_a = 0\ , \quad \alpha = 1, ..., n. \tag{7.4}$$

In other words, the only massless fields are those which are left massless by each of the potentials $V_\alpha(\Phi)$ separately i.e. which are the intersection of the massless field for each α. The true Goldstone fields will, of course, satisfy this criterion, and the isotropic part $V((\Phi, \Phi))$ of the potential will give a mass only to the polar Higgs field i.e. the Higgs field which is parallel to $\overset{o}{\Phi}$ in the representation. So the non-Goldstone, non-polar Higgs fields must get their masses from the non-isotropic terms in the potential i.e. the terms which depend on invariants of higher degree than the second. Let us see how the criterion works out for some examples.

8. APPLICATIONS OF THE PSEUDO-GOLDSTONE CRITERIUM

Application 1

The adjoint representation of $SU(n)$ with potential

$$V(\Phi) = h((\Phi, \Phi) - npq)^2 + g(\Phi \bigtriangledown \Phi - 2(p-q)\Phi)^2 \tag{8.1}$$

which is zero if, and only if, $\Phi = diag(p\mathbf{I}_q, -q\mathbf{I}_p)$, or one of its conjugates. According to (7.3) the mass-spectrum for fields other than the field parallel to $\overset{o}{\Phi}$ is given by

$$(\delta\Phi, M^2\ \delta\Phi) = 2g(\overset{o}{\Phi} \bigtriangledown \delta\Phi - (p-q)\delta\Phi)^2\ . \tag{8.2}$$

There are evidently three such $S(U(p) \times U(q))$-invariant mass-multiplets. namely those corresponding to the $SU(p)$ and $SU(q)$ adjoint algebras respectively, and the non-block-diagonal multiplet. For the first two the multiplet is both trace-orthogonal to $\overset{o}{\Phi}$ and commutes with it, so the product $\overset{o}{\Phi} \bigtriangledown \delta\Phi$ reduces to ordinary matrix multiplication. From (8.2) we then have

$$M_p^2 = M_q^2 = 2n^2 g \tag{8.3}$$

so these multiplets are certainly massive. The non-block-diagonal multiplet is still trace-orthogonal to $\overset{o}{\Phi}$, but since it does not commute with $\overset{o}{\Phi}$ the product $2(\overset{o}{\Phi} \bigtriangledown \delta\Phi)$ reduces only to the anti-commutator $\{\overset{o}{\Phi}, \delta\Phi\} = (p-q)\delta\Phi$. From (8.2) one then sees that for these fields $M^2 = 0$. But since the number of such fields is $2pq = dimSU(n)/dimS(U(p) \times U(q))$, one sees that they are just the true Goldstone fields.

Thus there are no pseudo-Goldstone fields in this example, and, in fact, the complete mass-spectrum is given by (8.3) and by the mass M_0^2 of the field in the $\overset{o}{\Phi}$ direction, which is easily computed to be

$$M^2(\overset{o}{\Phi}) = 2[npqh + g(p-q)^2] \,. \tag{8.4}$$

Application 2

The SA representation $(54+45)$ of $SO(10)$ with potential (5.4). In this case, the non-isotropic terms in the potential are evidently

$$h(S \wedge A, S \wedge A) + f(F,F) + g(G,G) \tag{8.5}$$

so, apart from the polar field, the mass-spectrum is given by

$$h(\overset{o}{A} \wedge \delta S + \overset{o}{S} \wedge \delta A)^2 + f(\frac{\partial F(\alpha)}{\partial S}\delta S)^2 + g(2\overset{o}{A} \nabla \delta A \; cos\beta + \frac{\partial F(\gamma)}{\partial S}\delta S \; sin\beta)^2 \,, \tag{8.6}$$

where $F(\theta) = S \nabla S \; cos\theta + S \; sin\theta$. Thus any massless field satisfies the three simultaneous conditions

$$2\overset{o}{S} \nabla \delta S = -tan\alpha \; \delta S \,, \quad 2\overset{o}{A} \nabla \delta A = -tan\beta \frac{\partial F(\gamma)}{\partial S}\delta S \,, \quad \overset{o}{A} \wedge \delta S + \overset{o}{S} \wedge \delta A = 0 \,. \tag{8.7}$$

The first equation in (8.7) is the pure S-condition of the previous example, and it admits only the true Goldstones for $SO(10)_S \to S(O(6) \times O(4))_S$. Since A and S are then block-diagonal the remaining equations split into

$$2\overset{o}{A} \nabla \delta A_d = 0 \,, \quad 2\overset{o}{A} \nabla \delta A_{od} = -tan\beta \frac{\partial F(\gamma)}{\partial S}\delta S_{od} \,, \quad \overset{o}{A} \wedge \delta S_{od} + \overset{o}{S} \wedge \delta A_{od} = 0 \tag{8.8}$$

(plus an identity) where d denotes diagonal and od off-diagonal. The first equation in (8.8) is just the pseudo-Goldstone equation for the A-sector alone, and for the breakdowns in question, $SO(6) \to U(3)$ and $SO(4) \to U(2)$, it is well-known to admit no pseudo-Goldstone fields. Thus there are no pseudo-Goldstones for $SO(10)_S$ and $SO(10)_A$ separately.

The possibility that the diagonal group $SO(10)_{S+A}$, which is the group we are really interested in, might have pseudo-Goldstone fields, is then eliminated by the last two equations in (8.8), since these equations show that δS_{od} and δA_{od} are not linearly independent.

Application 3

As a final application let us consider the 210-dimensional representation of $SO(10)$ and the potential

$$V(\Phi) = e_1(\Phi \vee \Phi)^2_{45} + e_2(\Phi \vee \Phi)^2_{210} + e_3(\Phi \vee \Phi)_{1050}(\Phi \vee \Phi)_{\overline{1050}} \tag{8.9}$$

($\Phi \in 210$) which has been investigated by the Naples group. This potential was chosen because each of the terms $(\overset{o}{\Phi} \vee \overset{o}{\Phi})_\alpha$ of the expansion $(\Phi \times \Phi)_s$ included in (8.9) is zero when $\Phi = \overset{o}{\Phi}$, where $\overset{o}{\Phi}$ is the $S(O(6) \times O(4))$ singlet in the 210. The question is whether the $S(O(6) \times O(4))$-invariant minimum point $\overset{o}{\Phi}$ is stable in the sense described above. From the computations of T. Tuzi[3)] one sees that there are four (non-Goldstone) $S(O(6) \times O(4))$ mass multiplets and that their masses are

$$M^2(15,1,3) = M^2(15,3,1) = \frac{4}{45}(e_2 + 5e_3)(\overset{o}{\Phi}, \overset{o}{\Phi}) \ ,$$

$$M^2(15,1,1) = \frac{4}{35}e_1(\overset{o}{\Phi}, \overset{o}{\Phi}) \ , \quad M^2(10,2,2) = \frac{1}{2}e_3(\overset{o}{\Phi}, \overset{o}{\Phi}) \tag{8.10}$$

respectively. Thus, as long as e_1 and e_3 are not zero, there are no pseudo-Goldstone fields and the system is stable. Note that, as far as the above stability is concerned, one could actually dispense with the constant e_2, and hence with the 210-term, in the potential.

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APPENDIX

Proof of Michel conjecture for adjoint representation of $SU(n)$.

Let us write the non-isotropic quartic potential $V(\Phi)$ for this representation in the traditional form

$$V(\Phi) = h(tr\Phi^2)^2 + g\ tr\Phi^4 + f\ tr\Phi^3 - m\ tr\Phi^2 \ , \tag{A1}$$

(where $g \neq 0$, because $g = 0, f \neq 0$ is trivial, and $g = f = 0$ is isotropic) and let us diagonalize (by an $SU(n)$ transformation) the matrix $\overset{o}{\Phi}$ which minimizes it. The problem is to show that $\overset{o}{\Phi}$ has only two distinct eigenvalues. We therefore suppose that there are at least three different eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and choose the basis so that they are the first three, $\overset{o}{\Phi} = diag(\lambda_1, \lambda_2, \lambda_3, ...)$. We consider the subgroup $S(U(3) \times U(n-3))$

which corresponds to this choice in an obvious way. Then we make an $S(U(3)\times U(n-3))$ decomposition of $\overset{o}{\Phi}$, namely $\overset{o}{\Phi}=\lambda+\mu$, where $\lambda=diag(\lambda_1-t,\lambda_2-t,\lambda_3-t,0,...,0)$, $\mu=diag(t,t,t,\lambda_4,\lambda_5,...)$ and $3t=\lambda_1+\lambda_2+\lambda_3$. Then since λ belongs to the adjoint representation of $SU(3)$ we have

$$tr\lambda^4=\frac{1}{2}(tr\lambda^2)^2 \quad \text{and} \quad -(tr\lambda^2)^{3/2}\leq 6\,tr\lambda^3\leq(tr\lambda^2)^{3/2}\,. \tag{A2}$$

Furthermore, in the second equation the equality sign is achieved if, and only if, two of the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ coincide, which we are assuming is not the case. If we now expand the potential $(A1)$ in terms of λ we obviously obtain

$$V(\Phi)=(f+4gt)tr\lambda^3+U(tr\lambda^2,t,\mu)\,. \tag{A3}$$

We can then regard t and $tr\lambda^3$ as two independent variables and if we consider the Hessian of $V(\Phi)$ with respect to them we see by inspection that it is

$$H=\begin{pmatrix} U_{tt} & 4g \\ 4g & 0 \end{pmatrix}\,, \quad \text{where} \quad U_{tt}=\frac{\partial^2 U}{\partial t^2}\,. \tag{A4}$$

But this matrix has a negative eigenvalue since $detH=-16g^2$. Hence $V(\Phi)$ cannot minimize on the open domain $36(tr\lambda^3)^2<(tr\lambda^2)^3$, contrary to hypothesis.

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