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Symmetric States of Composite Systems

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Abstract

Størmer proved a theorem on the integral decomposition of symmetric states on a C^*-algebra $\overset{*}{\otimes}\mathcal{B}$. Motivated by problems in statistical mechanics, we define symmetric states on a composite algebra $\mathcal{A} \otimes (\overset{*}{\otimes}\mathcal{B})$ and extend Størmer's theorem to this situation. Applications to spin-boson models are sketched.

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1 Introduction

In [1], Størmer proved a theorem characterizing the symmetric states of an infinite tensor product of C^*-algebras. The principal result of this paper is a generalization, dictated by the requirements of the statistical mechanics of composite systems, of Størmer's result. In [2], Størmer's Theorem was used to characterize the KMS and limiting Gibbs states of mean-field models; in §3, we sketch some results of this nature for composite systems which follow from the theorem of §2. We adopt, as far as possible , the notation of [1] and [2].

2 Symmetric States of Composite Systems

Let $\mathcal{B}, j = 1, 2, ...$ denote copies of a fixed matrix algebra M_n for some $n \geq 2$; let Λ be a finite ordered subset of N and denote by $\mathcal{B}_\Lambda$ the C^*-algebra $\otimes_{j\in\Lambda}\mathcal{B}_j$. For $\Lambda \subset \Lambda'$, there is an obvious canonical imbedding of $\mathcal{B}_\Lambda$ in $\mathcal{B}'_\Lambda$ given on generating elements by

$$\otimes_{j\in\Lambda}\, x_j \longrightarrow \otimes_{j\in\Lambda'} y_j \tag{2.1}$$

where

$$y_j = \begin{cases} x_j, & j \in \Lambda, \\ 1, & j \in \Lambda' \setminus \Lambda; \end{cases} \tag{2.2}$$

using the canonical imbedding, we define the C^*-algebra $\overset{*}{\otimes}\mathcal{B}$ to be the inductive limit of the family $\{\mathcal{B}_\Lambda : \Lambda \subset \mathrm{N}\}$. Let $\mathcal{S}$ denote the group of one-to-one mappingts of N onto itself leaving all but a finite number of integers fixed. There is a canonical action of $\mathcal{S}$ on $\overset{*}{\otimes}\mathcal{B}$ given by the action of an element π of $\mathcal{S}$ on each generating element:

$$\pi(x)_i = x_{\pi(i)} \tag{2.3}$$

A state ω of $\overset{*}{\otimes}\mathcal{B}$ is said to be *symmetric* if $\omega = \omega \circ \pi$ for all π in $\mathcal{S}$. A state ω of $\overset{*}{\otimes}\mathcal{B}$ is said to be a *product state* if, for each pair Λ, Λ' of disjoint finite subsets of N, we have

$$\omega(x_1 x_2) = \omega(x_1)\omega(x_2) \text{ for all } x_1 in \mathcal{B}_\Lambda \text{ and all } x_2 \text{ in } \mathcal{B}_{\Lambda'}.$$

A symmetric product state ω determines a state ρ of M_n and $\omega = \overset{*}{\otimes}\rho$; we shall find it convenient to denote $\overset{*}{\otimes}\rho$ by ω_ρ. It follows that, in the present setting, we have from [1]

STØRMER'S THEOREM

Let ω be a symmetric state of $\overset{*}{\otimes}\mathcal{B}$ with $\mathcal{B} = M_n$; then there exists a unique probability measure μ on D_n, the state space of M_n, such that

$$\omega = \int_{D_n} \mu(d\rho)\omega_\rho. \tag{2.4}$$

In statistical mechanics, one often meets the situation where a system is represented by the product of a C^*-algebra $\mathcal{A}$ and an infinite tensor product algebra $\overset{*}{\otimes}\mathcal{B}$. This motivates the following definitions: the action of $\mathcal{S}$ on $\overset{*}{\otimes}\mathcal{B}$ extends to an action on $\mathcal{A}\otimes(\overset{*}{\otimes}\mathcal{B})$ leaving $\mathcal{A}$ point-wise fixed and a state ω of $\mathcal{A}\otimes(\overset{*}{\otimes}\mathcal{B})$ is said to be *symmetric* if it is invariant under this action: $\omega = \omega\circ\pi$ for all π in $\mathcal{S}$; a state ω is said to be a *product state* if the restriction of ω to $\overset{*}{\otimes}\mathcal{B}$ is a product state and, on generating elements of $\mathcal{A}\otimes(\overset{*}{\otimes}\mathcal{B}\ )$ we have $\omega(xy) = \omega(x)\omega(y)$ for all x in $\mathcal{A}$ and all y in $\overset{*}{\otimes}\mathcal{B}$. It is clear that a symmetric product state determines a state ρ of M_n and a state η of $\mathcal{A}$ such that

$$\omega = \eta\otimes\omega_\rho = \eta\otimes(\overset{*}{\otimes}\rho). \tag{2.5}$$

We are now ready to state the theorem.

THEOREM

Let $\mathcal{A}$ be a separable C^*-algebra; let ω be a symmetric state of $\mathcal{A}\otimes(\overset{*}{\otimes}\mathcal{B})$ with $\mathcal{B} = M_n$. Then there exists a unique probability measure μ on D_n and a μ-measurable function $\rho\to\eta_\rho$ from D_n to the state space of $\mathcal{A}$ such that

$$\omega = \int_{D_n} \mu(d\rho)\eta_\rho\otimes\omega_\rho. \tag{2.6}$$

Proof:

For each symmetric state ω of $\mathcal{A}\otimes(\overset{*}{\otimes}\mathcal{B})$ and each positive element a of $\mathcal{A}$, the functional $x\to\omega(a\otimes x)$ is a symmetric positive linear functional on $\overset{*}{\otimes}\mathcal{B}$; by Størmer's Theorem , there exists a unique positive measure μ_a on D_n such that

$$\omega(a\otimes x) = \int_{D_n} \mu_a(d\rho)\omega_\rho(x) \tag{2.7}$$

for all x in $\overset{*}{\otimes}\mathcal{B}$. We show that the measure μ_a is absolutely continuous with respect to μ_1, which is the probability measure μ of the statement of the theorem. For each x in $\overset{*}{\otimes}\mathcal{B}$, we define a function $\tilde{x}$ on D_n by

$$\tilde{x}(\rho) = \omega_\rho(x); \tag{2.8}$$

we remark that this function is continuous. Now regard μ_a as a functional on $C(D_n)$; for elements of $C(D_n)$ of the form (2.8), we have

$$\begin{aligned} \mu_a(\tilde{x}) &= \int_{D_n} \mu_a(d\rho)\tilde{x}(\rho) \\ &= \int_{D_n} \mu_a(d\rho)\omega_\rho(x), && \text{by (2.8),} \\ &= \omega(a \otimes x), && \text{by (2.7).} \end{aligned}$$

For $x \geq 0$, we have

$$\omega(a \otimes x) \leq \| a \| \, \omega(x) \tag{2.9}$$

so that

$$\mu_a(\tilde{x}) \leq \| a \| \, \mu_1(\tilde{x}). \tag{2.10}$$

Denote by X the set $\{\tilde{x} : x \in \bigcup_{\Lambda \subset \mathbf{N}} \mathcal{B}_\Lambda\}$ and by X^+ the set $\{\tilde{x} \in X : x \geq 0\}$; to complete the proof of the theorem, we make use of the fact, whose proof we postpone, that the set X^+ is dense in $C^+(D_n)$. It follows from (2.10) that

$$\mu_a(f) \leq \| a \| \, \mu(f) \tag{2.11}$$

for all f in $C^+(D_n)$. Let E be a Borel subset of D_n and let 1_E denote the indicator function of E; by a corollary of Lusin's Theorem (see [3]) there is a uniformly bounded sequence $\{f_n : n = 1, 2, ...\}$ of positive continuous functions such that $f_n \to 1_E$ almost everywhere with respect to μ. Hence we have

$$\mu_a(1_E) \leq \| a \| \, \mu(1_E) \tag{2.12}$$

for every Borel subset E of D_n. It follows from the Radon-Nikodym Theorem that there is a unique element $\eta(.,a)$ of $L^1(D_n, \mu)$ such that

$$\mu_a(f) = \int_{D_n} \mu(d\rho)\eta(\rho, a)f(\rho) \tag{2.13}$$

for all f in $C(D_n)$. In particular taking $f = \tilde{x}$, we have

$$\omega(a \otimes x) = \int_{D_n} \mu(d\rho)\eta(\rho, a)\omega_\rho(x). \tag{2.14}$$

By the uniqueness of the μ_a and the linearity of $a \to \omega(a \otimes x)$ for all x, the map $a \to \mu_a$ is linear. By the uniqueness of the element $\eta(.,a)$, the map $\mu_a \to \eta(.,a)$ is well defined, hence the map $a \to \eta(.,a)$ is linear as a map from $\mathcal{A}$ to $L^1(D_n, \mu)$. Moreover $\eta(.,a)$ is positive since μ_a is positive, $\eta(.,\mathbf{1}) = 1$ and, from (2.12) ,we have $\eta(.,a) \leq \| a \|$. We wish to show that the map $a \to \eta(\rho, a)$ is a state of $\mathcal{A}$ for μ-almost all ρ; the properties of linearity, positivity, normalization and boundedness of η as an element of $L^1(D_n, \mu)$ hold almost everywhere, but on sets which may depend on the elements of the algebra. However, invoking the separability of $\mathcal{A}$, it is possible to show that there exists a Borel subset F of D_n, with $\mu(F) = 1$, on which the above properties hold for a countable dense subset of $\mathcal{A}$; on F, we may extend $\eta(\rho, .)$ to a state $\eta_\rho(.)$ of $\mathcal{A}$ by continuity.

It remains to prove that the set X^+ is dense in $C^+(D_n)$. Let f be an element of $C^+(D_n)$ and let $f^{1/2}$ denote its positive square root; we remark that the set X is an algebra under pointwise multiplication containing the constants and separating the points of D_n so that, by the Stone-Weierstrass Theorem, we can approximate $f^{1/2}$ by elements of X: given $\epsilon > 0$, there is an x in some $\mathcal{B}_\Lambda$ with $\Lambda = \{i_1, ..., i_m\}$ such that

$$\| f^{1/2} - \tilde{x} \| < \epsilon. \tag{2.15}$$

Let τ be a right-shift in N such that $\tau(i_1) > i_m$; then

$$\omega_\rho(x^*\tau(x)) = | \omega_\rho(x) |^2 . \tag{2.16}$$

It follows that

$$\lim_{M\to\infty} \frac{1}{M^2}\omega_\rho \left(\sum_{k=1}^{M}\sum_{\ell=1}^{M}(\tau^k x)^*.\tau^\ell x) = | \omega_\rho(x) |^2,\right) \tag{2.17}$$

so that we may choose M such that

$$\| f - \tilde{x_M^* x_M} \| < 3\epsilon \text{ with } x_M = \sum_{k=1}^{M} \tau^k x/M$$

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3 Comments and Applications

There is a straightforward generalization of the theorem of §2 which is possible, replacing M_n by a type I factor. For arbitrary $\mathcal{A}$ and $\mathcal{B}$, the extremal symmetric states are product states; it is a challenging problem to find conditions on a symmetric state sufficient to ensure that it has an integral decomposition analogous to that of the theorem of §2. In Størmer's setting, Hudson and Moody [4] proved that, when $\mathcal{B}$ is a type I factor, local normality of ω is sufficient; it would be interesting to investigate whether this remains true in our setting with $\mathcal{A}$ separable.

As far as the applications to statistical mechanics are concerned, it is necessary to use Hamiltonians which have the correct permutation invariance. In the case when $\mathcal{A}$ is trivial ($\mathcal{A}$ = C), the requirements of permutation invariance lead us to the usual mean-field models; as we remarked in the introduction, Størmer's Theorem and Hudson and Moody's generalization have been applied in this case to give a complete characterization of the equilibrium states. With $\mathcal{A}$ non-trivial , the theorem of §2 opens the possibility of carrying out such a programme for composite systems described by Hamiltonians of the type

$$H_\ell = H_\ell^0 \otimes \mathbf{1} + \sum_{j=1}^{\ell} \mathbf{1} \otimes A^{(j)} + \sum_{j=1}^{\ell} B_\ell \otimes C^{(j)}, \tag{3.1}$$

where H_ℓ^0 and B_ℓ are self-adjoint elements of $\mathcal{A}$ and

$$A^{(j)} = \mathbf{1} \otimes \ldots \otimes A \otimes \ldots \otimes \mathbf{1} \tag{3.2}$$

is an element of $\mathcal{B}_{\{1,\ldots,\ell\}}$ with A a self-adjoint element of $\mathcal{B}_j$ and $C^{(j)}$ is similarly defined.

This class of models includes the spin-boson models studied by Hepp and Lieb [5,6]. In such applications, the restriction that $\mathcal{A}$ be separable does not prove to be an obstacle to the study of equilibrium states. The boson part of the system is represented by the CCR algebra $\mathcal{A}$ which, as a C^*-algebra, is non-separable; one first proves that the equilibrium state is locally normal so that the GNS representation is on a separable Hilbert space and one can exploit this separability in the proof of the decomposition into product states. Defining the equilibrium states by means of the correlation inequalities (see Fannes and Verbeure [7]), we may repeat the arguments of [2] to obtain a characterization of the equilibrium states of the spin-boson models.

We sketch the argument: let ω_ℓ^β be the unique state of $\mathcal{A} \otimes \mathcal{B}_{\{1,\ldots,\ell\}}$ satisfying the correlation inequality

$$\beta\omega_\ell^\beta(x^*[H_\ell, x]) \geq \omega_\ell^\beta(x^*x) \ln \frac{\omega_\ell^\beta(x^*x)}{\omega_\ell^\beta(xx^*)}, \tag{3.3}$$

for all x in $\mathcal{A} \otimes \mathcal{B}_{\{1,\ldots,\ell\}}$.

Each ω_ℓ^β is symmetric because of the permutation-invariance of H_ℓ and the uniqueness of the state satisfying (3.3). Let ω^β be any limit-point of the sequence $\{\omega_\ell^\beta : \ell = 1, 2, \ldots\}$, then ω^β is symmetric and hence has the decomposition

$$\omega^\beta = \int_{D_n} \mu(d\rho)\eta_\rho^\beta \otimes \omega_\rho^\beta. \tag{3.4}$$

Under suitable conditions on the sequence $\{H_\ell : \ell = 1, 2, \ldots\}$ we can prove that, for almost all ρ, the following gap-equations hold:

$$\begin{aligned} &\lim_{\ell\to\infty} \beta\eta_\rho^\beta(y^*[H_\ell^0 + \ell\rho(c)B_\ell, y]) \\ &\qquad \geq \eta_\rho^\beta(y^*y) \ln \frac{\eta_\rho^\beta(y^*y)}{\eta_\rho^\beta(yy^*)}, \end{aligned} \tag{3.5}$$

$$\rho = \frac{e^{-\beta(A + \lim_{\ell\to\infty} \eta_\rho^\beta(B_\ell)C)}}{tr(e^{-\beta(A+\lim_{\ell\to\infty} \eta_\rho^\beta(B_\ell)C)})} \tag{3.6}$$

Details will appear elsewhere.

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