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THE STATISTICAL MECHANICS OF A BETHE ANSATZ-SOLUBLE MODEL

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ABSTRACT. The non-linear Schroedinger model is considered as an example of a model that is exactly soluble by means of the Bethe Ansatz. The theory of large deviations is applied to give a rigorous derivation of the thermodynamic formalism for this model, first proposed by Yang and Yang (1969) on heuristic grounds.

1. INTRODUCTION.

Nowadays several one-dimensional quantum mechanical models are known to be exactly soluble. One of the main techniques used is the so-called Bethe Ansatz method, invented by Bethe (1931) for determining the spectrum of the one-dimensional (isotropic) Heisenberg ferromagnet. The method consists in using a linear combination of travelling waves with permuted wave numbers as a trial wave function in the eigenvalue equation for the many-body Hamiltonian.
The model we shall be discussing here is the quantum non-linear Schroedinger model. It was discovered by Lieb and Liniger (1963) that its N-particle Hamiltonian can be diagonalised by means of the Bethe Ansatz. Subsequently Yang and Yang (1969) developed a thermodynamic formalism for this model using the exact eigenvalues obtained by Lieb and Liniger. The same formalism applies to other quantum statistical models described by Hamiltonians that can be diagonalised with the help of the Bethe Ansatz. (See e.g. the review by Lowenstein (1984).)

However significant the advance made by Yang and Yang, the way they arrive at their thermodynamic formalism is far from rigorous. The present note is a short sketch of a rigorous proof of the validity of the Yang-Yang formalism, the details of which can be found in Dorlas, Lewis and Pulé (1988). The main ingredient in the proof is the theory of large deviations developed by Varadhan (1966) and others. (See e.g. the book by Ellis (1985); for applications to other statistical mechanical models, see also Van den Berg, Lewis and Pulé (1988) and Cegła, Lewis and Raggio (1988).)

2. THE HAMILTONIAN AND ITS EIGENVALUES.

The Hamiltonian of the non-linear Schroedinger model in the $N-$particle space $L^2(\mathbf{R}^N)_{sym}$ is given by

$$\mathcal{H}_N = -\sum_{i=1}^{N} \partial_{x_i}^2 + 2c \sum_{1\le i<j\le N} \delta(x_i - x_j). \tag{2.1}$$

For a rigorous definition we restrict the system to a finite interval of length L and impose periodic boundary conditions. The corresponding Hamiltonian will be denoted by $\mathcal{H}_N^L$. Defined on a suitable domain it is essentially self-adjoint, and can be diagonalised with the help of the Bethe Ansatz. The eigenvalues can be labelled by an unordered set of N distinct Fermion momenta $\underline{k} = \{k_1, \ldots, k_N\}$, with $k_j = \frac{2\pi}{L} I_j$ and $I_j \in \mathbf{Z}$ if N is odd, while $I_j \in \mathbf{Z} + \frac{1}{2}$ if N is even. They are given by

$$\mathcal{E}_L(\underline{k}) = \sum_{j=1}^{N} \tilde{k}_j(\underline{k})^2, \tag{2.2}$$

where $\{\, \tilde{k}_j(\underline{k}) \,|\, j = 1, 2, \ldots, N\}$ is the unique solution of the set of implicit equations

$$\tilde{k}_j = k_j - \frac{1}{L} \sum_{i=1}^{N} \theta_c(\tilde{k}_j - \tilde{k}_i), \tag{2.3}$$

with

$$\theta_c(k) = 2\arctan(k/c). \tag{2.4}$$

3. THE MAIN RESULT AND LARGE DEVIATIONS.

The main result of the Yang-Yang formalism is a variational expression for the pressure in the thermodynamic limit. The pressure $p_L(\mu)$ for the system on a finite interval is defined by

$$\exp\{\beta L p_L(\mu)\} = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\underline{k}} \exp[-\beta \mathcal{E}_L(\underline{k})]. \tag{3.1}$$

In this formula β is the inverse temperature and μ is the chemical potential. The second sum is over all sets of N distinct Fermion momenta. We obtain the thermodynamic limit by considering a sequence of increasing intervals of length $L_l \to \infty$, $(l = 1, 2, \ldots)$ and calculating the limit $p(\mu) = \lim_{l\to\infty} p_l(\mu)$, where we have written $p_l(\mu)$ instead of $p_{L_l}(\mu)$. Our main result reads

Theorem.
The infinite volume pressure of the quantum non-linear Schroedinger model is given by

$$p(\mu) = \sup_{m\in E} \{\mu ||m|| - f[m]\}, \tag{3.2}$$

where E is the set of positive, bounded measures with finite second moment:

$$E = \{m \in \mathcal{M}_+^b \mid \int k^2 m(dk) < \infty\}, \tag{3.3}$$

and $f[m]$ is the free energy density. The latter is given by

$$f[m] = \int f_m(k)^2 \, m(dk) - \frac{1}{\beta} s[m], \tag{3.4}$$

where f_m is the unique solution of

$$f_m(k) = k - \int \theta_c \left(f_m(k) - f_m(k')\right) \, m(dk), \tag{3.5}$$

and $s[m]$ is the entropy density:

$$s[m] = \begin{cases} -\int \{\rho(k) \ln \rho(k) + (1 - \rho(k)) \ln(1 - \rho(k))\} \frac{dk}{2\pi}, & \text{if } m(dk) = \rho(k)\frac{dk}{2\pi} \\ & \text{with } 0 \leq \rho(k) \leq 1 \; ; \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that (3.5) is the same as (2.3) if $m = m_l$, where

$$m_l(dk) = \frac{1}{L_l} \sum_{i=1}^{N} \delta_{k_i}(dk) \tag{3.7}$$

is the distribution of Fermion momenta and $f_{m_l}(k_j) = \tilde{k}_j$. The first term in (3.4) is then exactly the energy $\mathcal{E}_{L_l}(\underline{k})$. The expression for the entropy is the same as it is in the case of a free-Fermion gas: the system behaves as a system of free Fermion-like quasi-particles. The expression (3.2) can be transformed into Yang & Yang's expression by writing everything in terms of the quasi-particle momenta $\tilde{k} = f_m(k)$.

We prove (3.2) by writing (3.1) as an integral w.r.t. the free-Fermion grand-canonical ensemble:

$$p_l(\mu) = p_l^0(\mu) + \frac{1}{\beta L_l} \ln \int_E e^{\beta L_l G[m]} \, \mathbf{K}_l^\mu[dm], \tag{3.8}$$

with

$$G[m] = \int (k^2 - f_m(k)^2) \, m(dk). \tag{3.9}$$

$p_l^0(\mu)$ is the free-Fermion pressure and $\mathbf{K}_l^\mu$ is the distribution of measures $m_l \in E$ in the free-Fermion grand-canonical ensemble. It is concentrated on the measures of the form (3.7). Next we show that the sequence of measures $\{\mathbf{K}_l^\mu | \, l = 1, 2, \ldots\}$ satisfies the *large deviation principle*, which means roughly that it behaves asymptotically like

$$\mathbf{K}_l^\mu[B] \sim \exp\{-\beta L_l \inf_{m \in B} I^\mu[m]\} \tag{3.10}$$

for $B \subset E$, where the function $I^\mu : E \to \mathbf{R}$ can be computed explicitly.
Then we apply Varadhan's theorem (Varadhan 1966),which is a generalisation of Laplace's method of steepest descent. It states that the limit of the second term in (3.8) as $l \to \infty$ exists and is given by

$$\lim_{l\to\infty} \frac{1}{\beta L_l} \ln \int_E e^{\beta L_l G[m]}\, \mathbf{K}_l^\mu[dm] = \sup_{l\to\infty} \{G[m] - I^\mu[m]\}, \tag{3.11}$$

provided certain technical conditions are fulfilled. Details about the rewriting of (3.1) in the form (3.8), and the conditions on the sequence of measures $\{\mathbf{K}_l^\mu\}$ and the function G can be found in Dorlas, Lewis and Pulé (1988).

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