LIMIT THEOREMS FOR STOCHASTIC PROCESSES ASSOCIATED WITH A BOSON GAS

M. van den Berg
Department of Mathematics, Heriot-Watt University
Riccarton, Edinburgh EH14 4AS, United Kingdom

J.T. Lewis
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Republic of Ireland

§1 INTRODUCTION

In this lecture, we discuss the density of particles having energy less than λ in a boson system as a stochastic process indexed by λ. The notation is that of [1] in this volume. Recall that the hamiltonian for the free boson gas is given by

$$H_\ell(\omega) = \sum_{j\geq 1} \lambda_\ell(j)\sigma_j(\omega), \tag{1.1}$$

where $0 = \lambda_\ell(1) \leq \lambda_\ell(2) \leq \ldots$. For a system in a region of volume V_ℓ, the grand canonical pressure $p_\ell(\mu)$ is defined for $\mu < 0$ by

$$p_\ell(\mu) = \frac{1}{\beta V_\ell} \ell n\left\{\sum_{\omega\in\Omega} e^{\beta(\mu N(\omega)-H_\ell(\omega))}\right\}. \tag{1.2}$$

In [2] in this volume, we recalled results (proved in [3]) on the existence of the pressure in the thermodynamic limit:

$$p(\mu) = \lim_{\ell\to\infty} p_\ell(\mu). \tag{1.3}$$

In order to discuss the phenomenon of boson condensation, we introduced in [3] the family of random variables $\{X_\ell(\cdot;\lambda):\lambda \geq 0\}$ defined by

$$X_\ell(\omega;\lambda) = \frac{1}{V_\ell} \sum_{\{j:\lambda_\ell(j)\leq\lambda\}} \sigma_j(\omega). \tag{1.4}$$

For the free boson gas, we have the following result:

THEOREM 1

Suppose that (S1) *and* (S2) *hold; then, for* ρ_c *finite,*

$$\lim_{\lambda\downarrow 0} \lim_{\ell\to\infty} \mathbb{E}_\ell^\rho [X_\ell(\lambda)] = (\rho-\rho_c)^+. \tag{1.5}$$

[Conditions (S1) and (S2) and the critical density ρ_c are defined in §2 and §3 of [2] in this volume. Here $\mathbb{E}_\ell^\rho[\cdot]$ denotes the expectation taken with respect to the grand canonical probability measure $\mathbb{P}_\ell^\mu[\cdot]$ with $\mu = \mu_\ell(\rho)$, defined in §3 of

[2]; it is the expectation at fixed mean density ρ.]

Proof:

From the definition of $X_\ell(\lambda)$, we have

$$\mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \int_{[0,\lambda)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda) = \rho - \int_{[\lambda,\infty)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda). \tag{1.6}$$

But, for $\mu < \lambda$, the sequence

$$\left\{ \int_{[\lambda,\infty)} p'(\mu|\lambda)dF_\ell(\lambda) : \ell = 1,2,\ldots \right\} \tag{1.7}$$

converges uniformly in μ on compacts to

$$\int_{[\lambda,\infty)} p'(\mu|\lambda)dF(\lambda). \tag{1.8}$$

Hence, by Proposition 2 of [2], we have for $\lambda > 0$:

$$\lim_{\ell\to\infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \rho - \int_{[\lambda,\infty)} p(\mu(\lambda)|\lambda)dF(\lambda). \tag{1.9}$$

But, by hypothesis, ρ_c is finite so that we may invoke the dominated convergence principle to conclude that

$$\lim_{\lambda\downarrow 0} \int_{[\lambda,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \int_{[0,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \begin{cases} \rho, & \rho < \rho_c, \\ \rho_c, & \rho \geq \rho_c. \end{cases}$$

Thus we have

$$\lim_{\ell\to\infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = (\rho - \rho_c)^+ \quad \blacksquare$$

In the free boson gas there is a second effect, discovered by M. Kac in 1971. We saw in §3 of [2] that the free-energy has a first-order phase-transition segment $[\rho_c,\infty)$; it follows that for $\rho > \rho_c$ there is no guarantee that the weak law of large numbers will hold for the distribution $\mathbb{K}_\ell^\rho = \mathbb{P}_\ell^\rho \circ X_\ell^{-1}$ of the number density $X_\ell = N/V_\ell$. In fact, there is no guarantee that for, $\rho > \rho_c$, the sequence $\{\mathbb{K}_\ell^\rho : \ell = 1,2,\ldots\}$ will converge; nevertheless, by the Helly Selection Principle, a subsequence will converge, but the limit distribution will depend on the detailed behaviour of the corresponding subsequence of the sequence $\{\lambda_\ell(\cdot) : \ell = 1,2,\ldots\}$. In other words, it is possible to have two sequences, $\{\lambda_\ell(\cdot) : \ell = 1,2,\ldots\}$ and $\{\hat{\lambda}_\ell(\cdot) : \ell = 1,2,\ldots\}$, each satisfying (S1) and (S2) and having the same integrated density of states $F(\cdot)$ but having limit distributions $\mathbb{K}^\rho$ and $\hat{\mathbb{K}}^\rho$ which are distinct for $\rho > \rho_C$. (For $\rho < \rho_C$, they must both be equal to δ_ρ, the degenerate distribution concentrated at ρ, by Theorem 1 of [1].) For example, Kac showed that in the standard example (described in §3 of [2]) the limit distribution is the exponential distribution supported on $[\rho_C,\infty)$ with mean ρ, for $\rho > \rho_C$; other examples are investigated in detail in [3]. We shall see in the next section that, in the mean-field model, this phenomenon disappears: there is no first-order phase-transition segment, the grand canonical pressure exists for all values of μ and is a differentiable function; the weak law of large numbers holds for X_ℓ for all values of the mean density ρ; nevertheless, condensation persists. In these circumstances it becomes interesting to regard $\lambda \longrightarrow X_\ell(\cdot;\lambda)$ as a stochastic process and to enquire about the convergence in distribution of a re-scaled, centred version of it. This we do in §3.

§2 THE MEAN FIELD-MODEL

To describe the mean-field model, we define a sequence of hamiltonians $\{\tilde{H}_\ell : \ell = 1,2,\ldots\}$ by

$$\tilde{H}_\ell(\omega) = H_\ell(\omega) + \frac{a}{2V_\ell} N^2(\omega) \tag{2.1}$$

with $a > 0$. The term $\frac{a}{2V_\ell}N^2$, which provides a crude caricature of the interaction, can be understood classically: it arises in an "index of refraction" approximation in which we imagine each particle to move through the system as if it were moving in a uniform optical medium and so receiving an increment of energy proportional to the density $X_\ell = N/V_\ell$; since a is positive the interaction is repulsive.

First, we compute the pressure $\tilde{p}_\ell(\mu)$, as explained in §4 of [1]: writing $u(x) = (\mu-\alpha)x - \frac{a}{2}x^2$, a straight-forward manipulation gives

$$\tilde{p}_\ell(\mu) = p_\ell(\alpha) + \frac{1}{\beta V_\ell}\ln \mathbb{E}_\ell^\alpha[e^{\beta V_\ell u(X_\ell)}] = p_\ell(\alpha) + \frac{1}{\beta V_\ell}\ln \int_{[0,\infty)} e^{\beta V_\ell u(x)}\mathbb{K}_\ell^\alpha[dx] \tag{2.2}$$

for each $\alpha < 0$, where $\mathbb{K}_\ell^\alpha = \mathbb{P}_\ell^\alpha \circ X_\ell^{-1}$. But $x \longrightarrow u(x)$ is continuous and bounded above and $\{\mathbb{K}_\ell^\alpha : \ell = 1,2,\ldots\}$ satisfies the Large Deviation Principle with rate-function $I^\alpha(x) = p(\alpha) + f(x) - \alpha x$, by Theorem 1 of [2]. Hence, by Varadhan's First Theorem, $\tilde{p}(\mu) = \lim_{\ell\to\infty}\tilde{p}_\ell(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_x\{u(x) - I^\alpha(x)\} = \sup_x\{\mu x - \tilde{f}(x)\}, \tag{2.3}$$

where the mean-field free-energy $\tilde{f}(\cdot)$ is given by

$$\tilde{f}(x) = f(x) + \frac{a}{2}x^2. \tag{2.4}$$

Thus we have proved:

THEOREM 2

Suppose that (S1) *and* (S2) *hold; then the mean-field pressure exists for all real* μ *and is given by*

$$\tilde{p}(\mu) = \sup_x\{\mu x - \tilde{f}(x)\}, \tag{2.5}$$

where $x \longrightarrow \tilde{f}(x)$ *is the mean-field free energy, given by* $\tilde{f}(x) = f(x) + \frac{a}{2}x^2$.

Next, we introduce the mean-field expectation functional $\tilde{\mathbb{E}}_\ell^\mu[\cdot]$ defined by

$$\tilde{\mathbb{E}}_\ell^\mu[\cdot] = \mathbb{E}_\ell^\alpha[\cdot e^{\beta M_\ell}]/\mathbb{E}_\ell^\alpha[e^{\beta M_\ell}], \tag{2.6}$$

and the associated probability measure $\tilde{\mathbb{P}}_\ell^\alpha[\cdot]$, where

$$M_\ell = V_\ell u(X_\ell). \tag{2.7}$$

COROLLARY

The mean-field pressure $\mu \longrightarrow \tilde{p}(\mu)$ *is differentiable for all values of* μ. *The sequence of distribution functions* $\{\tilde{\mathbb{K}}_\ell^\mu = \tilde{\mathbb{P}}_\ell^\mu \circ X_\ell^{-1}\}$ *converges weakly to the degenerate distribution* δ_ρ *concentrated at* $\rho = \tilde{p}'(\mu)$ *and satisfies the Large Deviation Principle with constants* $\{V_\ell\}$ *and rate-function* $\tilde{I}^\mu(x) = \tilde{p}(\mu) + \tilde{f}(x) - \mu x$.

Proof:

Since $x \longrightarrow f(x)$ is strictly convex for $0 \leq x < \rho_c$ and constant for $\rho_c \leq x < \infty$ and $x \longrightarrow \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$, the function $x \longrightarrow \tilde{f}(x) = f(x) + \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$; hence there is no first-order phase-transition segment; equivalently, $\mu \longrightarrow \tilde{p}(\mu)$, the Legendre transform of $x \longrightarrow \tilde{f}(x)$, is differentiable for $\mu < \infty$. It follows from Theorem 1 of [1] that $\tilde{\mathbb{K}}_\ell^\mu \to \delta_\rho$, where $\rho = \tilde{p}'(\mu)$, and from Theorem 4 of [1] that $\{\tilde{\mathbb{K}}_\ell^\mu : \ell = 1,2,\dots\}$ satisfies the Large Deviation Principle with constants $\{V_\ell\}$ and rate-function $\tilde{I}^\mu(\cdot)$ ■

Although the first-order phase-transition segment, which was present in the free energy function of the free-gas, has disappeared, the phenomenon of condensation persists:

THEOREM 3

Suppose that (S1) *and* (S2) *hold; then, for* ρ_C *finite, we have*

$$\lim_{\lambda\downarrow 0}\lim_{\ell\to\infty} \tilde{\mathbb{E}}^{\mu}_{\ell}[X_\ell(\lambda)] = (\rho-\rho_C)^+ , \tag{2.8}$$

where $\tilde{\mathbb{E}}^{\mu}_{\ell}[\cdot]$ *is the mean-field expectation functional and* $\rho = \tilde{p}'(\mu)$.

Proof:

First, we remark that an elementary exercise yields the following alternative formula for the mean-field pressure $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \inf_{\alpha<0}\left\{\frac{(\mu-\alpha)^2}{2a} + p(\alpha)\right\}, \tag{2.9}$$

where $p(\alpha)$ is the free-gas pressure. The idea of the proof of (2.8) is that we compute the cumulant generating function of $X_\ell(\lambda)$; since

$$V_\ell X_\ell(\lambda) = V_\ell X_\ell - \sum_{\{j:\lambda_\ell(j)>\lambda\}} \sigma_j \tag{2.10}$$

we get

$$\tilde{\mathbb{E}}^{\mu}_{\ell}\left[e^{\beta s V_\ell X_\ell(\lambda)}\right] = \tilde{\mathbb{E}}^{(s,\lambda),\mu}_{\ell}\left[e^{\beta s V_\ell X_\ell}\right],$$

where $\tilde{\mathbb{E}}^{(s,\lambda),\mu}_{\ell}[\cdot]$ is the mean-field expectation functional for which the free-gas hamiltonian has been modified by the addition of the term $\sum_{\{j:\lambda_\ell(j)>\lambda\}} s\sigma_j$. These considerations yield the formula

$$\lim_{\ell\to\infty} \tilde{\mathbb{E}}^{\mu}_{\ell}[X_\ell(\lambda)] = \frac{\partial}{\partial s}\tilde{p}(\mu+s;s,\lambda)\Big|_{s=0}, \tag{2.11}$$

where

$$\tilde{p}(\mu+s;s,\lambda) = \inf_{\alpha<0}\left\{\frac{(\mu+s-\alpha)^2}{2a} + p(\alpha;s,\lambda)\right\} ,$$

and

$$p(\alpha;s,\lambda) = \int_{[0,\lambda)} p(\alpha|\lambda)dF(\lambda) + \int_{[\lambda,\infty)} p(\alpha|s+\lambda)dF(\lambda).$$

A standard argument, using Griffith's Lemma, yields the result. ∎

§3 FLUCTUATIONS IN THE MEAN-FIELD MODEL

Fluctuations in $X_\ell = N/V_\ell$ in the mean-field model in the thermodynamic limit were studied for the standard example, described in §3 of [2] in this volume, by Davies [4], Wreszinski [5], Fannes and Verbeure [6] and Buffet and Pule [7]. The mean-field model in the general situation, where the only assumptions about the single-particle spectrum are that (S1) and (S2) hold, was investigated in [8]; we have summarized the results of [8] in §2 and now go on to investigate the fluctuations in X_ℓ. In fact, we do rather more; we regard $\lambda \longrightarrow X_\ell(\lambda)$ as a stochastic process and prove a central limit theorem:

THEOREM 4

Let $Z_\ell(\lambda) = V_\ell^{1/2}\{X_\ell(\lambda) - \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)]\}$; *then, for* $\mu < a\rho_C$, $Z_\ell(\lambda) \xrightarrow{(d)} Z(\lambda)$, *where* $Z(\lambda)$ *is gaussian with mean zero and covariance* $\Gamma(\lambda_1,\lambda_2)$ *given by*

$$\Gamma(\lambda_1,\lambda_2) = J^\mu_{\lambda_1\wedge\lambda_2} - \frac{aJ^\mu_{\lambda_1}J^\mu_{\lambda_2}}{1 + aJ^\mu_\infty} \qquad (3.1)$$

where

$$J^\mu_\lambda = \int_{[0,\lambda)} p''(\alpha(\mu)|\lambda)dF(\lambda) \qquad (3.2)$$

and $\alpha(\mu)$ *is the value of* α *at which* $\inf_{\alpha<0}\left\{\frac{(\mu-\alpha)^2}{2a} + p(\alpha)\right\}$ *is attained.*

Sketch of proof:

The result follows from a routine, but somewhat tedious, calculation of

$$\lim_{\ell\to\infty} \tilde{\mathbb{E}}_\ell^\mu\left[e^{\beta(s_1Z_\ell(\lambda_1)+s_2Z_\ell(\lambda_2))}\right]$$

along the lines of the proof of Theorem 3. ∎

It is interesting to identify the process $Z(\cdot)$ in terms of a standard process.

THEOREM 5

Let $B(\cdot)$ *be a* BM(1), *a brownian motion in* $\mathbb{R}^1$ *starting at zero; then, for* $\mu < a\rho_C$

$$Z(\lambda) \overset{(d)}{=} B(J^\mu_\lambda) - \frac{aJ^\mu_\lambda}{1 + aJ^\mu_\infty} B(J^\mu_\infty + 1/a). \qquad (3.3)$$

Proof:

A routine computation shows that the mean of the right-hand side of (3.3) is zero and the covariance is the same as that of $Z(\cdot)$, given by (3.1). Hence the two gaussian processes are equal in distribution. ■

The process (3.3) is a modification of a time-changed brownian bridge; it never reaches the point at which it is tied-down but, as a increases, that point comes closer to J^{μ}_{∞}. This shows how, as the strength of the interaction increases, the fluctuations in $Z(\infty)$ are damped down.

It is a little more difficult to deal with the case $\mu > a\rho_c$; we introduce

$$W_{\ell}(\lambda) = Z_{\ell}(\infty) - Z_{\ell}(\lambda) \tag{3.4}$$

and prove in analagous fashion:

THEOREM 6

For $\mu > a\rho_c$, $W_{\ell}(\lambda) \xrightarrow{(d)} W(\lambda)$, *where* $W(\lambda)$ *is a gaussian process with mean zero and covariance* $\Gamma(\lambda_1, \lambda_2)$ *given by*

$$\Gamma(\lambda_1, \lambda_2) = K^{\mu}_{\lambda_1 \vee \lambda_2}, \tag{3.5}$$

where

$$K^{\mu}_{\lambda} = \int_{[\lambda,\infty)} p''(0|\lambda)dF(\lambda). \tag{3.6}$$

In this case,

$$W(\lambda) \overset{(d)}{=} K^{\mu}_{\lambda} B\left[\frac{1}{K^{\mu}_{\ell}}\right]. \tag{3.7}$$

The method by which we discovered the representations may be of some interest. The stochastic differential equation satisfied by a process $(X_t)_{t\geq 0}$ with filtration $(\mathcal{F}_t)$ is discussed by Nelson [9]; see also McGill [10].

Suppose that a process $(X_t, \mathcal{F}_t)$ *satisfies the stochastic differential equation*

$$X_t = X_s + \int_s^t \sigma(u, X_u)dB(u) + \int_s^t \tau(u, X_u)du; \tag{3.8}$$

then

$$\tau(s,X_s) = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}[X_t - X_s | \mathcal{F}_s] \tag{3.9}$$

and

$$\sigma^2(s,X_s) = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}[X_t - X_s)^2 | \mathcal{F}_s]. \tag{3.10}$$

Assuming that the processes $Z(\lambda), W(\lambda)$ satisfy stochastic differential equations, the corresponding coefficients σ and τ can be computed using (3.9) and (3.10); this is a routine exercise starting from the expressions (3.1) and (3.5) for the covariances since the processes are gaussian. Obvious time-changes then give the stochastic differential equations for a brownian bridge and a brownian motion respectively.

REFERENCES

[1] J.T.Lewis: The Large Deviation Principle in Statistical Mechanics: an Expository Account (in this volume).

[2] M. van den Berg, J.T. Lewis, J.V. Pule: Large Deviations and the Boson Gas, (in this volume).

[3] M. van den Berg, J.T. Lewis and J.V. Pule: A General Theory of Bose-Einstein Condensation, Helv. Phys. Acta, 59, 1271-1288 (1986).

[4] E.B. Davies: The Thermodynamic Limit for an Imperfect Boson Gas, Commun. Math. Phys. 28, 69-86 (1972).

[5] W.F. Wreszinski: Normal Fluctuations in some Mean-Field Models in Quantum Statistical Mechanics, Helv. Phys. Acta, 46, 844-868 (1974).

[6] M. Fannes, A. Verbeure: The Imperfect Boson Gas, J. Math. Phys. 21, 1809-1818 (1980).

[7] E. Buffet, J.V. Pule: Fluctuation Properties of the Imperfect Boson Gas, J. Math. Phys. 24, 1608-1616 (1983).

[8] M. van den Berg, J.T. Lewis, P. de Smedt: Condensation in the Imperfect Boson Gas, J. Stat. Phys., 37, 697-707 (1984).

[9] E. Nelson: Dynamical Theories of Brownian Motion, New Jersey: Princeton University Press, 1967.

[10] P. McGill: Seminaires de Probabilités, XX, LNM 1204, Springer: Heidelberg 1986.

DIAS-STP-87-22

LIMIT THEOREMS FOR STOCHASTIC PROCESSES ASSOCIATED WITH A BOSON GAS

M. van den Berg
Department of Mathematics, Heriot-Watt University
Riccarton, Edinburgh EH14 4AS, United Kingdom

J.T. Lewis
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Republic of Ireland

§1 INTRODUCTION

In this lecture, we discuss the density of particles having energy less than λ in a boson system as a stochastic process indexed by λ. The notation is that of [1] in this volume. Recall that the hamiltonian for the free boson gas is given by

$$H_\ell(\omega) = \sum_{j\geq 1} \lambda_\ell(j)\sigma_j(\omega), \tag{1.1}$$

where $0 = \lambda_\ell(1) \leq \lambda_\ell(2) \leq \ldots$. For a system in a region of volume V_ℓ, the grand canonical pressure $p_\ell(\mu)$ is defined for $\mu < 0$ by

$$p_\ell(\mu) = \frac{1}{\beta V_\ell} \ell n\left\{\sum_{\omega\in\Omega} e^{\beta(\mu N(\omega)-H_\ell(\omega))}\right\}. \tag{1.2}$$

In [2] in this volume, we recalled results (proved in [3]) on the existence of the pressure in the thermodynamic limit:

$$p(\mu) = \lim_{\ell\to\infty} p_\ell(\mu). \tag{1.3}$$

In order to discuss the phenomenon of boson condensation, we introduced in [3] the family of random variables $\{X_\ell(\cdot;\lambda):\lambda \geq 0\}$ defined by

$$X_\ell(\omega;\lambda) = \frac{1}{V_\ell} \sum_{\{j:\lambda_\ell(j)\leq\lambda\}} \sigma_j(\omega). \tag{1.4}$$

For the free boson gas, we have the following result:

THEOREM 1

Suppose that (S1) *and* (S2) *hold; then, for* ρ_c *finite,*

$$\lim_{\lambda\downarrow 0} \lim_{\ell\to\infty} \mathbb{E}_\ell^\rho [X_\ell(\lambda)] = (\rho-\rho_c)^+. \tag{1.5}$$

[Conditions (S1) and (S2) and the critical density ρ_c are defined in §2 and §3 of [2] in this volume. Here $\mathbb{E}_\ell^\rho[\cdot]$ denotes the expectation taken with respect to the grand canonical probability measure $\mathbb{P}_\ell^\mu[\cdot]$ with $\mu = \mu_\ell(\rho)$, defined in §3 of

[2]; it is the expectation at fixed mean density ρ.]

<u>Proof</u>:

From the definition of $X_\ell(\lambda)$, we have

$$\mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \int_{[0,\lambda)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda) = \rho - \int_{[\lambda,\infty)} p'(\mu_\ell(\rho)|\lambda)dF_\ell(\lambda). \tag{1.6}$$

But, for $\mu < \lambda$, the sequence

$$\left\{ \int_{[\lambda,\infty)} p'(\mu|\lambda)dF_\ell(\lambda) : \ell = 1,2,\ldots \right\} \tag{1.7}$$

converges uniformly in μ on compacts to

$$\int_{[\lambda,\infty)} p'(\mu|\lambda)dF(\lambda). \tag{1.8}$$

Hence, by Proposition 2 of [2], we have for $\lambda > 0$:

$$\lim_{\ell\to\infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \rho - \int_{[\lambda,\infty)} p(\mu(\lambda)|\lambda)dF(\lambda). \tag{1.9}$$

But, by hypothesis, ρ_c is finite so that we may invoke the dominated convergence principle to conclude that

$$\lim_{\lambda\downarrow 0} \int_{[\lambda,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \int_{[0,\infty)} p'(\mu(\rho)|\lambda)dF(\lambda) = \begin{cases} \rho , & \rho < \rho_c, \\ \rho_c , & \rho \geqslant \rho_c. \end{cases}$$

Thus we have

$$\lim_{\ell\to\infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = (\rho - \rho_c)^+ \qquad \blacksquare$$

In the free boson gas there is a second effect, discovered by M. Kac in 1971. We saw in §3 of [2] that the free-energy has a first-order phase-transition segment $[\rho_c,\infty)$; it follows that for $\rho > \rho_c$ there is no guarantee that the weak law of large numbers will hold for the distribution $\mathbb{K}_\ell^\rho = \mathbb{P}_\ell^\rho \circ X_\ell^{-1}$ of the number density $X_\ell = N/V_\ell$. In fact, there is no guarantee that for, $\rho > \rho_c$, the sequence $\{\mathbb{K}_\ell^\rho : \ell = 1,2,\ldots\}$ will converge; nevertheless, by the Helly Selection Principle, a subsequence will converge, but the limit distribution will depend on the detailed behaviour of the corresponding subsequence of the sequence $\{\lambda_\ell(\cdot):\ell = 1,2,\ldots\}$. In other words, it is possible to have two sequences, $\{\lambda_\ell(\cdot):\ell = 1,2,\ldots\}$ and $\{\hat{\lambda}_\ell(\cdot):\ell = 1,2,\ldots\}$, each satisfying (S1) and (S2) and having the same integrated density of states $F(\cdot)$ but having limit distributions $\mathbb{K}^\rho$ and $\hat{\mathbb{K}}^\rho$ which are distinct for $\rho > \rho_C$. (For $\rho < \rho_C$, they must both be equal to δ_ρ, the degenerate distribution concentrated at ρ, by Theorem 1 of [1].) For example, Kac showed that in the standard example (described in §3 of [2]) the limit distribution is the exponential distribution supported on $[\rho_C, \infty)$ with mean ρ, for $\rho > \rho_C$; other examples are investigated in detail in [3]. We shall see in the next section that, in the mean-field model, this phenomenon disappears: there is no first-order phase-transition segment, the grand canonical pressure exists for all values of μ and is a differentiable function; the weak law of large numbers holds for X_ℓ for all values of the mean density ρ; nevertheless, condensation persists. In these circumstances it becomes interesting to regard $\lambda \longrightarrow X_\ell(\cdot;\lambda)$ as a stochastic process and to enquire about the convergence in distribution of a re-scaled, centred version of it. This we do in §3.

§2 THE MEAN FIELD-MODEL

To describe the mean-field model, we define a sequence of hamiltonians $\{\tilde{H}_\ell : \ell = 1,2,\ldots\}$ by

$$\tilde{H}_\ell(\omega) = H_\ell(\omega) + \frac{a}{2V_\ell} N^2(\omega) \tag{2.1}$$

with $a > 0$. The term $\frac{a}{2V_\ell} N^2$, which provides a crude caricature of the interaction, can be understood classically: it arises in an "index of refraction" approximation in which we imagine each particle to move through the system as if it were moving in a uniform optical medium and so receiving an increment of energy proportional to the density $X_\ell = N/V_\ell$; since a is positive the interaction is repulsive.

First, we compute the pressure $\tilde{p}_\ell(\mu)$, as explained in §4 of [1]: writing $u(x) = (\mu - \alpha)x - \frac{a}{2}x^2$, a straight-forward manipulation gives

$$\tilde{p}_\ell(\mu) = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \mathbb{E}_\ell^\alpha[e^{\beta V_\ell u(X_\ell)}] = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \int_{[0,\infty)} e^{\beta V_\ell u(x)} \mathbb{K}_\ell^\alpha[dx] \tag{2.2}$$

for each $\alpha < 0$, where $\mathbb{K}_\ell^\alpha = \mathbb{P}_\ell^\alpha \circ X_\ell^{-1}$. But $x \longrightarrow u(x)$ is continuous and bounded above and $\{\mathbb{K}_\ell^\alpha : \ell = 1,2,\ldots\}$ satisfies the Large Deviation Principle with rate-function $I^\alpha(x) = p(\alpha) + f(x) - \alpha x$, by Theorem 1 of [2]. Hence, by Varadhan's First Theorem, $\tilde{p}(\mu) = \lim_{\ell\to\infty} \tilde{p}_\ell(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_x\{u(x) - I^\alpha(x)\} = \sup_x\{\mu x - \tilde{f}(x)\}, \tag{2.3}$$

where the mean-field free-energy $\tilde{f}(\cdot)$ is given by

$$\tilde{f}(x) = f(x) + \frac{a}{2}x^2. \tag{2.4}$$

Thus we have proved:

THEOREM 2

Suppose that (S1) *and* (S2) *hold; then the mean-field pressure exists for all real* μ *and is given by*

$$\tilde{p}(\mu) = \sup_x\{\mu x - \tilde{f}(x)\}, \tag{2.5}$$

where $x \longrightarrow \tilde{f}(x)$ *is the mean-field free energy, given by* $\tilde{f}(x) = f(x) + \frac{a}{2}x^2$.

Next, we introduce the mean-field expectation functional $\tilde{\mathbb{E}}_\ell^\mu[\cdot]$ defined by

$$\tilde{\mathbb{E}}_\ell^\mu[\cdot] = \mathbb{E}_\ell^\alpha[\cdot e^{\beta M_\ell}]/\mathbb{E}_\ell^\alpha[e^{\beta M_\ell}], \tag{2.6}$$

and the associated probability measure $\tilde{\mathbb{P}}_\ell^\alpha[\cdot]$, where

$$M_\ell = V_\ell u(X_\ell). \tag{2.7}$$

COROLLARY

The mean-field pressure $\mu \longrightarrow \tilde{p}(\mu)$ *is differentiable for all values of* μ. *The sequence of distribution functions* $\{\tilde{\mathbb{K}}_\ell^\mu = \tilde{\mathbb{P}}_\ell^\mu \circ X_\ell^{-1}\}$ *converges weakly to the degenerate distribution* δ_ρ *concentrated at* $\rho = \tilde{p}'(\mu)$ *and satisfies the Large Deviation Principle with constants* $\{V_\ell\}$ *and rate-function* $\tilde{I}^\mu(x) = \tilde{p}(\mu) + \tilde{f}(x) - \mu x$.

Proof:

Since $x \longrightarrow f(x)$ is strictly convex for $0 \leq x < \rho_c$ and constant for $\rho_c \leq x < \infty$ and $x \longrightarrow \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$, the function $x \longrightarrow \tilde{f}(x) = f(x) + \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$; hence there is no first-order phase-transition segment; equivalently, $\mu \longrightarrow \tilde{p}(\mu)$, the Legendre transform of $x \longrightarrow \tilde{f}(x)$, is differentiable for $\mu < \infty$. It follows from Theorem 1 of [1] that $\tilde{\mathbb{K}}_\ell^\mu \to \delta_\rho$, where $\rho = \tilde{p}'(\mu)$, and from Theorem 4 of [1] that $\{\tilde{\mathbb{K}}_\ell^\mu : \ell = 1,2,\ldots\}$ satisfies the Large Deviation Principle with constants $\{V_\ell\}$ and rate-function $\tilde{I}^\mu(\cdot)$ ■

Although the first-order phase-transition segment, which was present in the free energy function of the free-gas, has disappeared, the phenomenon of condensation persists:

THEOREM 3

Suppose that (S1) *and* (S2) *hold; then, for* ρ_c *finite, we have*

$$\lim_{\lambda\downarrow 0}\lim_{\ell\to\infty} \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)] = (\rho-\rho_c)^+ , \tag{2.8}$$

where $\tilde{\mathbb{E}}_\ell^\mu[\cdot]$ *is the mean-field expectation functional and* $\rho = \tilde{p}'(\mu)$.

Proof:

First, we remark that an elementary exercise yields the following alternative formula for the mean-field pressure $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \inf_{\alpha<0}\left\{\frac{(\mu-\alpha)^2}{2a} + p(\alpha)\right\}, \tag{2.9}$$

where $p(\alpha)$ is the free-gas pressure. The idea of the proof of (2.8) is that we compute the cumulant generating function of $X_\ell(\lambda)$; since

$$V_\ell X_\ell(\lambda) = V_\ell X_\ell - \sum_{\{j:\lambda_\ell(j)>\lambda\}} \sigma_j \tag{2.10}$$

we get

$$\tilde{\mathbb{E}}_\ell^\mu\left[e^{\beta s V_\ell X_\ell(\lambda)}\right] = \tilde{\mathbb{E}}_\ell^{(s,\lambda),\mu}\left[e^{\beta s V_\ell X_\ell}\right],$$

where $\tilde{\mathbb{E}}_\ell^{(s,\lambda),\mu}[\cdot]$ is the mean-field expectation functional for which the free-gas hamiltonian has been modified by the addition of the term $\sum_{\{j:\lambda_\ell(j)>\lambda\}} s\sigma_j$. These considerations yield the formula

$$\lim_{\ell\to\infty} \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)] = \frac{\partial}{\partial s}\tilde{p}(\mu+s;s,\lambda)\Big|_{s=0}, \tag{2.11}$$

where

$$\tilde{p}(\mu+s;s,\lambda) = \inf_{\alpha<0}\left\{\frac{(\mu+s-\alpha)^2}{2a} + p(\alpha;s,\lambda)\right\} ,$$

and

$$p(\alpha;s,\lambda) = \int_{[0,\lambda)} p(\alpha|\lambda)dF(\lambda) + \int_{[\lambda,\infty)} p(\alpha|s+\lambda)dF(\lambda).$$

A standard argument, using Griffith's Lemma, yields the result. ■

§3 FLUCTUATIONS IN THE MEAN-FIELD MODEL

Fluctuations in $X_\ell = N/V_\ell$ in the mean-field model in the thermodynamic limit were studied for the standard example, described in §3 of [2] in this volume, by Davies [4], Wreszinski [5], Fannes and Verbeure [6] and Buffet and Pule [7]. The mean-field model in the general situation, where the only assumptions about the single-particle spectrum are that (S1) and (S2) hold, was investigated in [8]; we have summarized the results of [8] in §2 and now go on to investigate the fluctuations in X_ℓ. In fact, we do rather more; we regard $\lambda \longrightarrow X_\ell(\lambda)$ as a stochastic process and prove a central limit theorem:

THEOREM 4

Let $Z_\ell(\lambda) = V_\ell^{1/2}\{X_\ell(\lambda) - \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)]\}$; *then, for* $\mu < a\rho_c$, $Z_\ell(\lambda) \xrightarrow{(d)} Z(\lambda)$, *where* $Z(\lambda)$ *is gaussian with mean zero and covariance* $\Gamma(\lambda_1,\lambda_2)$ *given by*

$$\Gamma(\lambda_1,\lambda_2) = J^\mu_{\lambda_1\wedge\lambda_2} - \frac{aJ^\mu_{\lambda_1}J^\mu_{\lambda_2}}{1 + aJ^\mu_\infty} \quad , \tag{3.1}$$

where

$$J^\mu_\lambda = \int_{[0,\lambda)} p''(\alpha(\mu)|\lambda)dF(\lambda) \quad , \tag{3.2}$$

and $\alpha(\mu)$ *is the value of* α *at which* $\inf\limits_{\alpha<0}\left\{\frac{(\mu-\alpha)^2}{2a} + p(\alpha)\right\}$ *is attained.*

Sketch of proof:

The result follows from a routine, but somewhat tedious, calculation of

$$\lim_{\ell\to\infty}\tilde{\mathbb{E}}_\ell^\mu\left[e^{\beta(s_1Z_\ell(\lambda_1)+s_2Z_\ell(\lambda_2))}\right]$$

along the lines of the proof of Theorem 3. ■

It is interesting to identify the process $Z(\cdot)$ in terms of a standard process.

THEOREM 5

Let $B(\cdot)$ *be a* BM(1), *a brownian motion in* $\mathbb{R}^1$ *starting at zero; then, for* $\mu < a\rho_c$,

$$Z(\lambda) \overset{(d)}{=} B(J^\mu_\ell) - \frac{aJ^\mu_\lambda}{1 + aJ^\mu_\infty} B(J^\mu_\infty + 1/a). \tag{3.3}$$

Proof:

A routine computation shows that the mean of the right-hand side of (3.3) is zero and the covariance is the same as that of $Z(\cdot)$, given by (3.1). Hence the two gaussian processes are equal in distribution. ∎

The process (3.3) is a modification of a time-changed brownian bridge; it never reaches the point at which it is tied-down but, as a increases, that point comes closer to J^{μ}_{∞}. This shows how, as the strength of the interaction increases, the fluctuations in $Z(\infty)$ are damped down.

It is a little more difficult to deal with the case $\mu > a\rho_C$; we introduce

$$W_{\ell}(\lambda) = Z_{\ell}(\infty) - Z_{\ell}(\lambda) \tag{3.4}$$

and prove in analagous fashion:

THEOREM 6

For $\mu > a\rho_C$, $W_{\ell}(\lambda) \xrightarrow{(d)} W(\lambda)$, *where* $W(\lambda)$ *is a gaussian process with mean zero and covariance* $\Gamma(\lambda_1,\lambda_2)$ *given by*

$$\Gamma(\lambda_1,\lambda_2) = K^{\mu}_{\lambda_1 \vee \lambda_2}, \tag{3.5}$$

where

$$K^{\mu}_{\lambda} = \int_{[\lambda,\infty)} p''(0|\lambda)dF(\lambda). \tag{3.6}$$

In this case,

$$W(\lambda) \overset{(d)}{=} K^{\mu}_{\lambda} B\left[\frac{1}{K^{\mu}_{\ell}}\right]. \tag{3.7}$$

The method by which we discovered the representations may be of some interest. The stochastic differential equation satisfied by a process $(X_t)_{t\geq 0}$ with filtration $(\mathcal{F}_t)$ is discussed by Nelson [9]; see also McGill [10].

Suppose that a process $(X_t,\mathcal{F}_t)$ *satisfies the stochastic differential equation*

$$X_t = X_s + \int_s^t \sigma(u,X_u)dB(u) + \int_s^t \tau(u,X_u)du; \tag{3.8}$$

then

$$\tau(s,X_s) = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}[X_t - X_s | \mathcal{F}_s] \tag{3.9}$$

and

$$\sigma^2(s,X_s) = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}[X_t - X_s)^2 | \mathcal{F}_s]. \tag{3.10}$$

Assuming that the processes $Z(\lambda), W(\lambda)$ satisfy stochastic differential equations, the corresponding coefficients σ and τ can be computed using (3.9) and (3.10); this is a routine exercise starting from the expressions (3.1) and (3.5) for the covariances since the processes are gaussian. Obvious time-changes then give the stochastic differential equations for a brownian bridge and a brownian motion respectively.

REFERENCES

[1] J.T.Lewis: The Large Deviation Principle in Statistical Mechanics: an Expository Account (in this volume).

[2] M. van den Berg, J.T. Lewis, J.V. Pule: Large Deviations and the Boson Gas, (in this volume).

[3] M. van den Berg, J.T. Lewis and J.V. Pule: A General Theory of Bose-Einstein Condensation, Helv. Phys. Acta, 59, 1271-1288 (1986).

[4] E.B. Davies: The Thermodynamic Limit for an Imperfect Boson Gas, Commun. Math. Phys. 28, 69-86 (1972).

[5] W.F. Wreszinski: Normal Fluctuations in some Mean-Field Models in Quantum Statistical Mechanics, Helv. Phys. Acta, 46, 844-868 (1974).

[6] M. Fannes, A. Verbeure: The Imperfect Boson Gas, J. Math. Phys. 21, 1809-1818 (1980).

[7] E. Buffet, J.V. Pule: Fluctuation Properties of the Imperfect Boson Gas, J. Math. Phys. 24, 1608-1616 (1983).

[8] M. van den Berg, J.T. Lewis, P. de Smedt: Condensation in the Imperfect Boson Gas, J. Stat. Phys., 37, 697-707 (1984).

[9] E. Nelson: Dynamical Theories of Brownian Motion, New Jersey: Princeton University Press, 1967.

[10] P. McGill: Seminaires de Probabilités, XX, LNM 1204, Springer: Heidelberg 1986.