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THE FREE ENERGY OF THE SPIN-BOSON MODEL

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Abstract : For n spins $\frac{1}{2}$ coupled linearly to a boson field in a volume V_n, the existence of the specific free energy in the limit $n \to \infty$, $V_n \to \infty$ with n/V_n = const., is proved under specified conditions on the Hamiltonian. A variational expression is obtained for the limiting specific free energy, and a critical temperature is identified, above which the system behaves as if there were no coupling at all.

§1. Introduction, and main result

Consider the Hamiltonian

$$H_n = \sum_{\nu\geq 1} \omega_n(\nu) a^*_\nu a_\nu + V_n^{-\frac{1}{2}} \sum_{\nu\geq 1} \sum_{j=1}^{n} \{\lambda_n(j;\nu) a^*_\nu + \overline{\lambda_n(j;\nu)} a_\nu\} S^x_{(j)}$$

$$+ \sum_{j=1}^{n} \varepsilon_n(j) S^z_{(j)} \quad ,$$

for n spins $\frac{1}{2}$ - described by the spin operators $\{S^\alpha_{(j)} : j=1,2,\cdots,n;\ \alpha=x,y,z\}$, with $[S^x_{(j)}, S^y_{(k)}] = i\delta_{jk} S^z_{(j)}$ and cyclic permutations - interacting linearly with a countable number of bosonic degrees of freedom described by creation/annihilation operators $\{a^*_\nu, a_\nu : \nu\geq 1\}$, with $[a_\nu, a^*_{\nu'}] \subset \delta_{\nu,\nu'}$. The *strictly positive* bosonic frequencies $\omega_n(\nu)$ are assumed to satisfy

$$\sum_{\nu\geq 1} e^{-\beta\omega_n(\nu)} < \infty \quad , \text{ for } \beta > 0 \ ;$$

the coupling constants $\{\lambda_n(j;\nu) : \nu\geq 1,\ j=1,2,\cdots,n\}$ are complex numbers satisfying

$$\sum_{\nu\geq 1} |\lambda_n(j;\nu)|^2 < \infty \quad , \text{ for every } j=1,2,,\cdots,n \ ;$$

and the $\{\varepsilon_n(j) : j=1,2,\cdots,n\}$ are real.

The problem is to determine the specific free energy of the system in the thermodynamic limit $n \to \infty$, where V_n - the volume of the system - is proportional to n, that is to say $\rho = n/V_n$ - the density of the spins - is constant. This problem has been solved in a number of particular cases. Firstly, Hepp and Lieb [(8)], treated the case of 1 bosonic mode using a rotating-wave approximation for the coupling (Dicke Maser Model). These same authors then [(9)] removed the latter approximation and treated finitely many bosonic modes in the *homogeneous* case where the coupling constants and spin frequencies are independent of the spins: $\lambda_n(j;\nu) = \lambda_n(\nu)$, and $\varepsilon_n(j) = \varepsilon_n$ for every $j=1,2,\cdots,n$. Hepp and

Lieb, also obtain results on thermodynamic stability for the general (i.e. *heterogeneous*) model, leaving open the question of existence of the thermodynamic limit [9]. Subsequently, the "Approximating Hamiltonian Method" has been put to work on the Hamiltonian H_n and its variants [2,3,12]. The homogeneous case with countably many bosonic modes has been treated in detail [10] using large deviation methods developed in ref. 4.

Here, the problem is solved for the heterogeneous model using a method developed by Duffield and Pulè in their treatment of the B.C.S. model [6], supplemented with an idea of Bogoljubov (jr.) and Plechko [3]. It is shown that under certain specified conditions H_n is thermodynamically equivalent (in the sense that the difference of the specific free-energies vanishes in the thermodynamic limit) to the Hamiltonian

$$\tilde{H}_n = \sum_{\nu\geq 1} \omega_n(\nu) a_\nu^* a_\nu + \sum_{j=1}^{n} \epsilon_n(j) S^z_{(j)} - V_n^{-1} \sum_{j,k=1}^{n} \Lambda_n(j,k) S^x_{(j)} S^x_{(k)} ,$$

where the spin-boson interaction is replaced by an effective quadratic spin-spin interaction:

$$\Lambda_n(j,k) = \mathrm{Re} \sum_{\nu\geq 1} \omega_n(\nu)^{-1} \overline{\lambda_n(j;\nu)} \lambda_n(k;\nu) \quad , \ j,k = 1,2,\cdot\cdot,n .$$

Moreover, $\tilde{H}_n$ is thermodynamically equivalent to the Hamiltonian

$$\hat{H}_n(x) = \sum_{\nu\geq 1} \omega_n(\nu) a_\nu^* a_\nu + \sum_{j=1}^{n} \epsilon_n(j) S^z_{(j)} + \sum_{j,k=1}^{n} \Lambda_n(j,k) x_j (V_n x_k 1 - 2S^x_{(k)}) ,$$

if the real n-vector x is chosen so as to minimize the corresponding specific free energy.

The result is then the following:

<u>Theorem</u> <u>1</u>: *Suppose there exists real-valued continous functions* ϵ *on* $[0,1]$, *and* Λ *on* $[0,1]\times[0,1]$, *such that*

$$\lim_{n\to\infty} \sup_{j\in\{1,2,\cdot\cdot,n\}} |\epsilon_n(j) - \epsilon(j/n)| = 0 , \qquad \text{(C1)}$$

$$\lim_{n\to\infty} \sup_{j,k\in\{1,2,\cdots,n\}} |\Lambda_n(j,k)-\Lambda(j/n,k/n)|=0 \quad ; \tag{C2}$$

if

$$f^o = \lim_{\substack{n\to\infty \\ \rho=\text{const.}}} (-\beta V_n)^{-1} \log \operatorname{tr} \exp\{-\beta \sum_{\nu\geq 1} \omega_n(\nu) a_\nu^* a_\nu\} \quad , \tag{C3}$$

exists for some $\beta > 0$, *and if*

$$\lim_{n\to\infty} n^{-3/2} \sum_{\nu\geq 1} \omega_n(\nu)^{-\frac{1}{2}} \sum_{j=1}^{n} |\lambda_n(j;\nu)| = 0 \quad , \tag{C4}$$

then

$$\lim_{\substack{n\to\infty \\ \rho=\text{const.}}} (-\beta V_n)^{-1} \log \operatorname{tr} \exp\{-\beta H_n\} = f^o - \rho \sup_{\substack{r,s\in L^\infty_{\mathbb{R}}([0,1]) \\ |s| \leq r \leq 1}} \left\{ \int_0^1 \left[\beta^{-1} I(r(t)) \right.\right.$$

$$\left.\left. + \tfrac{1}{2}|\varepsilon(t)|[r(t)^2 - s(t)^2]^{\frac{1}{2}}\right] dt + \tfrac{1}{4}\rho \int_0^1 \int_0^1 \Lambda(t,u) s(t) s(u)\, dt\, du \right\} \quad ,$$

where $I(x) = -\frac{1}{2}(1+x)\log[\frac{1}{2}(1+x)] - \frac{1}{2}(1-x)\log[\frac{1}{2}(1-x)]$ for $0 \leq x \leq 1$.

This is proved in §3, after introducing notation in the following section 2. The solution of the variational problem, following Duffield and Pulè [6], is presented and briefly discussed in §4.

§2. Notation, definitions

It will be convenient to use Fock-space notation. For each $n=1,2,3...$, let $\mathscr{A}_n$ be a bounded region in $\mathbb{R}^d$ of volume (i.e. Lebesgue measure) V_n. Let $\mathfrak{h}_n$ be a positive *injective* selfadjoint operator on $L^2(\mathscr{A}_n)$ such that $\exp(-\beta\mathfrak{h}_n)$ is trace-class for $\beta > 0$. It follows that $\mathfrak{h}_n$ has a bounded inverse. Write $\mathfrak{K}_n$ for the n-fold tensor product of $\mathbb{C}^2$ and let $\underline{S}_{(j)}$ be a copy of of the Spin operator of magnitude $\frac{1}{2}$ acting on the j-th component of $\mathfrak{K}_n$ ($j=1,2,\ldots,n$). Let $\mathfrak{F}_n$ be the symmetric Fock space over $L^2(\mathcal{A}_n)$ and consider the Hamiltonian [1]

$$H_n = d\Gamma(\mathfrak{h}_n) + \sum_{j=1}^{n}\left\{(V_n)^{-\frac{1}{2}}(a^*(\lambda_n(j))+a(\lambda_n(j)))S^x_{(j)}+\varepsilon_n(j)S^z_{(j)}\right\} \quad (2.1)$$

acting on $\mathfrak{F}_n\otimes\mathfrak{K}_n$, where $\{\varepsilon_n(j)\}\subset\mathbb{R}$, $\{\lambda_n(j)\}\subset L^2(\mathcal{A}_n)$, $a(\cdot)$ is the familiar annihilation operator, and $d\Gamma$ denotes the second-quantization map. The quadratures formula (see ref. 5)

$$W[f]^* d\Gamma(\mathfrak{h})W[f] = d\Gamma(\mathfrak{h}) + a^*(\mathfrak{h}f)+a(\mathfrak{h}f) + \langle f,\mathfrak{h}f\rangle\cdot 1 , \quad (2.2)$$

valid for $f \in \mathrm{Dom}(\mathfrak{h})$ where $W[f]\equiv\exp\{\overline{a^*(f)-a(f)}\}$ is the unitary Weyl operator, enables one to write

$$H_n=\sum_{j=1}^{n}\left\{n^{-1}U_n(j)^* d\Gamma(\mathfrak{h}_n)U_n(j) + \varepsilon_n(j)S^z_{(j)} - \tfrac{1}{4}\rho\|\mathfrak{h}_n^{-\frac{1}{2}}\lambda_n(j)\|^2 1\right\} , \quad (2.3)$$

where the unitaries $U_n(j)$, $j=1,2,\cdot\cdot,n$, are given by

$$U_n(j) := W[\tfrac{1}{2}n(V_n)^{-\frac{1}{2}}\mathfrak{h}_n^{-1}\lambda_n(j)]P^+_{(j)} + W[\tfrac{1}{2}n(V_n)^{-\frac{1}{2}}\mathfrak{h}_n^{-1}\lambda_n(j)]^* P^-_{(j)}, \quad (2.4)$$

where $P^{\pm}_{(j)}$ is the spectral projection of $S^x_{(j)}$ to the eigenvalue $\pm\frac{1}{2}$. Formula (2.3) can now be used to prove the self-adjointness of H_n.

Two free energy densities are associated with H_n:

$$\exp\{-\beta V_n f_n\} = \mathrm{tr}_{\mathfrak{F}_n\otimes\mathfrak{K}_n}(\exp\{-\beta H_n\}) , \quad (2.5)$$

$$\exp\{-\beta V_n f^o_n\} = \mathrm{tr}_{\mathfrak{F}_n}(\exp\{-\beta d\Gamma(\mathfrak{h}_n)\}) . \quad (2.6)$$

Of interest is the limit $n \to \infty$, such that V_n diverges but

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[1] Tensor notation for operators is not used, i.e. $S_{(j)}=1\otimes S_{(j)}$, $a(\circ)=a(\circ)\otimes 1$ etc.

$\rho=n/V_n$, remains *constant*.

The Hamiltonian (2.1) has the following symmetry. Let the self-adjoint, unitary operator L_n on $\mathfrak{F}_n \otimes \mathfrak{K}_n$ be given by $L_n = \Gamma(-1)(\prod_{j=1}^{n} 2S^z_{(j)})$; then $L_n S^z_{(j)} L_n = S^z_{(j)}$, and $L_n S^x_{(j)} L_n = -S^x_{(j)}$ for every $j=1,2,\cdots,n$, and $L_n d\Gamma(\circ) L_n = d\Gamma(\circ)$, $L_n a(\circ) L_n = -a(\circ)$. In particular, L_n commutes with H_n.

Consider the Hamiltonian $H_n(h)$, $h \in \mathbb{R}^n$, defined by

$$H_n(h) = H_n + \sum_{j=1}^{n} h_j S^x_{(j)} \quad , \tag{2.7}$$

where the symmetry of H_n implemented by L_n is *broken* if the external field vector h is non zero. The free energy density associated with $H_n(h)$ is written $f_n(h)$, and is a concave function of each of the n components of h. Expectation values with respect to the canonical state associated with $H_n(h)$ are denoted by $\langle\circ\rangle_h$.

The $(n\times n)$-matrix Λ_n is defined by its matrix elements

$$\Lambda_n(j,k) \equiv \mathrm{Re}\langle\lambda_n(j), \mathfrak{h}_n^{-1}\lambda_n(k)\rangle_{L^2(\mathscr{A}_n)} \quad , \quad j,k \in \{1,2,\cdots,n\} \quad ; \tag{2.8}$$

it is readily seen that Λ_n is *positive semi-definite* and the multiplicity of the eigenvalue 0 is equal to n minus the number of vectors in $\{\lambda_n(j): j=1,2,\cdots,n\}$ which are real-linearly independent.

§3. The proofs

Introduce a bosonic Hamiltonian $H^b_n(x)$, $x \in \mathbb{R}^n$, on $\mathfrak{F}_n$ by

$$H^b_n(x) = d\Gamma(\mathfrak{h}_n) + V_n \sum_{j=1}^{n} x_j \Big\{ V_n^{-\frac{1}{2}}(a^*(\lambda_n(j)) + a(\lambda_n(j))) + \sum_{k=1}^{n} \Lambda_n(j,k) x_k 1 \Big\} \quad , \tag{3.1}$$

and two spin Hamiltonians $\tilde{H}^s_n(h)$ and $\hat{H}^s_n(h;x)$, $h,x \in \mathbb{R}^n$, on $\mathfrak{K}_n$ by

$$\tilde{H}^s_n(h) = \sum_{j=1}^{n} \left\{\epsilon_n(j)S^z_{(j)} + h_j S^x_{(j)} - V_n^{-1} \sum_{k=1}^{n} \Lambda_n(j,k)S^x_{(j)}S^x_{(k)}\right\}, \tag{3.2}$$

$$\hat{H}^s_n(h;x) = \sum_{j=1}^{n} \left\{\epsilon_n(j)S^z_{(j)} + \left\{h_j - 2\sum_{k=1}^{n} \Lambda_n(j,k)x_k\right\}S^x_{(j)}\right\} + V_n x\Lambda_n x 1. \tag{3.3}$$

Write $\tilde{f}^s_n(h)$, and $\hat{f}^s_n(h;x)$ for the free energy densities associated with (3.2) and (3.3) respectively. Expectation values with respect to a canonical state will be written as angular brackets indexed by the corresponding Hamiltonian or distinctive parameters characterizing it.

<u>Lemma 1</u> :

$$(-\beta V_n)^{-1}\log \operatorname{tr}_{\mathfrak{J}_n} \exp(-\beta H^b_n(x)) = f^o_n \ , \text{ for every } x \in \mathbb{R}^n \ ;$$

$$\hat{f}^s_n(h;x) = x\Lambda_n x$$

$$- (V_n\beta)^{-1} \sum_{j=1}^{n} \log\left\{2\cosh\left\{\tfrac{1}{2}\beta\left[\epsilon_n(j)^2 + (h_j - 2\sum_{k=1}^{n}\Lambda_n(j,k)x_k)^2\right]^{\frac{1}{2}}\right\}\right\} .$$

<u>Proof</u>: An application of (2.2) shows that (3.1) is unitarily equivalent to $d\Gamma(\mathfrak{h}_n)$ for every $x \in \mathbb{R}^n$ (see the proof of Lemma 2A). Up to the constant term $V_n x\Lambda_n x 1$, the Hamiltonian (3.3) is the sum of n pairwise commuting operators

$$\epsilon_n(j)S^z + (h_j - 2\sum_{k=1}^{n} \Lambda_n(j,k)x_k)S^x \ ,$$

on $\mathbb{C}^2$, each of which has $\pm\frac{1}{2}[\epsilon_n(j)^2 + (h_j - 2\sum_{k=1}^{n} \Lambda_n(j,k)x_k)^2]^{\frac{1}{2}}$ as its eigenvalues.■

<u>Lemma</u> <u>2A</u> : $\tilde{f}_n^s(h) - \inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x) \le f_n^o + \tilde{f}_n^s(h) - f_n(h)$.

<u>Proof</u>: Equivalently,

$$f_n^o + \inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x) - f_n(h) \ge 0 \quad . \qquad (*)$$

By the first part of Lemma 1, $f_n^o+\hat{f}_n^s(h;x)$ is the specific free energy associated with the Hamiltonian $\hat{H}_n(h;x) = H_n^b(x) + \hat{H}_n^s(h;x)$; by Bogoljubov's inequality (see ref. 7 for a proof)

$$f_n^o + \hat{f}_n^s(h;x) - f_n(h) \ge V_n^{-1}\langle \hat{H}_n(h;x)-H_n(h)\rangle_{\hat{H}_n(h;x)} \quad . \qquad (**)$$

Now by (3.1),(3.2) and (2.7), the right-hand side of (**) is given by

$$\sum_{j=1}^{n}\left\{\left[V_n^{-\frac{1}{2}}\langle a^*(\lambda_n(j))+a(\lambda_n(j))\rangle_{H_n^b(x)} + 2\sum_{k=1}^{n}\Lambda_n(j,k)x_k\right]\right.$$

$$\left.\times\left[x_j - V_n^{-1}\langle S^x_{(j)}\rangle_{\hat{H}_n^s(h;x)}\right]\right\} \quad .$$

By (2.2), $H_n^b(x)=W[-V_n^{\frac{1}{2}}\sum_{j=1}^{n} x_j\mathfrak{h}_n^{-1}\lambda_n(j)]d\Gamma(\mathfrak{h}_n)W[V_n^{\frac{1}{2}}\sum_{j=1}^{n} x_j\mathfrak{h}_n^{-1}\lambda_n(j)]$. Using the formula $W[f]^*a(g)W[f]=a(g)+\langle g,f\rangle 1$ and (2.8),

$$\langle a^*(\lambda_n(k))+a(\lambda_n(k))\rangle_{H_n^b(x)} = \langle W[V_n^{\frac{1}{2}}\sum_{j=1} x_j\mathfrak{h}_n^{-1}\lambda_n(j)]$$

$$\cdot\{a^*(\lambda_n(k))+a(\lambda_n(k))\}W[-V_n^{\frac{1}{2}}\sum_{j=1} x_j\mathfrak{h}_n^{-1}\lambda_n(j)]\rangle_{d\Gamma(\mathfrak{h}_n)}$$

$$= - V_n^{\frac{1}{2}}\sum_{j=1}^{n} x_j\left[\overline{\langle\lambda_n(k),\mathfrak{h}_n^{-1}\lambda_n(j)\rangle} + \langle\lambda_n(k),\mathfrak{h}_n^{-1}\lambda_n(j)\rangle\right]$$

$$+ \langle a^*(\lambda_n(k))+a(\lambda_n(k))\rangle_{d\Gamma(\mathfrak{h}_n)} = -2V_n^{\frac{1}{2}}\sum_{j=1}^{n}\Lambda_n(j,k)x_j \quad .$$

Thus, the right-hand side of (**) is zero for every $x\in\mathbb{R}^n$; (*) follows by taking the infimum with respect to x.■

Bogoljubov's inequality also gives an upper bound on $f_n^o+\tilde{f}_n^s(h)-f_n(h)$; this involves

$$V_n^{-3/2}\sum_{\nu\geq 1}\sum_{j=1}^{n}\langle\{\lambda_n(j;\nu)a_\nu^* + \overline{\lambda_n(j;\nu)}a_\nu\}S^x_{(j)}\rangle_h \quad . \qquad (3.4)$$

Bogoljubov and Plechko [3] have devised an alternative method which avoids the problem of estimating (3.4). Fix an arbitrary n, and consider an arbitrary finite number N of boson modes with strictly positive frequencies $\{\omega_n(\nu):1\leq\nu\leq N\}$, and associated coupling constants $\{\lambda_n(j;\nu):1\leq\nu\leq N,\ j=1,2,\cdots,n\}$. The Hamiltonian $H_n(h;N)$ is that obtained from $H_n(h)$ by considering only these N modes, and the associated specific free energy will be written $f_n(h;N)$; accordingly, write $f_n^o(N)$, and $\tilde{f}_n^s(h;N)$.

Let $\mathbb{A}=\{\nu:1\leq\nu\leq N,\ \lambda_n(j;\nu)=0$ for every $j=1,2,\cdots,n\}$, and $\mathbb{B}=\{1,2,\cdots,N\}\backslash\mathbb{A}$. For any set $\tau=\{\tau_\nu:\nu\in\mathbb{B}\}$ of real numbers in the open interval (0,1), one has the identity

$$H_n(h;N) = \sum_{\nu\in\mathbb{A}}\omega_n(\nu)a_\nu^*a_\nu + \sum_{\nu\in\mathbb{B}}(1-\tau_\nu)\omega_n(\nu)a_\nu^*a_\nu + \tilde{H}_n^s(h;N;\tau)$$

$$+ \sum_{\nu\in\mathbb{B}}\tau_\nu\omega_n(\nu)b_\nu(\tau)^*b_\nu(\tau) \quad , \qquad (3.5)$$

where

$$\tilde{H}_n^s(h;N;\tau)=\sum_{j=1}^{n}\left\{\epsilon_n(j)S^z_{(j)}+h_jS^x_{(j)}-V_n^{-1}\sum_{k=1}^{n}\Lambda_n^N(j,k;\tau)S^x_{(j)}S^x_{(k)}\right\} \quad , \qquad (3.6)$$

$$\Lambda_n^N(j,k;\tau) = \mathrm{Re}\sum_{\nu\in\mathbb{B}}(\tau_\nu\omega_n(\nu))^{-1}\overline{\lambda_n(j;\nu)}\lambda_n(k;\nu) \quad , \qquad (3.7)$$

$$b_\nu(\tau) = a_\nu + V_n^{-\frac{1}{2}}(\tau_\nu\omega_n(\nu))^{-1}\sum_{j=1}^{n}\lambda_n(j;\nu)S^x_{(j)} \quad . \qquad (3.8)$$

Let $f_n^o(N;\tau)$ be the specific free energy of $\sum_{\nu\in\mathbb{A}}\omega_n(\nu)a_\nu^*a_\nu + \sum_{\nu\in\mathbb{B}}(1-\tau_\nu)\omega_n(\nu)a_\nu^*a_\nu$, and write $\tilde{f}_n^s(h;N;\tau)$ for that of (3.6). Since the last term in (3.5) is positive, $f_n^o(N;\tau)+\tilde{f}_n^s(h;N;\tau)\leq f_n(h;N)$ by

Bogoljubov's inequality. Thus

$$f_n^o(N) + \tilde{f}_n^s(h;N) - f_n(h;N) \le \{f_n^o(N) - f_n^o(N;\tau)\}$$

$$+ \{\tilde{f}_n^s(h;N) - \tilde{f}_n^s(h;N;\tau)\} \quad . \tag{3.9}$$

Using Bogoljubov's inequality, and the familiar formula for $f_n^o(N;\tau)$

$$f_n^o(N) - f_n^o(N;\tau) \le V_n^{-1} \sum_{\nu\in\mathbb{B}} \tau_\nu \omega_n(\nu) \langle a_\nu^* a_\nu \rangle_{(N;\tau)}$$

$$= -\sum_{\nu\in\mathbb{B}} \tau_\nu \{\partial f_n^o/\partial\tau_\nu\}(N;\tau) = V_n^{-1} \sum_{\nu\in\mathbb{B}} \tau_\nu \omega_n(\nu) (e^{\beta(1-\tau_\nu)\omega_n(\nu)} - 1)^{-1}$$

$$\le (\beta V_n)^{-1} \sum_{\nu\in\mathbb{B}} \tau_\nu (1-\tau_\nu)^{-1} \quad . \tag{3.10}$$

Also using Bogoljubov's inequality and $-\frac{1}{2}1 \le S^x \le \frac{1}{2}1$,

$$\tilde{f}_n^s(h;N) - \tilde{f}_n^s(h;N;\tau) \le V_n^{-2} \sum_{\nu\in\mathbb{B}} \Big\{ (\tau_\nu^{-1} - 1)\omega_n(\nu)^{-1}$$

$$\cdot \mathrm{Re} \sum_{j,k=1}^{n} \overline{\lambda_n(j;\nu)} \lambda_n(k;\nu) \langle S_{(j)}^x S_{(k)}^x \rangle_{(h;N;\tau)} \Big\}$$

$$\le (2V_n)^{-2} \sum_{\nu\in\mathbb{B}} (1-\tau_\nu)\tau_\nu^{-1} \omega_n(\nu)^{-1} \Big[\sum_{j=1}^{n} |\lambda_n(j;\nu)| \Big]^2 \quad . \tag{3.11}$$

Inserting (3.10) and (3.11) into (3.9),

$$\{f_n^o(N) + \tilde{f}_n^s(h;N)\} - f_n(h;N) \le (\beta V_n)^{-1} \sum_{\nu\in\mathbb{B}} \tau_\nu (1-\tau_\nu)^{-1}$$

$$+ (2V_n)^{-2} \sum_{\nu\in\mathbb{B}} (1-\tau_\nu)\tau_\nu^{-1} \omega_n(\nu)^{-1} \Big[\sum_{j=1}^{n} |\lambda_n(j;\nu)| \Big]^2 \quad . \tag{3.12}$$

The infimum of the right hand side of (3.12) with respect to τ is assumed at

$$\tau_\nu = \frac{\beta^{\frac{1}{2}}\ \omega_n(\nu)^{-\frac{1}{2}} \sum_{j=1}^{n} |\lambda_n(j;\nu)|}{2V_n^{\frac{1}{2}} + \beta^{\frac{1}{2}}\ \omega_n(\nu)^{-\frac{1}{2}} \sum_{j=1}^{n} |\lambda_n(j;\nu)|} , \tag{3.13}$$

which lies in (0,1) by virtue of the definition of $\mathbb{B}$. Thus,

$$f_n^o(N)+\tilde{f}_n^s(h;N)-f_n(h;N) \le V_n^{-1}(\beta V_n)^{-\frac{1}{2}} \sum_{\nu\ge1}^{N} \omega_n(\nu)^{-\frac{1}{2}} \sum_{j=1}^{n} |\lambda_n(j;\nu)| \quad . \tag{3.14}$$

For fixed n, it follows that $f_n^o(N)$, $\tilde{f}_n^s(h;N)$ and $f_n(h;N)$ converge to f_n^o, $\tilde{f}_n^s(h)$ and $f_n(h)$ respectively, as $N \to \infty$, so that the following result is proved.

<u>Lemma 2B</u> : $f_n^o + \tilde{f}_n^s(h) - f_n(h) \le V_n^{-1}(\beta V_n)^{-\frac{1}{2}} \sum_{\nu\ge1} \omega_n(\nu)^{-\frac{1}{2}} \sum_{j=1}^{n} |\lambda_n(j;\nu)|$.

The limit of $\tilde{f}_n^s(h)$ has been recently obtained by Duffield and Pulè [6] in their analysis of the B.C.S. model. Their result, which combines large deviation methods with Berezin-Lieb bounds, is the following

<u>Theorem 2</u> (Duffield & Pulè) : *If conditions (C1) and (C2) are satisfied, and there exists a real-valued continous function* h *on* [0,1] *such that*

$$\lim_{n\to\infty} \sup_{j\in\{1,2,\cdots,n\}} |h_j - h(j/n)| = 0 \quad , \tag{C0}$$

then

$$\tilde{f}^s(h) = \lim_{\substack{n\to\infty \\ \rho=\text{const.}}} \tilde{f}_n^s(h) = \rho \inf_{\substack{r,s\in L_{\mathbf{R}}^\infty([0,1]) \\ |s| \le r \le 1}} \left\{ \int_0^1 \left[-\beta^{-1} I(r(t)) + \tfrac{1}{2} h(t)s(t) - \tfrac{1}{2}|\varepsilon(t)|[r(t)^2 - s(t)^2]^{\frac{1}{2}} \right] dt - \tfrac{1}{2}\rho \int_0^1 \int_0^1 \Lambda(t,t')s(t)s(t')dtdt' \right\} .$$

Remark 1: The proofs of ref. 6 apply without change under the slightly stronger assumptions: $h_j=h(j/n)$, $\varepsilon_n(j)=\varepsilon(j/n)$, and $\Lambda_n(j,k)=\Lambda(j/n,k/n)$; but can be adapted to accomodate (C0)-(C2).

$\inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x)$ is discussed in Appendix A; one has the following result:

Lemma 3: *Under the assumptions (C0)-(C2),*

$$\lim_{\substack{n\to\infty \\ \rho=\text{const.}}} \inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x) = \tilde{f}^s(h) \quad .$$

Proof : Let $M_n=\inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x)$; by Lemma A1, setting $s_j=r_j\sin(\vartheta_j)$,

$$M_n = \inf_{|s_j|\le r_j\le 1} \left\{ V_n^{-1} \sum_{j=1}^{n} \left\{ -\beta^{-1} I(r_j) - \tfrac{1}{2}|\varepsilon_n(j)|[r_j^2-s_j^2]^{\frac{1}{2}} + \tfrac{1}{2}h_j s_j \right\} \right.$$

$$\left. - \tfrac{1}{4}V_n^{-2} \sum_{j=1}^{n} \sum_{k=1}^{n} \Lambda_n(j,k)s_j s_k \right\} \quad .$$

Define L_n by replacing $\varepsilon_n(j)$, h_j, and $\Lambda_n(j,k)$ in the above expression for M_n, by $\varepsilon(j/n)$, $h(j/n)$, and $\Lambda(j/n,k/n)$ respectively, where $\varepsilon(\circ)$, $h(\circ)$, and $\Lambda(\circ,\circ)$ are the functions given by conditions (C0)-(C2). As in Theorem 3 of ref. 6, one proves that $L_n \to \tilde{f}^s(h)$ as $n \to \infty$ with ρ=const..Now,

$$|M_n-L_n| \le \sup_{|s_j|\le r_j\le 1} \left| V_n^{-1} \sum_{j=1}^{n} \left\{ \tfrac{1}{2}(|\varepsilon(j/n)|-|\varepsilon_n(j)|)[r_j^2-s_j^2]^{\frac{1}{2}} \right. \right.$$

$$\left. \left. + \tfrac{1}{2}(h_j-h(j/n))s_j \right\} + \tfrac{1}{4}V_n^{-2} \sum_{j=1}^{n}\sum_{k=1}^{n} (\Lambda(j/n,k/n)-\Lambda_n(j,k))s_j s_k \right|$$

$$\le \tfrac{1}{2}\rho n^{-1} \sum_{j=1}^{n} \left\{ ||\varepsilon(j/n)|-|\varepsilon_n(j)|| + |h_j-h(j/n)| \right\}$$

$$+ \frac{1}{4}\rho^2 n^{-2} \sum_{j=1}^{n} \sum_{k=1}^{n} |\Lambda(j/n,k/n)-\Lambda_n(j,k)| \quad ,$$

so that, by (C0)-(C2), $M_n-L_n \to 0$ as $n \to \infty$ with ρ=const.∎

<u>Remark 2</u> : One can prove $\lim_{n\to\infty} \{\tilde{f}_n^s(h)-\inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x)\}=0$, directly by the "Approximating Hamiltonian Method" using an idea of ref. 1; one has to assume that n^{-1}[number of non-zero eigenvalues of Λ_n] $\to$ 0 as $n \to \infty$; moreover, the positivity of Λ_n is used [(11)].

The proof of Theorem 1 is obtained combining Lemmas 2A, 2B and 3, and Theorem 2.

One can recover the results of ref. 10 which are valid for the homogeneous case: $\varepsilon_n(j)=\varepsilon_n$, $\lambda_n(j;\nu)=\lambda_n(\nu)$, and $h_j=h$, for all $j=1,2,\cdots,n$ [2]. Condition (C0) is trivially met; conditions (C1) and (C2) demand the existence of real numbers ε, and Λ (≥ 0) such that $\varepsilon_n \to \varepsilon$, and $\langle\lambda_n,\mathfrak{h}_n^{-1}\lambda_n\rangle_{L^2(\mathcal{A}_n)} \to \Lambda$.

<u>Lemma 4</u> : *In the homogeneous case*

$$\tilde{f}^s(h) = -\rho \sup_{0\leq z,u\leq 1} \{\beta^{-1}I(u) + \tfrac{1}{2}|h|u(1-z^2)^{\frac{1}{2}} + \tfrac{1}{2}|\varepsilon|uz + \tfrac{1}{4}\rho\Lambda u^2(1-z^2)\} \ .$$

<u>Proof</u>: By Theorem 2, choosing $r(t)=r$ and $s(t)=s$ a.e.,

$$-\tilde{f}^s(h)/\rho \geq \sup_{|s|\leq r\leq 1} \{\beta^{-1}I(r) - \tfrac{1}{2}hs + \tfrac{1}{2}|\varepsilon|[r^2-s^2]^{\frac{1}{2}} + \tfrac{1}{4}\rho\Lambda s^2\}$$

$$= \sup_{0\leq x,r\leq 1} \{\beta^{-1}I(r) + \tfrac{1}{2}|h|rx + \tfrac{1}{2}|\varepsilon|r[1-x^2]^{\frac{1}{2}} + \tfrac{1}{4}\rho\Lambda r^2x^2\} \ .$$

For r and s in $L^\infty_{\mathbb{R}}([0,1])$ with $|s|\leq r\leq 1$, (all integrals are over [0,1])

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[2] Condition (C4) is not needed for the results of ref. 10.

$$\int[r(t)^2-s(t)^2]^{\frac{1}{2}}dt = \int[r(t)-s(t)]^{\frac{1}{2}}[r(t)+s(t)]^{\frac{1}{2}}\,dt$$

$$\le \left[\int[r(t)-s(t)]dt\cdot\int[r(t)+s(t)]\,dt\right]^{\frac{1}{2}} = \left[\{\int r(t)dt\}^2-\{\int s(t)dt\}^2\right]^{\frac{1}{2}} ,$$

by the Schwarz inequality; since I is concave,

$$-\tilde{f}^s(h)/\rho \le \sup_{\substack{r,s\in L^\infty_{\mathbb{R}}([0,1]) \\ |s|\le r\le 1}} \Big\{\beta^{-1}I(\int r(t)dt) - \tfrac{1}{2}h\int s(t)dt + \tfrac{1}{4}\rho\Lambda\{\int s(t)dt\}^2$$

$$+ \tfrac{1}{2}|\varepsilon|\left[\{\int r(t)dt\}^2-\{\int s(t)dt\}^2\right]^{\frac{1}{2}} \Big\}$$

$$= \sup_{|s|\le r\le 1} \{\beta^{-1}I(r) - \tfrac{1}{2}hs + \tfrac{1}{2}|\varepsilon|[r^2-s^2]^{\frac{1}{2}} + \tfrac{1}{4}\rho\Lambda s^2\} \quad .\blacksquare$$

§4. The phase transition

The variational problem determining $\tilde{f}^s(h)$, and thus $f(h)$, is

$$\mathcal{F}(h) = \sup_{\substack{r,s\in L^\infty_{\mathbb{R}}([0,1]) \\ |s|\le r\le 1}} \Big\{ \int_0^1 \Big[\beta^{-1}I(r(t)) + \tfrac{1}{2}|\varepsilon(t)|[r(t)^2-s(t)^2]^{\frac{1}{2}}$$

$$- \tfrac{1}{2}h(t)s(t)\Big]dt + \tfrac{1}{4}\rho\int_0^1\int_0^1 \Lambda(t,t')s(t)s(t')dtdt' \Big\} \quad . \tag{4.1}$$

For $\Lambda(t,t') \ge 0$ (and h=const.) this problem [3], is solved by Duffield and Pulè [(6)]; most of their arguments apply to the case of arbitrary Λ.

Notice that if h=0 and (r,s) is a maximizer for (4.1), then so

---

[3] Our kernel need not be positive; it defines a positive operator. $\Lambda(t,t') > 0$ is used in the uniqueness results of ref. 6.

is $(r,-s)$. The function I is concave, with derivative $-\text{arctanh}$. The r-variation can be done as in ref. 6; for $s \in L^\infty_{\mathbb{R}}([0,1])$ with $|s| \le 1$, let $r_s:[0,1] \to \mathbb{R}$ be defined (a.e.) to be 1 where $|s|=1$, and otherwise as the *largest* zero in the interval $[|s(t)|,1]$ of the function

$$x \to \tfrac{1}{2}\beta|\varepsilon(t)|x - [x^2-s(t)^2]^{\frac{1}{2}}\text{arctanh}(x) \quad ^4 \quad ; \tag{4.2}$$

then, if $\mathcal{B}$ denotes the unit ball of $L^\infty_{\mathbb{R}}([0,1])$, one has

$$\mathcal{F}(h)= \sup_{s\in\mathcal{B}} \{\mathcal{V}(s;h)\} \quad , \tag{4.3}$$

where

$$\mathcal{V}(s;h) = \int_0^1 \Big[\beta^{-1}I(r_s(t)) + \tfrac{1}{2}|\varepsilon(t)|[r_s(t)^2-s(t)^2]^{\frac{1}{2}} - \tfrac{1}{2}h(t)s(t)\Big]dt$$

$$+ \tfrac{1}{4}\rho \int_0^1 \int_0^1 \Lambda(t,t')s(t)s(t')dtdt' \quad . \tag{4.4}$$

For $h=0$, one has inversion symmetry $\mathcal{V}(s;0)=\mathcal{V}(-s;0)$. Let K be the selfadjoint, integral operator on $L^2_{\mathbb{R}}([0,1])$ defined by the kernel Λ; K is compact. Consider the continous function g_β on $[0,1]$ given by

$$g_\beta(t) = \begin{cases} (\beta/2)^{\frac{1}{2}} & , \text{ if } \varepsilon(t)=0 \\ \left[\dfrac{\tanh(\frac{1}{2}\beta|\varepsilon(t)|)}{|\varepsilon(t)|}\right]^{\frac{1}{2}} & , \text{ if } \varepsilon(t)\neq 0 \end{cases} \quad , \tag{4.5}$$

and let G_β be the (bounded, positive) operator on $L^2_{\mathbb{R}}([0,1])$ of multiplication by g_β. Let $U^\rho_\beta=\rho G_\beta K G_\beta$, i.e.

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[4] Notice that $r_0(t)=\tanh(\frac{1}{2}\beta|\varepsilon(t)|)$ a.e., that $r_{-s}=r_s$, and that $r_s=|s|$ on the set where $\varepsilon(t)=0$.

$$\{U^\rho_\beta\psi\}(t) = \rho g_\beta(t) \int_0^1 g_\beta(t')\Lambda(t,t')\psi(t')dt' \quad . \tag{4.6}$$

Define $\Phi^\rho_\beta(s;t)$ (a.e.) by

$$\Phi^\rho_\beta(s;t) = \rho\{Ks\}(t) - \begin{cases} 2\beta^{-1}\operatorname{arctanh}(s(t)) & , \ \varepsilon(t)=0 \\ |\varepsilon(t)|s(t)/[r_s(t)^2-s(t)^2]^{\frac{1}{2}} & , \ \varepsilon(t)\neq 0 \end{cases} \tag{4.7}$$

and notice that $\Phi^\rho_\beta(-s;\circ)=-\Phi^\rho_\beta(s;\circ)$.

The solution of (4.1) for h=0 is obtained from the following two results which will be proved in Appendix B by adjusting the arguments of ref. 6 :

<u>Theorem</u> <u>3</u>: *If* $\|U^\rho_\beta\| \leq 1$, *then*

$$\mathcal{F}(0) = \mathcal{V}(0;0) = \beta^{-1}\int_0^1 \log[2\cosh(\tfrac{1}{2}\beta\varepsilon(t))]dt \quad .$$

<u>Theorem</u> <u>4</u>: *If* $\|U^\rho_\beta\| > 1$, *then there exists a non-zero* $s_*\in\mathcal{B}$ *such that* $\mathcal{F}(0)=\mathcal{V}(s_*;0)=\mathcal{V}(-s_*;0)$. s_* *and* $-s_*$ *are solutions of the Euler-Lagrange equation* $\Phi^\rho_\beta(s;\circ)=0$. *Moreover*

$$\mathcal{F}(0) = \mathcal{V}(\pm s_*;0) = \beta^{-1}\int_0^1 \log[2\cosh(\tfrac{1}{2}\beta\{\varepsilon(t)^2+k_\beta(t)^2\}^{\frac{1}{2}})]dt$$

$$- \tfrac{1}{4}\int_0^1 \frac{\tanh(\tfrac{1}{2}\beta\{\varepsilon(t)^2+k_\beta(t)^2\}^{\frac{1}{2}})}{\{\varepsilon(t)^2+k_\beta(t)^2\}^{\frac{1}{2}}} k_\beta(t)^2\,dt \quad ,$$

where $k_\beta \neq 0$ *satisfies*

$$k_\beta(t) = \rho\int_0^1 \Lambda(t,t') \frac{\tanh(\tfrac{1}{2}\beta\{\varepsilon(t')^2+k_\beta(t')^2\}^{\frac{1}{2}})}{\{\varepsilon(t')^2+k_\beta(t')^2\}^{\frac{1}{2}}} k_\beta(t')dt' \quad .$$

Remark 3 : Most likely, s_* and $-s_*$ are the *only* non-zero solutions of the Euler-Lagrange equation if K is positive, but I am unable to prove this.

The map $\beta \to \|U_\beta^\rho\|$ is strictly increasing with $\lim_{\beta \downarrow 0} \|U_\beta^\rho\| = 0$, so that one can identify a possibly infinite critical reciprocal temperature β_c such that if $\beta < \beta_c$ then $\|U_\beta^\rho\| < 1$, and if $\beta > \beta_c$ then $\|U_\beta^\rho\| > 1$. For $\beta \le \beta_c$, $\tilde{f}^s$ - and thus f - is independent of the interaction: the system is thermodynamically equivalent to a non-interacting system of bosons and spins. Qualitatively, the results are identical to those of refs. 9 and 10.

As an illustration, in the homogeneous case, one has

$$\|U_\beta^\rho\| = \rho\Lambda \begin{cases} \frac{1}{2}\beta & , \text{ if } \varepsilon=0 \\ \tanh(\frac{1}{2}\beta|\varepsilon|)/|\varepsilon| & , \text{ if } \varepsilon\neq 0 \end{cases} ,$$

and thus, as in ref. 10,

$$\beta_c = \begin{cases} 2\operatorname{arctanh}(|\varepsilon|/\rho\Lambda)/|\varepsilon| & , \text{ if } \varepsilon\neq 0 \text{ and } |\varepsilon| < \rho\Lambda \\ +\infty & , \text{ if } \varepsilon\neq 0 \text{ and } |\varepsilon| \ge \rho\Lambda \\ 2/(\rho\Lambda) & , \text{ if } \varepsilon=0 \end{cases} .$$

Finally, one can proceed as in ref. 6, to obtain the thermodynamic limit of the equilibrium expectation of the average spin-polarization in x-direction when $h(t)=\hbar$ (by symmetry this limit is zero for $\hbar=0$); and then consider the limit $\hbar \to 0$. The result is qualitatively the same as that for the homogeneous case (10), namely: the limit is zero for $\beta \le \beta_c$, and *not* zero if $\beta > \beta_c$ with different sign depending on whether $\hbar\uparrow 0$ or $\hbar\downarrow 0$.

Appendix A: Discussion of $\inf_{x\in\mathbb{R}^n} \tilde{f}_n^s(h;x)$

Lemma A1 : *Let* I *on* [0,1] *be defined as in Theorem 1. Then,*

$$\inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x) = \inf_{\substack{r_j\in[0,1]\\ \vartheta_j\in[0,2\pi]}} \left\{ V_n^{-1} \sum_{j=1}^{n} \left\{-\beta^{-1} I(r_j) + \tfrac{1}{2}\epsilon_n(j) r_j \cos(\vartheta_j) \right.\right.$$

$$\left.\left. + \tfrac{1}{2} h_j r_j \sin(\vartheta_j) - \tfrac{1}{4} V_n^{-1} \sum_{k=1}^{n} \Lambda_n(j,k) r_j r_k \sin(\vartheta_j) \sin(\vartheta_k) \right\}\right\}$$

$$= \inf_{\substack{r_j\in[0,1]\\ \vartheta_j\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]}} \left\{ V_n^{-1} \sum_{j=1}^{n} \left\{-\beta^{-1} I(r_j) - \tfrac{1}{2}|\epsilon_n(j)| r_j \cos(\vartheta_j) \right.\right.$$

$$\left.\left. + \tfrac{1}{2} h_j r_j \sin(\vartheta_j) - \tfrac{1}{4} V_n^{-1} \sum_{k=1}^{n} \Lambda_n(j,k) r_j r_k \sin(\vartheta_j) \sin(\vartheta_k) \right\}\right\} .$$

<u>Proof</u> : One verifies that for a and b real,

$$\inf_{\substack{r\in[0,1]\\ y^2+z^2=1}} \{-\beta^{-1} I(r) + \tfrac{1}{2} arz + \tfrac{1}{2} bry\} = -\beta^{-1}\log(2\cosh(\tfrac{1}{2}\beta[a^2+b^2]^{\frac{1}{2}})) .$$

Thus, by Lemma 1,

$$\hat{f}_n^s(h;x) = V_n^{-1} \inf_{\substack{r_j\in[0,1]\\ z_j^2+y_j^2=1}} \sum_{j=1}^{n} \left\{-\beta^{-1} I(r_j) + \tfrac{1}{2}\epsilon_n(j) r_j z_j \right.$$

$$\left. + \tfrac{1}{2} r_j y_j \left[h_j - 2\sum_{k=1}^{n} \Lambda_n(j,k) x_k\right]\right\} + x\Lambda_n x .$$

The variation over $x \in \mathbb{R}^n$ can be done explicitely (for this, it is convenient to diagonalize Λ_n); it follows that

$$\inf_{x\in\mathbb{R}^n} \hat{f}_n^s(h;x) = V_n^{-1} \inf_{\substack{r_j\in[0,1]\\ z_j^2+y_j^2=1}} \sum_{j=1}^{n} \left\{-\beta^{-1} I(r_j) + \tfrac{1}{2}\epsilon_n(j) r_j z_j \right.$$

$$+ \tfrac{1}{2}h_j r_j y_j - \tfrac{1}{4}V_n^{-1} \sum_{k=1}^{n} r_j r_k y_j y_k \Lambda_n(j,k) \Big\} \quad ,$$

which proves the first claim upon setting $z_j = \cos(\vartheta_j)$, $\vartheta_j \in [0,2\pi]$. The second claim is obvious.■

<u>Appendix</u> <u>B</u> : Solution of the variational problem following Duffield and Pulè [6].

Write $\mathcal{S}$ for $\mathcal{S}(0)$, and $\mathcal{V}(s)$ for $\mathcal{V}(s;0)$.

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>3</u> : This is a minor adjustment of the corresponding result of ref. 6, to accomodate the fact that our variation is over $\mathcal{B}$ and not its positive part. Let A be the support of ε. For arbitrary $s \in \mathcal{B}$ and $0 < p < 1$, put $F(p)=\mathcal{V}(ps)$. F is differentiable with derivative (integrals with unspecified domain are over [0,1])

$$F'(p) = \tfrac{1}{2}p\rho \iint\Lambda(t,t')s(t)s(t')dtdt'$$

$$- \tfrac{1}{2}p \int_A |\varepsilon(t)|s(t)^2[r_{ps}(t)^2-p^2s(t)^2]^{-\frac{1}{2}}dt$$

$$-\beta^{-1}\int_{A^c}\operatorname{arctanh}(p|s(t)|)|s(t)|dt \quad .$$

Using the inequalities

$$|s(t)|\operatorname{arctanh}(p|s(t)|) \ge p\, s(t)^2 \quad ,$$

$$[r_s(t)^2-s(t)^2]^{\frac{1}{2}} \le \tanh(\tfrac{1}{2}\beta|\varepsilon(t)|) \quad ,$$

one obtains $F'(p) \le \tfrac{1}{2}p\langle\bar{s},(U_\beta^\rho-1)\bar{s}\rangle_{L^2_{\mathbb{R}}([0,1])}$, where $\bar{s}(t)=s(t)/g_\beta(t)$. The assumption $\|U_\beta^\rho\|\le 1$ implies $F'(p) \le 0$, so that $\mathcal{V}(ps) \le \mathcal{V}(0)$, and by continuity $\mathcal{V}(s) \le \mathcal{V}(0)$. $\mathcal{V}(0)$ can be computed using $r_0(t)=\tanh(\tfrac{1}{2}\beta|\varepsilon(t)|)$.■

The proof of Theorem 4 is broken up into a series of lemmas all of which have their origins in ref. 6.

<u>Lemma B1</u> : *There exists* $s \in \mathcal{B}$ *such that* $\mathcal{S}(h)=\mathcal{V}(s;h)$.

<u>Proof</u> : See Theorem 5 of ref. 6.■

<u>Lemma B2</u> : *If* $\|U_\beta^\rho\| > 1$ *then* $\mathcal{S} > \mathcal{V}(0)$.

<u>Proof</u> : Let $s \in \mathcal{B}$ with $\mathcal{V}(s)=\mathcal{S}$. Since U_β^ρ is compact, $\|U_\beta^\rho\|$ is an eigenvalue; let ξ be a corresponding eigenvector. Define $\xi_n \in L_{\mathbb{R}}^\infty([0,1])$ by

$$\xi_n(t) = \begin{cases} \xi(t) & , \text{ if } |\xi(t)| \leq n \\ 0 & , \text{ otherwise} \end{cases} \quad , \text{ a.e.} \quad .$$

It follows that $\langle \xi_n, (U_\beta^\rho - 1)\xi_n \rangle_{L_{\mathbb{R}}^2([0,1])} \to \|U_\beta^\rho\| - 1$ (>0 !) as $n \to \infty$. Choose m such that $\langle \xi_m, (U_\beta^\rho - 1)\xi_m \rangle_{L_{\mathbb{R}}^2([0,1])} > 0$, and let $\hat{s}=\xi_m g_\beta$. The proof then proceeds as in Lemma 3 of ref. 6 ■

<u>Lemma B3</u> : *If* $s \in \mathcal{B}$ *and* $\mathcal{S} = \mathcal{V}(s)$, *then* $\{t\in[0,1]:|s(t)|=1\}$ *has zero measure.*

<u>Proof</u> : Proceed as in the proof of Lemma 2 of ref. 6, with the set $\{t\in[0,1]:|s(t)|=1\}$.■

<u>Lemma B4</u> : *If* $s \in \mathcal{B}$, *and* $\mathcal{S} = \mathcal{V}(s)$, *then* $\Phi_\beta^\rho(s;\circ)=0$.

<u>Proof</u> : This is an adaptation of the proof of Theorem 6 of ref. 6. Let $0 < \delta < 1$, and take $\xi \in L_{\mathbb{R}}^\infty([0,1])$ with essential support contained in $A_\delta \equiv \{t\in[0,1]:|s(t)|<1-\delta\}$. For $|p|$ sufficently small, $s_p = s[1+p\xi]$ lies in $\mathcal{B}$. Let $F(t)=\mathcal{V}(s_p)$. Taking the derivative at $p=0$, one obtains

$$\frac{1}{2}\int_{A_\delta} \xi(t)s(t)\Phi_\beta^\rho(s;t)dt = 0 \quad . \tag{*}$$

Now take $\xi = s\Phi_\beta^\rho(s;\circ)$ on A_δ, and $\xi=0$ on A_δ^c ; (*) implies that $s\Phi(s;\circ)=0$ on A_δ. Since δ was arbitrary, Lemma B3 implies that $s\Phi_\beta^\rho(s;\circ)=0$. Thus, $\Phi_\beta^\rho(s:\circ)=0$ on B, the essential support of s; but by the definition of $\Phi_\beta^\rho(s;\circ)$, $\Phi_\beta^\rho(s;\circ)=0$ on B^c.■

The first part of Theorem 4 follows from Lemmas B2-B4; the rest of the claim follows as in ref. 6.

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