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Renormalization and the Continuum Limit *

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Abstract

It is explained how the renormalization transformation can be used to take the continuum limit of a lattice field. It is shown that, by rescaling, the problem can be formulated on a fixed lattice $\mathbf{Z}^d$. The procedure is illustrated by two examples: the one-dimensional Euclidean free field and a hierarchical model with ϕ^4 -interaction.

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1. The general renormalization procedure

In the following we shall adopt as a definition of a (Euclidean) scalar field theory a generalized random field on some space $\mathcal{F}$ of functions, i.e. a *linear* mapping $\phi : \mathcal{F} \longrightarrow L^0(E,\mu)$, where $L^0(E,\mu)$ is the set of random variables on a topological space E with probability measure μ . We do not concern ourselves here with the Osterwalder-Schrader axioms. In the case of a d-dimensional lattice field $\mathcal{F}$ is a class of functions $f : \mathbf{Z}^d \longrightarrow \mathbf{C}$, and in the case of a continuum field $\mathcal{F}$ is a class of functions $f : \mathbf{R}^d \longrightarrow \mathbf{C}$.

We want to study the continuum limit of a lattice field ϕ on $\mathbf{Z}^d$. We therefore rescale the lattice $\mathbf{Z}^d$ with a factor $\delta > 0$ to obtain fields φ_δ on finer and finer lattices $\delta\mathbf{Z}^d$,and hope to be able to give a meaning to the limiting field $\varphi = \lim_{\delta\downarrow 0} \varphi_\delta$. In general it will be necessary to rescale the parameters defining the fields φ_δ in order to obtain a meaningful limit. This defines a transformation of parameters which is called the renormalization transformation. As a function of $\delta > 0$ these transformations obviously form a multiplicative 1-parameter semigroup, which is (erroneously) called the renormalization group.

In order to arrive at a suitable procedure to obtain a continuum limit let us assume for the moment that the continuum field φ is already given. Then we can obtain lattice fields φ_δ by coarse-graining, i.e. by averaging over lattice blocks

$$\square_\delta(\underline{x}) = \{\underline{x}_i - \frac{1}{2}\delta \le \underline{u}_i < \underline{x}_i + \frac{1}{2}\delta, i = 1, ..., d\} \tag{1.1}$$

for $\underline{x} \in \delta\mathbf{Z}^d$.
Explicitly,

$$\varphi_\delta(\underline{x}) = \delta^{-d}\varphi(1_{\square_\delta(\underline{x})}) \tag{1.2}$$

where 1_A is the indicator function of the set A.
For $f \in \mathcal{F}_\delta$ this becomes

$$\varphi_\delta(f) = \sum_{\underline{x}\in\delta\mathbf{Z}^d} f(\underline{x})\varphi(1_{\square_\delta(\underline{x})}) \tag{1.3}$$

The lattice fields φ_δ satisfy

$$\varphi_{L\delta}(\underline{x}) = L^{-d} \sum_{\underline{y}\in\delta\mathbf{Z}^d\cap\square_{L\delta}(\underline{x})} \varphi_\delta(\underline{y}). \tag{1.4}$$

Conversely, given a sequence of lattice fields φ_n on $L^{-n}\mathbf{Z}^d$, we can define φ by the limit

$$\varphi(f) = \lim_{n\to\infty} \varphi_n(f) = \lim_{n\to\infty} \sum_{\underline{x}\in L^{-n}\mathbf{Z}^d} L^{-d} f(\underline{x})\varphi_n(\underline{x}). \tag{1.5}$$

If the φ_n satisfy (1.4) then this limit certainly exists for functions f of the form

$$f = \sum_{\underline{x} \in L^{-m} \mathbf{Z}^d} f(\underline{x}) 1_{\square_m(\underline{x})} \tag{1.6}$$

for some positive integer m.
Although it is desirable that this limit exist for a class of smooth functions f , we do not go into that problem here and concentrate on obtaining a sequence φ_n satifying (1.4). Rescaling we can reduce the latter problem to a fixed unit lattice $\mathbf{Z}^d$. Indeed, given fields ϕ_n on $\mathbf{Z}^d$ satisfying

$$\phi_n(x) = L^{-d} \sum_{y \in B_L(x)} \phi_{n+1}(y), \tag{1.7}$$

with

$$B_L(x) = \{y \in \mathbf{Z}^d | -\frac{L}{2} \leq y_i - Lx_i < \frac{L}{2}, i = 1, ..., d\}, \tag{1.8}$$

we can put

$$\varphi_n(\underline{x}) = \phi_n(L^n \underline{x}) \qquad (\underline{x} \in L^{-n} \mathbf{Z}^d). \tag{1.9}$$

We shall obtain ϕ_n by averaging a field ϕ on $\mathbf{Z}^d$ with a restricted number of parameters that can vary with n. The eventual continuum field then also depends on a finite number of parameters and thus, in principle, has predictive power. We define the averaging procedure by

$$(M\phi)_x = L^{-d+\sigma} \sum_{y \in B_L(x)} \phi_y, \tag{1.10}$$

where σ is an adjustable parameter, to be fixed later.
We denote the original field ϕ, but with parameters depending on m, by $\phi_{(m)}$. The fields ϕ_n can be defined by

$$(\phi_n)_x = L^{n\sigma} \lim_{m \to \infty} (M^{m-n} \phi_{(m)}). \tag{1.11}$$

One easily checks that these fields satisfy (1.7). Notice also that the limit depends on σ ,so that this is not a superfluous parameter. In fact only one particular choice of σ leads to a non-trivial limit.

2. The one-dimensional Euclidean free field.

Let us now illustrate the above procedure by a simple example : a one-dimensional Euclidean free field. We can define a Gaussian measure γ_M on $\mathcal{S}'(\mathbf{Z})$ by its covariance

$$C_{xy} = \int \delta_x . \delta_y \; d\gamma, \tag{2.1}$$

where $\delta_x(\phi) = \phi(x)$. We put

$$C_{xy} = (-\Delta_1 + M^2)^{-1}_{xy}, \tag{2.2}$$

where Δ_1 is the lattice Laplacian,

$$(\Delta_1 f)(x) = f(x+1) + f(x-1) - 2f(x). \tag{2.3}$$

By Fourier transformation we find

$$\begin{aligned} C_{xy} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ip(x-y)}}{4\sin^2\frac{p}{2} + M^2}dp \\ &= \frac{e^{-\omega|x-y|}}{2M\sqrt{1+(M/2)^2}} \qquad if \quad x \neq y, \end{aligned} \tag{2.4}$$

with

$$\omega = 2 \ \text{arsinh}\frac{M}{2}. \tag{2.5}$$

The field ϕ is simply given by

$$\phi(f)(F) = \langle F, f\rangle \quad ; f \in \mathcal{S}(\mathbf{Z}), F \in \mathcal{S}'(\mathbf{Z}), \tag{2.6}$$

where $\langle .,. \rangle$ are the duality brackets.
Heuristically,

$$\gamma(d\phi) = \frac{1}{Z}\exp[-\frac{1}{2}\langle\phi, (-\Delta_1 + M^2)\phi\rangle] \prod_{x\in\mathbf{Z}} d\phi_x. \tag{2.7}$$

(Because of the simple defining relation for the field we write ϕ instead of F with an abuse of notation). Z is a normalization factor.

Let us calculate the covariance C' of the renormalized field $\phi' = M\phi$.

$$\begin{aligned} C'_{xy} &= \int \phi'_x.\phi'_y \ \gamma(d\phi) \\ &= L^{-2+2\sigma} \sum_{u\in B_L(x)} \sum_{v\in B_L(y)} C_{uv} \\ &= L^{-2+2\sigma}\frac{e^{-\omega L|x-y|}}{2M\sqrt{1+(M/2)^2}}\left\{\frac{\sinh\frac{\omega L}{2}}{\sinh\frac{\omega}{2}}\right\}^2 \qquad (x \neq y). \end{aligned} \tag{2.8}$$

Clearly $(M_L)^n = M_{L^n}$, so that

$$C^{(n)}_{xy} = L^{-2n+2n\sigma}\frac{e^{-\omega L^n|x-y|}}{2M\sqrt{1+(M/2)^2}}\left\{\frac{\sinh\frac{\omega L^n}{2}}{\sinh\frac{\omega}{2}}\right\}^2. \tag{2.9}$$

Obviously $C^{(n)}_{\underline{x}\underline{y}} \to 0$ unless we let $\omega = \omega_n$ depend on n so that $\omega_n L^n \to const.$ We shall take $M_n = L^{-n} M_0$, so that, by (2.5), $\omega_n L^n \to M_0$. Formula (2.9) then contains a factor $L^{-2n+2n\sigma+n+2n}$,so that we have to take $\sigma = -\frac{1}{2}$.
The covariance C_n of the field ϕ_n defined by (1.11) thus becomes

$$\begin{aligned} C_{n;xy} &= L^{2n\sigma} \lim_{m\to\infty} \left(C^{(m-n)}_{M_n} \right)_{xy} \\ &= \frac{e^{-M_0 L^{-n}|x-y|}}{2M_0} \left\{ \frac{\sinh \frac{1}{2} M_0 L^{-n}}{\frac{1}{2} M_0} \right\}^2 . \end{aligned} \tag{2.10}$$

Rescaling we find the covariance of the fields φ_n on $L^{-n}\mathbf{Z}$,

$$\int \varphi_n(\underline{x})\varphi_n(\underline{y})\gamma(d\varphi_n) = \frac{e^{-M_0|\underline{x}-\underline{y}|}}{2M_0} \left\{ \frac{\sinh \frac{1}{2} M_0 L^{-n}}{\frac{1}{2} M_0} \right\}^2 . \tag{2.11}$$

Taking the limit $n \to \infty$ of

$$\int \varphi_n(f)\varphi_n(g)\gamma_n(d\varphi_n) = \sum_{\underline{x},\underline{y}\in L^{-n}\mathbf{Z}} L^{-2n} f(\underline{x})g(\underline{y}) \int \varphi_n(\underline{x})\varphi_n(\underline{y})\gamma_n(d\varphi_n)$$

we find the covariance of the continuum field,

$$\mathcal{C}(f,g) = \int_{\mathbf{R}} d\underline{x} \int_{\mathbf{R}} d\underline{y} f(\underline{x})g(\underline{y}) \frac{e^{-M_0|\underline{x}-\underline{y}|}}{2M_0}. \tag{2.12}$$

The kernel

$$\mathcal{C}(\underline{x},\underline{y}) = \frac{e^{-M_0|\underline{x}-\underline{y}|}}{2M_0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(\underline{x}-\underline{y})}}{p^2 + M_0^2} dp \tag{2.13}$$

is the usual Euclidean free field "propagator" or Green's function.

In d dimensions one finds $\sigma = \frac{d-2}{2}$. This is called the canonical dimension of the scalar field φ. If one considers fields with (self-)interaction it may be necessary to change σ. One then speaks of an anomalous dimension.

All this may seem like a complicated way of replacing Δ_1 by

$$\Delta_n f(\underline{x}) = L^{2n}[f(\underline{x} + L^{-n}) + f(\underline{x} - L^{-n}) - 2f(\underline{x})]$$

and taking the limit

$$\sum_{\underline{x}\in L^{-n}\mathbf{Z}^d} L^{-n} f(\underline{x})((-\Delta_n + M_0^2).g)(\underline{x}) \longrightarrow \int d\underline{x} \quad f(\underline{x})((-\Delta + M_0^2).g)(\underline{x}).$$

For models with interaction, however, i.e. models with non- quadratic terms in the Hamiltonian or in other words a non- Gaussian measure μ instead of γ , serious singularities appear. By the renormalization group method described above these singularities are broken up into contributions from different scales, which all have approximately the same form. We illustrate this by a hierarchical model in which the massless quadratic part of the Hamiltonian is replaced by a hierarchical analogue.

3. The hierarchical model.

We replace the massless quadratic part $-\Delta_1$ of the Hamiltonian by the expression

$$H_0 = \sum_{k=0}^{\infty} L^{-(2+d)k} \Gamma_k \tag{3.1}$$

The kernels Γ_k are defined by

$$\Gamma_k(x,y) = \Gamma(x^{(k)}, y^{(k)}) \tag{3.2}$$

with

$$\begin{aligned} \Gamma(x,y) &= 1 - L^{-d} && \text{if } x = y \\ &= -L^{-d} && \text{if } x \neq y \text{ but } x^{(1)} = y^{(1)} \\ &= 0 && \text{if } x^{(1)} \neq y^{(1)} \end{aligned} \tag{3.3}$$

$x^{(1)}$ is the label of the block containing x , i.e. $x \in B_L(x^{(1)})$, and more generally, $x^{(k+1)} \in B_L(x^{(k)})$.
The corresponding covariance is

$$C^H_{xy} = \sum_{k=0}^{\infty} L^{-2\sigma k} \Gamma(x^{(k)}, y^{(k)}) \tag{3.4}$$

with $\sigma = \frac{d-2}{2}$.
One easily shows that

$$\left(C^H\right)'_{xy} = L^{-2\sigma} C^H_{x^{(1)} y^{(1)}} + \Gamma(x,y). \tag{3.5}$$

Heuristically one can define a measure μ by

$$\begin{aligned} \mu(d\phi) &= \frac{1}{Z} \exp[-\frac{1}{2}\langle\phi, H_0\phi\rangle - \sum_{x\in \mathbf{Z}^d} v(\phi_x)] \prod_{x\in\mathbf{Z}^d} d\phi_x \\ &= \frac{1}{Z'} \exp[-\sum_{x\in\mathbf{Z}^d} v(\phi_x)] \gamma^H(d\phi) \end{aligned} \tag{3.6}$$

with the quartic interaction potential

$$v(\phi_x) = \frac{1}{2} r \phi_x^2 + \frac{1}{4} g \phi_x^4. \tag{3.7}$$

It can be shown that there actually exists a Gibbs measure for the Hamiltonian

$$H(\phi) = \frac{1}{2}\langle \phi, H_0 \phi \rangle + \sum_{x \in \mathbf{Z}^d} v(\phi_x) \tag{3.8}$$

if $g \geq 0$. For details see [1]. Alternatively one can work in a finite volume Λ_N but prove the convergence independently of N. (Cf.[2]).

The simplifying property of the hierarchical model is that ϕ' can be described on a measure space with a measure μ' of the same form as μ :

$$\mu'(d\phi') = \frac{1}{Z'} \exp[-\sum_{x \in \mathbf{Z}^d} v'(\phi'_x)] \gamma^H(d\phi') \tag{3.9}$$

with

$$e^{-v'(\phi'_x)} = \frac{\int \exp[-\sum_{y \in B_L(x)} v(L^{-\sigma}\phi'_x + \xi_y)] \gamma_\Gamma(d\xi)}{\int \exp[-\sum_{y \in B_L(x)} v(\xi_y)] \gamma_\Gamma(d\xi)}. \tag{3.10}$$

The integrals appearing in formula (3.10) are finite dimensional. Nevertheless this transformation is far from simple. However, one can make a Taylor expansion in g. Replacing v by the Wick-ordered expression

$$v(\phi) = \frac{1}{2} r : \phi^2 : + \frac{1}{4} g : \phi^4 : \tag{3.11}$$

and retaining terms up to second order one finds

$$v'(\phi') = \frac{1}{2} r' : \phi'^2 : + \frac{1}{4} g' : \phi'^4 : + O_3 \tag{3.12}$$

with

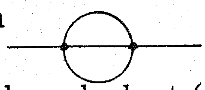
$$\begin{aligned} r' &= L^2(r - 3arg - 6cg^2) \\ g' &= L^{4-d}(g - 9ag^2) \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} a &= 1 - L^{-d} \\ c &= 3L^{2-d}(1 - L^{-d})\langle \phi^2 \rangle + (1 - L^{-d})(1 - 2L^{-d} + 2L^{-2d}) \end{aligned} \tag{3.14}$$

For $d = 3$ one finds that $v_{(m)}^{(m-n)}$ converges as $m \longrightarrow \infty$ if we put

$$\begin{aligned} r_m &= L^{-2m}(r_0 + 6mcg_0^2) \\ g_m &= L^{-m} g_0 \end{aligned} \tag{3.15}$$

Apart from the scaling factors L^{-2m} and L^{-m} there appears a non-trivial *mass-renormalization* term $6mcg_0^2$. It corresponds to the primitive divergent Feynman diagram 

A closer look at (3.13) shows that the substitution (3.15) might well be sufficient to all orders of perturbation theory. Indeed the mass-renormalization term appears because $g_m = O(L^{-m})$ while $r_m = O(L^{-2m})$. Using techniques developed by Gawedzki and Kupiainen [2] , this can actually be proved to be the case: see [1].

REFERENCES

[1]. T. C. Dorlas : On some aspects of renormalization group theory and hierarchical models, Thesis University of Groningen, 1987

[2]. K. Gawedzki & A. Kupiainen : Triviality of ϕ_4^4 and all that in a hierarchical model approximation, J.Stat.Phys. 29 (1982), 683.