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LARGE DEVIATIONS AND THE BOSON GAS

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§1 Introduction

In this lecture we review some large deviation results for probability distributions associated with the free boson gas and discuss briefly their application to models of an interacting boson gas. In §2 we describe the probabilistic setting; in §3 we review results on the free boson gas which we shall require; in §4, §5, §6 and §7 we summarize large deviation results in increasing order of sophistication; in §8 we sketch some applications.

§2 The Probabilistic Setting

Our ultimate aim is to compute thermodynamic functions for certain models of an interacting boson gas. The physical relevance of these calculations will not be discussed here; we shall concentrate on the probabilistic aspects of the investigation.

The probability space Ω on which the models are defined is the space of terminating sequences of non-negative integers: an element ω of Ω is a sequence

$$\{w(j) \in N : j = 1, 2, \ldots\}$$

satisfying $\sum_{j\geq 1} w(j) < \infty$.

The basic random variables, *the occupation numbers*, are the evaluation maps $\sigma_j : \Omega \to N$ given by

$$\sigma_j(\omega) = \omega(j) \tag{2.1}$$

The sequence $\{H_l : l = 1, 2 \ldots\}$ of free-gas hamiltonians is defined by

$$H_l(\omega) = \sum_{j \geq 1} \lambda_l(j)\sigma_j(\omega), \tag{2.2}$$

where $\{\lambda_l(j) : j = 1, 2, \ldots\}$is an ordered sequence of real numbers associated with a region Λ_l of some Euclidean space R^d:

$$0 = \lambda_l(1) \leq \lambda_l(2) \leq \ldots \tag{2.3}$$

The total number of particles $N(\omega)$ is defined by

$$N(\omega) = \sum_{j \geq 1} \sigma_j(\omega). \tag{2.4}$$

As in §2 of [1], we are in a position to define for $\mu < 0$, the grand canonical measure $P_l^\mu[\,\cdot\,]$ on Ω and the grand canonical pressure $p_l(\mu)$:

$$P_l^\mu[\omega] = \frac{e^{\beta(\mu N(\omega) - H_l(\omega))}}{e^{\beta V_l p_l(\mu)}} \tag{2.5}$$

where

$$p_l(\mu) = (\beta V_l)^{-1} \ln\Big(\sum_{\omega \epsilon \Omega} e^{\beta(\mu N(\omega) - H_l(\omega))}\Big). \tag{2.6}$$

Because of (2.3), both (2.5) and (2.6) hold for all $\mu < 0$. The mean particle number density $\mathbf{E}_l^\mu[X_l]$, where $X_l = N/V_l$ and $\mathbf{E}_l^\mu[\,\cdot\,]$, denotes the expectation with respect to the probability measure $P_l^\mu[\,\cdot\,]$ is given by

$$\mathbf{E}_l^\mu[X_l] = p_l'(\mu). \tag{2.7}$$

Using an identity known to Euler, we have

$$\exp \beta V_l p_l(\mu) = \prod_{j \geq 1} (1 - e^{\beta(\mu - \lambda_l(j))})^{-1}, \tag{2.8}$$

so that we write

$$p_l(\mu) = V_l^{-1} \sum_{j \geq 1} p(\mu | \lambda_l(j)). \tag{2.9}$$

where the partial pressure $p(\mu|\lambda)$ is given by

$$p(\mu|\lambda) = \beta^{-1} \ln(1 - e^{\beta(\mu - \lambda)})^{-1}. \tag{2.10}$$

Lemma 1. *For each $\mu < 0$, the occupation numbers are independent, geometrically distributed random variables:*

$$P_l^\mu[\sigma_j \geq m] = e^{m\beta(\mu - \lambda_l(j))}. \tag{2.11}$$

Proof: For $\alpha_j \leq 0$, $j = 1, 2, \ldots$, we have

Silly proof.

$$\mathbf{E}_l^\mu[e^{\beta(\sum_{j\leq 1} \alpha_j \sigma_j)}] = \prod_{j\geq 1} \frac{(1 - e^{\beta(\mu - \lambda_l(j))})}{(1 - e^{\beta(\mu + \alpha_j - \lambda_l(j))})}.$$

□

It is convenient to introduce the distribution function

$$F_l(\lambda) = (V_l)^{-1} \sharp\{j : \lambda_l(j) \leq \lambda\}; \tag{2.12}$$

with respect to this, (2.9) can be rewritten as

$$p_l(\mu) = \int_{[0,\infty)} p(\mu|\lambda) dF_l(\lambda); \tag{2.13}$$

the mean particle density is given by

$$\mathbf{E}_l^\mu[X_l] = \int_{[0,\infty)} p'(\mu|\lambda) dF_l(\lambda). \tag{2.14}$$

We note that, for each l, $\mu \mapsto p_l(\mu)$ is a convex function defined on $(-\infty, 0)$; we define

$$p_l(0) = \lim_{\mu \uparrow 0} p_l(\mu) = +\infty \tag{2.15}$$

and

$$p_l(\mu) = +\infty, \quad \mu > 0. \tag{2.16}$$

Then each p_l is a closed convex function defined on the whole of R; its essential domain is

$$\text{dom} \;\; p_l = (-\infty, 0).$$

In order to prove the existence of the pressure in the thermodynamic limit, it is necessary to make some assumptions about the $\lambda_l(j)$ and the V_l; putting $\phi_l(\beta) = \int_{[0,\infty)} e^{-\beta\lambda} dF_l(\lambda)$, we formulate conditions:

(S1)

$$\phi(\beta) = \lim_{l \to \infty} \phi_l(\beta)$$

exists for all β in $(0, \infty)$

(S2) *$\phi(\beta)$ is non-zero for at least one value of $\beta \in (0, \infty)$.*

These conditions are weak restrictions on the sequences; their verification in a particular instance can involve some hard analysis.

§3 Results Concerning the Free Boson Gas

In this section we review some results on the general theory of the free boson gas; the proofs can be found in [2].

Proposition 1. *Suppose that (S1) and (S2) hold; then the following limits exist.*

(1) $$p(\mu) = \lim_{l\to\infty} p_l(\mu), \qquad \mu < 0,$$

(2) $$F(\lambda) = \lim_{l\to\infty} F_l(\lambda).$$

They are related by

$$p(\mu) = \int_{[0,\infty)} p(\mu|\lambda)dF(\lambda).$$

Moreover, we have

$$p'(\mu) = \int_{[0,\infty)} p'(\mu|\lambda)dF(\lambda).$$

The standard example is the following one: let $h_l = -\frac{1}{2}\Delta$ in Λ_l with Dirichlet conditions on $\partial\Lambda_l$ where $\{\Lambda_l : l = 1,2...\}$ is a sequence of dilations of a convex region in R^d which eventually fills out the whole of R^d; let $\varepsilon_l(1) = \varepsilon_l(2) \leq ...$ be the eigenvalues of h_l and put $\lambda_l(j) = \varepsilon_l(j) - \varepsilon_l(1)$; then $(S1)$ and $(S2)$ hold and $F(\lambda) = C_d\lambda^{d/2}$.

Next we define the critical density ρ_c:

if $\lambda \to p'(0\ |\lambda)$*is integrable on* $[0,\infty)$ *with respect to* F, *put*

$$\rho_c = \int_{[0,\infty)} p'(0\ |\lambda)dF(\lambda); \tag{3.1}$$

put $\rho_c = \infty$ *otherwise.*

It follows from the dominated convergence principle that if ρ_c is finite then

$$\rho_c = \lim_{\mu\uparrow 0}\int_{[0,\infty)} p'(\mu|\lambda)dF(\lambda) = \lim_{\epsilon\downarrow 0}\int_{[\epsilon,\infty)} p'(0|\lambda)dF(\lambda). \tag{3.2}$$

Clearly, if $F(\lambda) \sim \lambda^\sigma$ with $\sigma > 1$ then ρ_c is finite; if ρ_c is finite then $F(\lambda) \to 0$ as $F \downarrow 0$. (In fact, we have the more precise estimate: for $\varepsilon > 0, F(\varepsilon) < \beta\varepsilon e^{\beta\varepsilon}\rho_c$). Note that in the standard example, ρ_c is finite if and only if $d > 2$.

Again it is convenient to follow the standard conventions for convex functions in extending p to the whole of R: we define $p(0)$ by $p(0) = \lim_{\mu\uparrow 0} p(\mu)$ and put $p(\mu) = +\infty,\ \ \mu > 0$. Since p is convex and differentiable for $\mu < 0$, $p'_-(0) = \lim_{\mu\uparrow 0} = \rho_c$. Define

$p'_+(0)$ to be $+\infty$ and $p'_-(\mu) = p'_+(\mu) = +\infty$ for $\mu > 0$. Then p is a closed convex function on the whole of $\mathbf{R}$.

The sub-differential ∂p is given by

$$(\partial p)(\mu) = \begin{cases} p'(\mu), & \mu < 0; \\ [\rho_c, \infty), & \mu = 0. \end{cases} \tag{3.3}$$

For fixed l, the function $\mu \mapsto p'_l(\mu)$ is strictly increasing on $(-\infty, 0)$ and $p'_l(\mu) \to 0$ as $\mu \to -\infty$ while $p'_l(\mu) \to \infty$ as $\mu \to 0$ since $\lambda_l(1) = 0$. It follows that the equation

$$p'_l(\mu) = \rho \tag{3.4}$$

has a unique solution $\mu_l(\rho)$ in $(-\infty, 0)$, for each ρ in $(0, \infty)$. On the other hand, for $\rho_c < \infty$, the function $\mu \mapsto p'(\mu)$ increases from zero to ρ_c as μ ranges through $(-\infty, 0)$. It is convenient to define $\mu(\rho)$ for ρ in $(0, \infty)$ to be the unique root of

$$p'(\mu) = \rho \tag{3.5}$$

if $\rho < \rho_c$ and to be zero if $\rho \geq \rho_c$.

Defining

$$\pi_l(\rho) = (p_l \circ \mu_l)(\rho),$$

so that $\pi_l(\rho)$ is the pressure at mean density ρ and $= (p \circ \mu)(\rho)$, we have

Proposition 2. *Suppose that* $(S1)$ *and* $(S2)$ *hold; then*

(1) $$\lim_{l \to \infty} \mu_l(\rho) = \mu(\rho),$$

(2) $$\lim_{l \to \infty} \pi_l(\rho) = \pi(\rho),$$

(3) $$f(x) \equiv \sup(\mu x - \rho(\mu)) = x\mu(x) - \pi(x).$$

Thus we have a first-order phase-transition when ρ_c is finite; the first-order phase-transition segment is $[\rho_c, \infty)$.

§4 Large Deviations of the Particle Number Density

Let $K_l^\mu = P_l^\mu \circ X_l^{-1}$ be the distribution function of the particle number density $X_l = N/V_l$. It follows from Theorem 1 of [1] that, for $\mu < 0$, $\{K_l^\mu\}$ converges weakly to the degenerate distribution δ_ρ concentrated at $\rho = p'(\mu)$. It follows from Theorem 2 of [1] that the Large Deviation upperbound (LD3) holds for $\mu < 0$ with rate-function $I^\mu(\,\cdot\,)$ given by

$$I^\mu(x) = p(\mu) + f(x) - \mu x. \tag{4.1}$$

However, the existence of the pressure is not sufficient to ensure that the Large Deviation lowerbound holds for an arbitrary open subset of $[0,\infty)$ when ρ_c is finite; although $\operatorname{ran} \partial p = [0,\infty)$, the existence of the first-order phase-transition segment $[\rho_c,\infty)$ prevents an application of Theorem 3 of [1] to the whole of $[0,\infty)$. Nevertheless, as we shall see, special features of the free boson gas enable establish the Large Deviation lowerbound (LD4).

Theorem 1. *Suppose that (S1) and (S2) hold; then, for $\mu < 0$, the sequence*

$$\{I_l^\mu = P_l^\mu \circ X_l^{-1} : l = 1,2,\ldots\}$$

satisfies the Large Deviation Principle with constants $\{V_l : l = 1,2,\ldots\}$ and rate function $I^\mu(\,\cdot\,)$ given by

$$I^\mu(x) = \begin{cases} p(\mu) + f(x) - \mu x, & x \geq 0, \\ \infty, & x < 0. \end{cases} \tag{4.2}$$

Proof:

It was proved in §9 of [1] in this volume that (LD1), (LD2) hold and in §6 that (LD3) holds; it remains to prove that, for each open subset G of $[0,\infty)$:

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^\mu[G] \geq -\inf_G I^\mu(x). \tag{4.3}$$

Let y be an arbitrary point of G; choose $\delta > 0$ so that $B_y^\delta = (y-\delta, y+\delta) \subset G$ and t_l such that $p_l'(\mu + t_l) = y$. Then, as in §8 of [1], we have

$$K_l^\mu[G] \geq e^{\beta V_l \{p_l(\mu+t_l) - p_l(\mu) - t_l y - \delta|t_l|\}} K_l^{\mu+t_l}[B_y^\delta].$$

By Proposition 2, $\mu + t_l \to \mu(y)$ and $p_l(\mu + t_l) \to p(\mu(y))$ so that

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^\mu[G] \geq p(\mu(y)) - p(\mu) - (\mu(y) - \mu)y$$

$$-\delta|\mu(y) - \mu| + \liminf_{l\to\infty} \frac{1}{\beta V_l} ln K_l^{\mu+t_l}[B_y^\delta].$$

We now have to distinguish two cases: if $y < \rho_c$ then $\mu(y) < 0$ and we make use of the fact that $\mu \mapsto p(\mu)$ is differentiable for $\mu < 0$; then, by Theorem 1 of [1], for all l sufficiently large $K_l^{\mu+t_l}[B_y^\delta] \geq \frac{1}{2}$ and hence

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^{\mu+t_l}[B_y^\delta] = 0; \tag{4.4}$$

on the other hand, if $y \geq \rho_c$ we have $\mu(y) = 0$ and we must proceed differently.

Lemma 2. *Let N_1 and N_2 be independent non-negative integer-valued random variables with means m_1 and m_2 respectively. Suppose that N_1 is geometrically distributed and that $\delta \geq 1$; then*

$$P[N_1 + N_2 \in B^{\delta}_{m_1+m_2}] \geq \frac{1}{m_1+m_2}\left(\frac{m_1}{m_1+1}\right)^{m_1+m_2+2}. \tag{4.5}$$

Proof:

The interval $B^1_{m_1+m_2} = (m_1+m_2-1, m_1+m_2+1)$ contains a unique integer $n_0 \geq m_1+m_2$. Now

$$P[N_1+N_2 \in B^{\delta}_{m_1+m_2}] = \sum_{m \in B^{\delta}_{m_1+m_2}} \sum_{n=0}^{m} P[N_1 = m-n]P[N_2 = n]$$

$$\geq \sum_{n=0}^{n_0} P[N_1 = n_0 - n]P[n_2 = n]. \tag{4.6}$$

Since N_1 is geometrially distributed, $n \mapsto P[N_1 = n]$ is a decreasing function so that

$$\sum_{n=0}^{n_0} P[N_1 = n_0 - n]P[N_2 = n] \geq P[N_1 = n_0]P[N_2 \leq n_0] \tag{4.7}$$

Now

$$P[N_1 = n_0] = \frac{1}{m_1+1}\left(\frac{m_1}{m_1+1}\right)^{n_0}$$
$$\geq \frac{1}{m_1+1}\left(\frac{m_1}{m_1+1}\right)^{m_1+m_2+1} \tag{4.8}$$

and

$$P[N_2 \leq n_0] \geq P[N_2 \leq m_1+m_2] \geq \frac{m_1}{m_1+m_2} \tag{4.9}$$

by Markov's Inequality. Hence

$$P[N_1+N_2 \in B^{\delta}_{m_1+m_2}] \geq \frac{1}{m_1+m_2}\left(\frac{m_1}{m_1+1}\right)^{m_1+m_2+2}. \tag{4.10}$$

□

Returning to the proof of Theorem 1, it follows from Lemma 1 that σ_1 is geometrically distributed; applying Lemma 2 with $N_1 = \sigma_1$ and $N_2 = N - \sigma_1$, we have $\dfrac{m_1}{m_1+1} = e^{\beta(\mu+t_l)}$ and $m_1 + m_2 = V_l y$; thus

$$K_l^{\mu+t_l}[B_y^{\delta}] \geq \frac{1}{V_l y} e^{\beta(\mu+t_l)(V_l y+2)} \tag{4.11}$$

for $V_l \geq \frac{1}{\delta}$. It follows that

$$\liminf_{l\to\infty} \frac{l}{\beta V_l} \ln K_l^{\mu+t_l}[B_y^\delta] = 0 \tag{4.12}$$

since, for $y \geq \rho_c$, $\mu + t_l \to 0$. Thus we have, in both cases,

$$\begin{aligned}\liminf_{l\to\infty} \frac{l}{\beta V_l} \ln K_l^\mu[G] &\geq -p(\mu) - f(y) + \mu y \\ &= -I^\mu(y)\end{aligned} \tag{4.13}$$

for all y in G, since δ was arbitrary. Hence

$$\begin{aligned}\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln\ K_l^\mu[G] &\geq \sup_G(-I^\mu(y)) \\ &= -\inf_G I^\mu(y)\end{aligned} \tag{4.14}$$

□

§5 The Large Deviations of a Vector-valued Random Variable.

The Large Deviation result established in §4 enables us to apply Varadhan's Theorems to suitable functions of $X_l = N/V_l$; to deal with functions of the $m+1$ variables $\sigma_1/V_l, \ldots, \sigma_m/V_l$, N/V_l we prove a Large Deviation result for the sequence of probability distributions of a vector-valued random variable.

Define the vector-valued random variable $X_l : \Omega \to R^{m+1}$ by

$$\begin{aligned} X_l^{(1)}(\omega) &= V_l^{-1}\sigma_1(\omega), \\ &\vdots \\ X_l^{(m+1)}(\omega) &= V_l^{-1} \sum_{j>m} \sigma_j(\omega). \end{aligned}$$

In order to prove a Large Deviation result for $K_l^\mu = P_l^\mu o X_l^{-1}$, it is necessary to make a further hypothesis about the single-particle spectrum. First, we define the cumulant generating function $C_l^\mu[\,\cdot\,]$ by

$$C_l^\mu[t] = \frac{1}{\beta V_l} \ln \mathbf{E}_l^\mu[e^{\beta V_l <t, X_l>}]. \tag{5.1}$$

Lemma 3. *Suppose that (S 1) and (S 2) hold and that* $\lim_{l\to\infty} \lambda_l(j) = \lambda(j)$ *exists for* $j = 1\cdots, m+1$; *then the cumulant generating function*

$$C^\mu[t] = \lim_{l\to\infty} C_l^\mu[t]$$

exists for all t in R^{m+1} *and is given by*

$$C^{\mu}[t] = \begin{cases} p(\mu + t_{m+1} - \lambda(m+1)) - p(\mu), & t \in \mathcal{D}_{\mu}; \\ \infty, & \textit{otherwise}. \end{cases} \tag{5.2}$$

where

$$\mathcal{D}_{\mu} = \{t : t_j + \lambda(j) < -\alpha, j = 1 \cdots, m+1\} \tag{5.3}$$

Proof:

Put

$$p_l^{(j)}(\mu) = \frac{-1}{\beta V_l} \ln(1 - e^{\beta(\mu - \lambda_l(k))}) \tag{5.4}$$

for $1 \leq j \leq m$ and put

$$p_l^{(m+1)}(\mu) = \frac{-1}{\beta V_l} \sum_{j>m} \ln(1 - e^{\beta(\mu - \lambda_l(j))}). \tag{5.5}$$

Since $\lambda_l(j) \to \lambda(j)$ as $l \to \infty$, $p_l^{(j)}(\mu + t_k)$ is defined, for all l sufficiently large, for $\mu < \lambda(j) - t_j$. On the set $\mathcal{D}_{\mu}$, we have, by Proposition 1,

$$\lim_{l \to \infty} p_l^{(m+1)}(\mu + t_{m+1}) \to p(\mu + t_{m+1} - \lambda(m+1)), \tag{5.6}$$

while for $l \leq j \leq m$,

$$\lim_{l \to \infty} p_l^{(j)}(\mu + t_j) \to 0. \tag{5.7}$$

It follows that

$$\lim_{l \to \infty} C_l^{\mu}[t] = p(\mu + t_{m+1} - \lambda(m+1)) - p(\mu) \quad , \quad t \in \mathcal{D}_{\mu}; \tag{5.8}$$

put $C^{\mu}[t] = \infty$ for t in the complement of $\mathcal{D}_{\mu}$. Then $t \mapsto C^{\mu}[t]$ is a closed proper convex function on R^{m+1} with dom $C^{\mu} = \mathcal{D}_{\mu}$; put

$$I^{\mu}[x] = \sup_{t \in R^{m+1}} \{< x, t > -C^{\mu}[t]\}. \tag{5.9}$$

Theorem 2. *Suppose that (S1) and (S2) hold and that* $\lim_{l\to\infty} \lambda_l(j) = \lambda(j)$ *exists for* $l \leq j \leq m+1$; *then, for* $\mu < 0$, *the sequence*

$$\{K_l^{\mu} = P_l^{\mu} \circ X_l^{-1} : l = 1, 2, \cdots\}$$

satisfies the Large Deviation Principle with constants $\{V_l : l = 1, 2, \cdots\}$ *and rate-function* $I^{\mu}[\,\cdot\,]$.

Proof.

The proof that (LD1) and (LD2) holds follows, as in §9 of [1], by the fact that $I^{\mu}[\,\cdot\,]$ is the Legendre transform of $C^{\mu}\,[\,\cdot\,]$. to prove that (LD3) holds, we follow Ellis [3] and adapt to our situation Gartners's Lemma:

Let K be a non-empty closed subset of R^{m+1} define $I_\mu[K] = \inf_K I^\mu[x]$. If $0 < I^\mu[K] < \infty$ then there exists a finite set $\tau^{(1)}, \ldots, \tau^{(r)}$ *of non-zero vectors in* R^{m+1} *such that, for* $c = I^\mu[K] - \epsilon$, $\epsilon > 0$,

$$K \subset \cup_{j=1}^{r} H_+^\mu(\tau^{(j)}; c), \tag{5.10}$$

where $H_+^\mu(\tau; c) = \{x :< x, \tau > -C^\mu[t] \geq c\}$ *if* $I^\mu[K] = +\infty$ *then, for each* $R > 0$, *there exists a finite set* $\tau^{(j)}, \cdots, \tau^{(r)}$ *of non-zero vectors in* R *such that*

$$K \subset \cup_{j=1}^{r} H_+^\mu(\tau^{(j)}; R). \tag{5.11}$$

First suppose that K is such that $0 < I^\mu[K] < \infty$; then

$$\begin{aligned} K_l^\mu[K] &\leq \sum_{j=1}^{r} K_l^\mu[H_+^\mu(\tau^{(j)}; c)] \\ &= \sum_{j=1}^{r} K_l^\mu[\{x :< x, \tau^{(j)} > \quad \geq \quad c^\mu[\tau^{(j)}] + c\}]. \end{aligned} \tag{5.12}$$

But by Markov's Inequality,

$$\begin{aligned} K_l^\mu[\{x :< x, \tau^{(j)} > \geq C^\mu[\tau^{(j)}] + c\}] &\leq e^{-\beta V_l\{c^\mu[\tau^{(j)}]+c\}} \int_{R^{m+1}} e^{\beta V_l < x, \tau^{(j)} >} K_l^\mu[dx] \\ &= e^{-\beta V_l\{C^\mu[\tau^{(j)}]+c-c_l^\alpha[\tau^{(j)}]\}} \end{aligned} \tag{5.13}$$

hence

$$\limsup_{l \to \infty} \frac{1}{\beta V_l} \ln K_l^\mu[K] \leq -I^\mu[K] \tag{5.14}$$

since $C_l^\mu[t] \to C^\mu[t]$ and $\epsilon > 0$ was arbitrary. Now suppose that $I^\mu[K] = +\infty$; then

$$\limsup_{l \to \infty} \frac{1}{\beta V_l} \ln K_l^\mu[K] \leq -R$$

for each $R > 0$ and the result follows. To prove that (LD4) holds for an arbitrary open set G of R^{m+1}, let y be an arbitrary point of G and choose $\delta > 0$ such that

$$\prod_{j=1}^{m+1} (y_j - \delta, y_j + \delta) \subset G. \tag{5.15}$$

Then

$$K_l^\mu[G] \geq \prod_{j=1}^{m+1} K_l^{(j),\mu}[(y_j - \delta, y_j + \delta)] \tag{5.16}$$

where $K_l^{(j),\mu}$ is determined by

$$\int_{[0,\infty)} e^{\beta V_l t_j x} K_l^{(j),\mu}[dx] = e^{\beta V_l \{p_l^{(j)}(\mu+t_j)-p_l^{(j)}(\mu)\}}. \tag{5.17}$$

Now

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^{(m+1),\mu}[(y_{m+1}-\delta, y_{m+1}+\delta)] \geq -I^{(m+1),\mu}(y_o), \tag{5.18}$$

where

$$I^{(m+1),\mu}(x) = \begin{cases} p(\mu)+f(x)-(\mu-\lambda(m+1))x, & x \geq 0; \\ \infty, & x < 0, \end{cases} \tag{5.19}$$

by the reasoning which established Theorem 1. For $1 \leq j \leq m$,

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^{(j),\mu}[(y_j-\delta, y_j+\delta)] \geq -I^{(j),\mu}(y_j) \tag{5.20}$$

by direct calculation, where

$$I^{(j),\mu}(x) = \begin{cases} -(\mu-\lambda(j))x, & x \geq 0, \\ \infty, & x < 0, \end{cases} \tag{5.21}$$

since σ_j is geometrically distributed (Lemma 1). Hence

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^{\mu}[G] \geq -\sum_{j=1}^{m+1} I^{(j),\mu}(y_j) = -I^{\mu}(y) \tag{5.22}$$

and since y was an arbitrary point of G

$$\liminf_{l\to\infty} \frac{1}{\beta V_l} \ln K_l^{\mu}[G] \geq \sup_G(-I^{\mu}(y)) = -\inf_G I^{\mu}(y). \tag{5.23}$$

□

§6 A Large Deviation Result for a Banach Space-valued Random Variable

Let $X_l : \Omega \longrightarrow l^1_+$ be defined by

$$X_l^{(0)}(\omega) = V_l^{-1}N(\omega), \qquad X_l^{(j)}(\omega) = V_l^{-1}\sigma_j(\omega), \qquad j \geq 1;$$

then$K_l^{\mu} = P_l^{\mu} \circ X_l^{-1}$ is a probability measure on $l^1_+ = \{x_j \geq 0 : \sum_{j\geq 0} x_j < \infty\}$. We regard l^1_+ as the positive cone of the real Banach space l^1 ; equipped with the norm topology, l^1 is a complete separable metric space (a Polish space). However, for our purposes, the weak$\star$-topology on l^1 (the $\sigma(l^1, c_0)$ topology induced by the space c_0 of real sequences converging to zero) is the appropriate one for our purposes. The space l^1 equipped with the $\sigma(l^1, c_o)$ topology is not metrizable; nevertheless, the theory of large deviations is still applicable since the σ-field of Borel subsets of l^1 is the same in both the norm topology and in the $\sigma(l^1, c_0)$topology (see Azencott [4] for a full discussion of this point and Yamasaki [5] for the measure theory).

Notice also that each of the measures K_l^μ is supported on the convex set $\{x \in l_+^1 : x_o = \sum_{j\geq 1} x_j\}$ since $N(\omega) = \sum_{j\geq 1} \sigma_j(\omega)$.

The proofs of the results in this section are more technical and we will not give them here.

Lemma 4. *Suppose that (S1) and (S2) hold and that* $\lim_{l\to\infty} \lambda_l(j) = 0$ *for* $j = 1, 2, \ldots$ *then, for* $\mu < 0$, *we have for each* t *in* c_0

$$C^\mu[t] = \lim_{l\to\infty} C_l^\mu[t] = \begin{cases} p(\mu + t_o) - p(\mu), & t \in \mathcal{D}_\mu, \\ \infty, & otherwise, \end{cases} \tag{6.1}$$

where

$$C_l^\mu[t] = \frac{1}{\beta V_l} \ln \mathbf{E}^{\mu}_l[e^{\beta\{t_o N + t_1\sigma_1 + t_2\sigma_2 + \cdots\}}]$$

and

$$\mathcal{D}_\mu = \{t \in c_o : t_o + \mu < 0, t_o + \sup_{j\geq 1} t_j + \alpha < 0\}. \tag{6.2}$$

Let $I^\mu[x] = \sup_{t\epsilon c_o}\{< x, t > -C^\mu[t]\}$; then a straightforward caculation yields

$$I^\mu[x] = \begin{cases} p(\mu) + f(x_o - \sum_{j\geq 1} x_j) - \mu x_o, & x \in \mathcal{D}_\mu^*, \\ \infty, & \text{otherwise}, \end{cases} \tag{6.3}$$

where

$$\mathcal{D}_\mu^* = \{x \in l_+^1 : x_o \geq \sum_{j\geq 1} x_j\}. \tag{6.4}$$

Theorem 3. *Suppose that (S1) and (S2) hold and that* $\lim_{l\to\infty} \lambda_l(j) = 0$ *for* $j = 1, 2, \ldots$; *then, for* $\mu < 0$, *the sequence*

$$\{K_l^\mu = P_l^\mu \circ X_l^{-1} : l = 1, 2, \ldots\}$$

of probability measures on l_+^1 *satisfies the Large Deviation Principle with constants* $\{V_l\}$ *and rate-function* $I^\mu[\,\cdot\,]$.

§7. A Large Deviation Result for the Occupation Measure

We introduce a measure-valued random variable

$$L_l(\omega; B) = \frac{1}{V_l}\sum_{j\geq 1}\sigma_j(\omega)\delta_{\lambda_l(j)}[B]$$

, where $\delta_\lambda[B] = 1$ if λ is in B and is zero otherwise. Then L_l maps Ω into the space $E = M_b^+(R^+)$ of positive bounded measures on the positive real line. Let $K_l^\mu = P_l^\mu \circ L_l^{-1}$ be the induced probability measure on E; in terms of this we can express the expectation of a functional of L_l as an integral over E. For example,

$$E_l^\mu[e^{-\beta a N^2/2V_l}] = \int_E e^{\beta V_l G(m)} K_l^\mu[dm]$$

where $G(m) = -\frac{a}{2}\|m\|^2$ and $\|m\| = \int_{[0,\infty)} m(d\lambda)$. But even in this simplest of examples there is a difficulty in applying Varadhan's Theorem (supposing that we have established a Large Deviation result for $\{K_l^\mu\}$. It is this: in order to prove a Large Deviation result, we have to make use of the weak$\star$-topology on E determined by $C_o(R^+)$, the continuous functions vanishing at infinity; but the function $m \mapsto \|m\|$ is not continuous in this topology and Varadhan's Theorem does not apply. We get around this difficulty as follows: we introduce a *cut-off* T and prove a Large Deviation result for $K_l^\mu = P_l^\mu \circ L_l^{-1}$ where now

$$L_l(\omega; B) = V_l^{-1}\sum_{\{j:\lambda_j(j)\leq T\}}\sigma_j(\omega)\delta_{\lambda_l(j)}[B]; \tag{7.1}$$

then we prove an estimate for $P_l^\mu[X_l^T \geq \delta]$ where

$$X_l^T(\omega) = V_l^{-1}\sum_{\{j:\lambda_l(j)>T\}}\sigma_j(\omega). \tag{7.2}$$

We state these results without proof:

Theorem 4. *Suppose that (S1) and (S2) hold; then, for* $\mu < 0$, *the sequence* $\{K_l^\mu = P_l^\mu \circ L_l^{-1}\}$ *of probability measures on* $M_b^+([O,T])$ *satisfies the Large Deviation Principle with constants* $\{V_l\}$ *and rate-function* $I^\mu[\,\cdot\,]$, *given by*

$$I^\mu[m] = \sup_{C([O,T])}\{< m, t > -C^\mu[t]\} \tag{7.3}$$

where

$$C^\mu[t] = \begin{cases} \int_{[O,T]}\{p(\mu + t(\lambda)|\lambda) - p(\mu|x)\}dF(\lambda), & \sup_{[O,T]}\{t(\lambda)-\lambda\} < -\mu, \\ \infty, & \textit{otherwise.}\end{cases} \tag{7.4}$$

Lemma 5. *For $\delta > 0$ and T such that*

$$\int_{[T,\infty)} p'(\mu|\lambda)dF(\lambda) < \frac{\delta}{2},$$

we have

$$\liminf_{l\to\infty} P_l^\mu[X_l^t \leq \delta] \geq 1 - e^{-\delta/2} \tag{7.5}$$

§8 Applications

In this section, we sketch some applications to the statistical mechanics of models of the interacting boson gas. In [6] in this volume, we used Theorem 1 of §4 to prove the existence of the pressure in the mean-field model.

In the same way, Theorem 2 of §5 has been used to prove the existence of the pressure in the Huang-Yang-Luttinger model; details will be found in [7]. Let $H_l(\cdot)$ be the hamiltonian of the free boson gas in the region Λ_l; the m-level H-Y-L model has hamiltonian

$$H_l^{(m)}(\omega) = H_l(\omega) + \frac{a}{2V_l}\{2N(\omega)^2 - \sum_{j=1}^{m}\sigma_j(\omega)^2\} \tag{8.1}$$

with $a > 0$. It was introduced in [8] and discussed also by Thouless [9]. Using Varadhan's Theorem and Theorem 2 we can prove

Theorem 5. *Suppose that (S1) and (S2) hold and that* $\inf\{\lambda : F(\lambda) > 0\} = 0$. *Then the pressure*

$$p^{(m)}(\mu) = \lim_{l\to\infty} p_l^{(m)}(\mu)$$

in the H-Y-L model with hamiltonian (8.1) exists for all real values of μ and is given by

$$p^{(m)}(\mu) = \sup_{\{0\leq x_1\leq x_0\}}\{\mu x_0 - f(x_0 - x_1) - \frac{a}{2}(2x_0^2 - x_1^2)\}. \tag{8.2}$$

Remarks:

(1) The results is independent of m for $m \geq 1$, so that it is reasonable to conjecture that the same result holds for the pressure p_l in the H-Y-L model with hamiltonian $H_l^{(\infty)}$; we hoped to prove this using Theorem 3 of §6, but, so far, technical difficulties have prevented us.

(2) No explicit assumption is made concerning the existence of $\lim_{l\to\infty}\lambda_l(j)$, while the $\lambda(j) = \lim_{l\to\infty}\lambda_l(j)$, $j = 1,\ldots,m+1$, occur explicitly in the statement of Theorem 2. The reason is that $\inf\{\lambda : F(\lambda) > 0\} = 0$ implies that $\lambda(j) = 0$ for $j = 1, 2, \ldots$.

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are equal.

The H-Y-L model is a special case of the diagonal model [9] for which the hamiltonian is

$$H_l^D(\omega) = H_l(\omega) + \frac{a}{2V_l}\{2N(\omega)^2 - \sum_{j=1}^{\infty}\sigma_j(\omega)^2\}$$

$$+\frac{1}{2V_l}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u(\lambda_l(i), \lambda_l(j))\sigma_i(\omega)\sigma_j(\omega). \tag{8.3}$$

The last two terms in this hamiltonian have different asymptotic behaviour for large l. To understand the effect of each of these two terms we study them separately. Therefore we consider the regularized hamiltonian:

$$H_l^R(\omega) = H_l(\omega) + \frac{1}{2V_l}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v(\lambda_l(i), \lambda_l(j))\sigma_i(\omega)\sigma_j(\omega). \tag{8.4}$$

If we assume that $v : R \to R$ is continuous, bounded and positive then we can use Theorem 4 and Lemma 5 of §7 to obtain the following result which is proved in [10].

Theorem 6. *Suppose that (S1)and (S2) hold; then the pressure $p^R(\mu)$ corresponding to the sequence of hamiltonians $\{H_L^R\}$ is given by*

$$p^R(\mu) = \sup_{m\in M_b^+(R+)}\{\mu\|m\| - f^R[m]\} \tag{8.5}$$

where

$$f^R[m] = \int_{[0,\infty)}\lambda m(d\lambda) + \frac{1}{2}\int_{[0,\infty)} m(d\lambda)\int_{[0,\infty)} m(d\lambda')v(\lambda,\lambda')$$

$$-\beta^{-1}\int_{[0,\infty)} s(\rho(\lambda))dF(\lambda) \tag{8.6}$$

and

$$s(x) = (1+x)\ln(l+x) - x\ln x; \tag{8.7}$$

here

$$m(d\lambda) = m_s(d\lambda) + \rho(\lambda)dF(\lambda) \tag{8.8}$$

is the Lebesgue decomposition of m with respect to $dF(\lambda)$.

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