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Towards Time - Dynamics for Bosonic Systems in Quantum Statistical Mechanics

A.G. Shuhov[1], Yu.M. Suhov[1,2], and A.V. Teslenko[1]

Abstract:

Consider a one-dimensional lattice boson system with the Hamiltonian in a finite box Λ, $H_\Lambda = K_\Lambda + U_\Lambda$. Here K_Λ is the kinetic energy (the discrete Laplacian) and U_Λ is the potential energy corresponding to a finite-range pair interaction. For a class of states φ of the infinite system, we prove the existence of the limit $\varphi_t(A) = \lim_{\Lambda \nearrow \mathbb{Z}^1} \varphi(e^{itH_\Lambda} A e^{-itH_\Lambda})$ for any $t \in \mathbb{R}^1$ and any local observable A. Thereby a family $\{\varphi_t, t \in \mathbb{R}^1\}$ of locally normal states is determined which describes the time-evolution of the initial state φ.

Key words and phrases: one-dimensional lattice boson system, diagonal state, time-evolution, path integrals.

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[1] Institute for Problems of Information Transmission,
USSR Academy of Sciences, 19 Yermolova St.,
GSP - 4 Moscow, 101447 USSR

[2] School of Theoretical Physics,
Dublin Institute for Advanced Studies,
10, Burlington Road,
Dublin 4,
Ireland

1. Introduction

The problem of constructing time-dynamics for an infinitely extended system is a major one in statistical mechanics, both classical and quantum. So far, in quantum statistical mechanics this problem has been solved in a satisfactory way for a particular class of systems only, namely, for quantum spin systems. Here Robinson's theorem [1] (see also [2]) asserts that, given a "reasonable" interaction potential of a general form, there exists the corresponding strongly continuous $*$- automorphism group of the quasilocal C* - algebra. This provides, in particular, an elegant definition of an equilibrium state via the KMS boundary condition.

However, for other types of quantum systems of interest, e.g., for interacting particle systems in a Euclidean space, the problem remains open (some results, both of a positive and a negitive character, are available for free systems; see, e.g., [3], [4]). A wide-spread opinion is that the "traditional" C* -algebras (the CAR, the CCR and the quasilocal algebras) are not appropriate for this purpose; by analogy with the classical case, it is believed that trouble may arise from singularities whichoccur in a system with infinitely many degrees of freedom , such as uncontrolled "collapces" or "accelerations" of particles.

From this point of view, it seems natural to consider the time-evolution of a state rather than the time-dynamics on an algebra of observables. One such version based on time-dependent Green's functions was elaborated in [5], [6] for equilibrium states and in [7] in a general set-up. However, to check the assumptions on an initial state φ which were formulated in [7] is not an easy matter.

The present paper deals with an alternative approach to the problem. For an appropriate initial state φ we construct directly the time-evolved state φ_t by setting

$$\varphi_t(A) = \lim \varphi(e^{itH_\Lambda} A e^{-itH_\Lambda}) \tag{1.1}$$

for any local observable A. We consider a system of interacting bosons on the one-dimensional lattice $\mathbb{Z}^1$.

The (formal) Hamiltonian H of the infinite system is

$$H = -\tfrac{1}{2} \sum_{j \in \mathbb{Z}^1} a_j^+ (\Delta a)_j + \sum_{j, j' \in \mathbb{Z}^1} \Phi(|j-j'|)\, n_j\, n_{j'}, \tag{1.2}$$

where Δ stands for the second difference operator,

$$(\Delta a)_j = \tfrac{1}{2}\,(a_{j+1} + a_{j-1} - 2a_j), \; j \in \mathbb{Z}^1,$$

a_k^+ and a_k are respectively, the creation and annihilation operators at a point $k \in \mathbb{Z}^1$, Φ is a real-valued function on $\mathbb{Z}_+^1$, the set of non-negative integers, with bounded support and $n_k = a_k^+ \, a_k$ is the particle number operator at a point $k \in \mathbb{Z}^1$. In principle, one can admit infinite values for the potential Φ (hard-core type interaction), but to emphasize the "bosonic" character of the system under consideration, we shall assume that $|\Phi(r)| < \infty$ for any $r \in \mathbb{Z}_+^1$.

Given a finite "box" Λ (an interval, or a segment of the lattice $\mathbb{Z}^1$), one can consider the finite-volume version H_Λ of the Hamiltonian (1.2); more precisely, one takes the self-adjoint extension corresponding to a standard boundary condition on $\partial\Lambda$, and for definiteness we shall deal with the Dirichlet condition. Then the * - automorphism group

$$A \mapsto e^{itH_\Lambda} A e^{-itH_\Lambda}, \; t \in \mathbb{R}^1, \tag{1.3}$$

determines the time-dynamics of an observable A. For a class of states φ which we call diagonal, we establish the existence of the limit (1.1), as $\Lambda \nearrow \mathbb{Z}^1$, for any local observable A. This yields the family $\{\varphi_t, \, t \in \mathbb{R}^1\}$ of locally normal states of the C* -algebras of the infinite system which describes the time-evolution of the initial state φ.

In a separate paper we shall prove that the family $\{\varphi_t\}$ provides a unique solution of the infinite-volume Liouville equation as well as a solution to the corresponding BBGKY hierarchy.

The condition of diagonality on the initial state is used to simplify technicalities; we hope to drop it in a later publication.

The method of proof relies heavily on the concrete form (1.2) of the Hamiltonian as well as on the one-dimensional character of the system. The problem of the recovering dynamics of continuous systems of interacting quantum particles by passing to the limit when the lattice spacing tends to zero is reserved for future study.

In conclusion we remark that the main idea of this paper is inspired by Sinai's approach to the construction of a cluster-dynamics for classical systems (see [8] , [9]).

2. Preliminaries and results [1]

For any $k \in \mathbb{Z}^1$ consider a copy $\mathcal{H}_k$ of the separable Hilbert space $\ell_2(\mathbb{Z}^1_+)$ with standard orthonormal basis $\{e^{(k)}_s, s \in \mathbb{Z}^1_+\}$. Letting $\mathcal{B}_k$ be the C* -algebra of bounded operators in $\mathcal{H}_k$, we set $\mathcal{B} = \otimes_j \mathcal{B}_j$. The creation and annihilation operators a^+_k and a_k act in $\mathcal{H}_k$ as $a^+_k e^{(k)}_s = (s+1)^{1/2} e^{(k)}_{s+1}$ and $a_k e^{(k)}_s = s^{1/2} e^{(k)}_{s-1}$, $s \in \mathbb{Z}^1_+$ ($a_k e^{(k)}_0$ is set to be zero), $k \in \mathbb{Z}^1_+$. Given a finite lattice interval $\Lambda \subset \mathbb{Z}^1$, the Hamiltonian H_Λ acts in the Hilbert space $\mathcal{H}_\Lambda = \otimes_{k\in\Lambda} \mathcal{H}_k$ and is defined via (1.2) with the Dirichlet boundary condition. Hence, $e^{\pm itH_\Lambda}$ is unitary and (1.3) defines the *-automorphism group on $\mathcal{B}$, the time-dynamics in the volume Λ.

A locally normal state φ of the C* -algebra $\mathcal{B}$ is determined by its values $\varphi(E^{(\Lambda^0)}_{x,x'})$; here Λ^0 is an arbitrary bounded interval of the lattice, x and x′ stand for occupation number configurations in Λ^0, i.e., for functions $\Lambda^0 \to \mathbb{Z}^1_+$ (we shall use notation $x, x' \dot{\subset} \Lambda^0$) and $E^{(\Lambda^0)}_{x,x'}$ is the corresponding matrix unit in the standard orthonormal basis $\{e^{(\Lambda^0)}_y, y \dot{\subset} \Lambda^0\}$ in $\mathcal{H}_{\Lambda^0}$:

$$E^{(\Lambda^0)}_{x,x'} e^{(\Lambda^0)}_y \begin{cases} = e^{(\Lambda^0)}_x, \text{ provided that } y = x', \\ = 0, \text{ otherwise.} \end{cases} \qquad (2.1)$$

A locally normal state φ is called gauge-invariant if and only if $\varphi(E^{(\Lambda^0)}_{x,x'}) = 0$ provided $|x| \neq |x'|$ where $|y|$ is the total particle number in y: $|y| = \sum_j y(j)$. Likewise, φ is called diagonal if and only if $\varphi(E^{(\Lambda^0)}_{x,x'}) = 0$ provided $x \neq x'$.

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[1] A detailed C* -algebra background for this section may be found in [10] . For details of a probabilistic character see, e.g., [11] .

Diagonal states are in one-to-one correspondence with probability measures on the measurable space $(\mathcal{M},\mathfrak{M})$ where $\mathcal{M}$ is the space of occupation number configurations in $\mathbb{Z}^1$, i.e., of functions $X : \mathbb{Z}^1 \to \mathbb{Z}^1_+$, and $\mathfrak{M}$ is the σ- algebra of subsets of $\mathcal{M}$ generated by "cylinders" $B_x^{(\Lambda^0)}$:

$$B_x^{(\Lambda^0)} = \{ X \in \mathcal{M} : X \upharpoonright \Lambda^0 = x \}.$$

The correspondence between a diagonal state φ and the measure on $(\mathcal{M},\mathfrak{M})$, which we denote by the same symbol φ, is established by the formula

$$\varphi(E_{x,x}^{(\Lambda^0)}) = \varphi(B_x^{(\Lambda^0)}).$$

A simple, but useful, example of a probability measure on $(\mathcal{M},\mathfrak{M})$ (and hence, of a diagonal state) is the product (Bernoulli) measure $\times_j p^{(j)}$ where $p^{(j)}(=p)$ is a fixed probability distribution on $\mathbb{Z}^1_+$ with $p(0) > 0$ (e.g., a geometric distribution : $p(s) = (1-q)\, q^s$, $s \in \mathbb{Z}^1_+$, where $q \in (0,1)$).

A Bernoulli measure has the following property which will play an important role in the sequel : let $\{ S_m, m \in \mathbb{Z}^1_+ \}$ be a monotone increasing sequence of positive integers and set

$$I_m = [-S_{m+1}, S_{m+1}], \qquad (2.2)$$
$$I_m^- = [-S_{m+1}-1, -S_m],\ I_m^+ = [S_m, S_{m+1}-1]$$

(all the space intervals considered here and below are on the lattice $\mathbb{Z}^1$). Denote $\gamma_{m+1} = S_{m+1} - S_m$ and consider the event (from $\mathfrak{M}$) that neither on I_m^- nor on I_m^+ can one find a subinterval of length greater than or equal α_m, $\alpha_m < \gamma_m$, which is free of particles. The probability of that event does not exceed

$$2\,(1-p(0)^{\alpha_m})^{\gamma_m/\alpha_m} \leqslant 2 \exp(-p(0)^{\alpha_m}\gamma_m/\alpha_m).$$

If we assume that $\gamma_m/\alpha_m \; p(0)^{\alpha_m} \nearrow \infty$ as $m \nearrow \infty$ so that

$$\sum_{m \geqslant 1} \exp(-p(0)^{\alpha_m}\gamma_m/\alpha_m) < \infty,$$

then, by the Borel-Cantelli Lemma, for φ-a.e. $X \in \mathcal{M}$ one can find $m_0 = m_0(X)$ with the property that for any $m \geqslant m_0$ there exist subintervals $J^{\pm} \subset I_m^{\pm}$ of length greater than or equal to α_m such that $X(j) = 0$ for $j \in J^- \cup J^+$. Moreover, the following bound holds:

$$\varphi(m_0(X) \geq \bar{m}) \leq 2 \sum_{m \geq \bar{m}} \exp(-\gamma_m/_{\alpha_m} p(0)^{\alpha_m}), \quad \bar{m} \in \mathbb{Z}^1_+ . \tag{2.3}$$

Now consider an arbitary measure φ on $(\mathcal{M},\mathfrak{M})$. We shall suppose that the foregoing conditions hold for φ with $S_m = 2^{m+1}$, $\gamma_m = 2^m$ and $\alpha_m = [m^\delta]$, where $\delta \in (0.1)$ is fixed. In that case we denote the property under consideration by (d*). In probabilistic terms property (d*) means that, for a measure φ,

$$\varphi\Big(\bigcup_{l=1}^{\infty} \bigcap_{m \geq l} \mathcal{M}(m)\Big) = \lim_{l \to \infty} \varphi\Big(\bigcap_{m \geq l} \mathcal{M}(m)\Big) = 1 \tag{2.4}$$

Here $\mathcal{M}$(m) stands for the event

$$\{ X \in \mathcal{M} : \exists \text{ intervals } J^{\pm} = J^{\pm}(X,m) \subseteq I^{\pm}_m, \text{ of length} \geq [m^\delta], \text{ such that } X(j) = 0 \ \forall j \in J^- \cup J^+ \}. \tag{2.5}$$

In what follows we shall think of $J^{\pm}$ as the longest intervals possessing the properties listed (if such intervals are not unique, we take the left-most (resp., the right-most) of them). So far we have checked that the property (d*) holds for Bernoulli measures but the class of probability measures on $(\mathcal{M},\mathfrak{M})$ for which (d*) holds is in fact much larger; it includes positively recurrent Markov chains and DLR measures corresponding to classical superstable interactions (not necessary of a finite range).

Another property of an initial state which will be used heavily in the sequel is again inspired by the example of Bernoulli measures. This property is denoted below by (d**) and is a combination of the two conditions, (d_1^{**}) and (d_2^{**}).

(d_1^{**}) There exists constants c, $\sigma, b > 0$ and a value s $\in \mathbb{Z}^1_+$ such that for any bounded interval $\Lambda \subset \mathbb{Z}^1$ with $|\Lambda| \geq s_0$ we have

$$\varphi(\Pi^{(\Lambda)}_{\geq s}) \leq c \int_{\bar{s}}^{\infty} du \exp(-u^2/2), \tag{2.6}$$

where

$$\bar{s} = (\sigma |\Lambda|^{1/2})^{-1} (s - b|\Lambda|)_+$$

and $\Pi^{(\Lambda)}_{\geqslant s}$ is the orthogonal projection in $\mathcal{H}_\Lambda$ onto the subspace generated by occupation number configurations $x \dot{\subset} \Lambda$ with $|x| \geqslant s$.

(d_2^{**}) There exist constants $c_1 > 0$ and $\rho \in (0,1)$ such that for any pair of finite occupation number configurations y, y' we have:

$$\varphi(\bar{B}_{y+y'}) \leqslant c_1^{R(y)} \rho^{|y|} \varphi(\bar{B}_{y'}), \tag{2.7}$$

where $R(y) = \sum_j \chi(y(j) \geqslant 1)$ and we have used the notation

$$\bar{B}_x = \{X \in \mathcal{M} : X \geqslant x\} ;$$

the sum $y + y'$ means here and below summation of functions.

As before, the conditions (d_1^{**}) and (d_2^{**}) hold for a large class of probability measures on $(\mathcal{M}, \mathfrak{M})$. Property ($d_1^{**}$) establishes a kind of limit theorem estimation while (d_2^{**}) gives a kind of stability bound, both are natural from the point of view of statistical mechanics.

Remark: We have stated properties (d^*) and (d^{**}) in a form chosen to emphasize their probabilistic character; as will be seen from what follows, this is not necessary. For instance, the inequality (2.6) may be replaced with the following relation: for any bounded interval $\Lambda^0 = [v^0, v^1] \subset \mathbb{Z}^1$

$$\lim_{s \to \infty} \varphi(\Pi^{(\Lambda^0([s^{1/2}]))}_{\geqslant s}) = 0,$$

where $\Lambda^0(r) = [v^0 - r, v^1 + r]$, $r \in \mathbb{Z}^1_+$.

We are now able to formulate the results of the paper:

<u>Theorem</u> 1. Suppose that a diagonal state φ has property (d^*). Then, for any bounded interval $\Lambda^0 \subset \mathbb{Z}^1$, any pair of occupation number configurations $x, x' \dot{\subset} \Lambda^0$ and any $t \in \mathbb{R}^1$ the following limit exists:

$$\varphi_t(E^{(\Lambda^0)}_{x,x'}) = \lim_{\Lambda \nearrow \mathbb{Z}^1} \varphi_{\Lambda,t}(E^{(\Lambda^0)}_{x,x'}), \tag{2.8}$$

where

$$\varphi_{\Lambda,t}(A) = \varphi(e^{itH_\Lambda} A e^{-itH_\Lambda}), \; A \in \mathcal{B}. \; \triangleleft^{1)} \tag{2.9}$$

1) The symbol $\triangleleft$ indicates the end of a statement. The end of a proof is indicated by the symbol $\square$.

Theorem 2 Suppose that a diagonal state φ has property (d**). Then, for any bounded interval $\Lambda^0 \subset \mathbb{Z}^1$ and any $t \in \mathbb{R}^1$ the following relation holds:

$$\lim_{s\to\infty} \sup_{\Lambda \supset \Lambda^0} \varphi_{\Lambda,t}(\Pi_{\geq s}^{(\Lambda^0)}) = 0. \quad \triangleleft \tag{2.10}$$

From these two statements we derive immedately the following result :

Theorem 3 Suppose that a diagonal state φ has both properties (d*) and (d**). Then, given $t \in \mathbb{R}^1$, for any local $A \in \mathfrak{B}$ the following limit exists:

$$\varphi_t(A) = \lim_{\Lambda \nearrow \mathbb{Z}^1} \varphi_{\Lambda,t}(A), \tag{2.11}$$

where $\varphi_{\Lambda,t}(A)$ is defined by (2.9). The limit (2.11) determines a locally normal state φ_t of the C*-algebra $\mathfrak{B}$. $\triangleleft$

Proof of Theorem 3 (given Theorems 1 and 2): Let A be localized in an interval $\Lambda^0 \subset \mathbb{Z}^1$. Writing

$$\varphi_{\Lambda,t}(A) = \varphi_{\Lambda,t}(\Pi_{\geq s}^{(\Lambda^0)} A \Pi_{\geq s}^{(\Lambda^0)}) + \varphi_{\Lambda,t}((\mathbb{1} - \Pi_{\geq s}^{(\Lambda^0)}) A \Pi_{\geq s}^{(\Lambda^0)}) +$$

$$+ \varphi_{\Lambda,t}(\Pi_{\geq s}^{(\Lambda^0)} A (\mathbb{1} - \Pi_{\geq s}^{(\Lambda^0)})) + \varphi_{\Lambda,t}((\mathbb{1} - \Pi_{\geq s}^{(\Lambda^0)}) A (\mathbb{1} - \Pi_{\geq s}^{(\Lambda^0)})),$$

we estimate from above the first term in the RHS by $\varphi_{\Lambda,t}(\Pi_{\geq s}^{(\Lambda^0)})$ and the second and third ones by $(\varphi_{\Lambda,t}(\mathbb{1} - \Pi_{\geq s}^{(\Lambda^0)}) \varphi_{\Lambda,t}(\Pi_{\geq s}^{(\Lambda^0)}))^{1/2}$. Theorem 2 asserts that all these estimates may be made small uniformly in Λ whereas Theorem 1 says that the fourth term has a limit when $\Lambda \nearrow \mathbb{Z}^1$. Hence, the whole RHS of (2.11) tends to a limit.

The problem of verifying that φ_t is a locally normal state is reduced to checking the equality

$$\sum_{x \dot{\subset} \Lambda^0} \varphi_t\left(E_{x,x}^{(\Lambda^0)}\right) = 1 \tag{2.12}$$

for any bounded interval $\Lambda^0 \subset \mathbb{Z}^1$. To do this, we again use Theorem 2 for estimating the difference

$$1 - \varphi_{\Lambda,t}\left(\mathbb{1} - \Pi_{\geqslant s}^{(\Lambda^0)}\right)$$

and then Theorem 1 for doing the same for

$$1 - \sum_{x \dot{\subset} \Lambda^0 : |x| < s} \varphi_t\left(E_{x,x}^{(\Lambda^0)}\right).$$

These estimates vanish when $s \to \infty$. This yields the assertion of Theorem 3. □

The proofs of Theorems 1 and 2 are based on the representation of the matrix elements $(e^{\pm itH_\Lambda})_{y,y'}$ in terms of integrals over the paths of a Markov jump process on $\mathbb{Z}^1$. Precisely, let P_j^τ, $j \in \mathbb{Z}^1$, $\tau \geqslant 0$, denote the path distribution up to time τ of the process starting at the point j which, after spending the mean one exponential time at a given site, jumps to one of the nearest neighbour sites, each with probability $\frac{1}{2}$.

Suppose a triple of occupation number configurations z, $y, y' \dot{\subset} \Lambda$ with $|y| = |y'|$ is given. By $\mathcal{O}_{y,y'}$ we denote the set of all matchings between y and y'. Every matching is identified with a function $\Gamma: \Lambda^0 \times \Lambda^0 \to \mathbb{Z}_+^1$ with the following properties: (i) $\Gamma(j,j') = 0$ provided $y(j)y'(j') = 0$, (ii) $\sum_{k \in \Lambda^0} \Gamma(j,k') = y(j)$ and $\sum_{k \in \Lambda^0} \Gamma(k,j') = y(j')$ for any $j, j' \in \Lambda^0$. Given $\tau \geqslant 0$, let P_y^τ denote the product measure $P_y^\tau = \times_{j: y(j) \geqslant 1} (P_j^\tau)^{\times y(j)}$. This describes the development up to time τ of $|y|$ copies of the process starting from the points occupied by y. Futhermore, given a matching $\Gamma \in \mathcal{O}_{y,y'}$ denote by W_Γ^τ the set of families $\Omega = \{\omega\}$ of paths which start at the points occupied by y and arrive, at the epoch τ, at the corresponding points occupied by y'. Next, given a family $\Omega = \{\omega\}$, we set

$$N(\Omega) = \sum_{\omega \in \Omega} N(\omega), \quad \chi_\Lambda(\omega) = \prod_{\omega \in \Omega} \chi_\Lambda(\omega), \tag{2.13}$$

$$\mathcal{V}(\Omega|z) = \sum_{\substack{\omega,\omega'\in\Omega:\\ \omega\neq\omega'}} \mathcal{V}(\omega,\omega') + \sum_{\substack{\omega\in\Omega,\\ k\in\Lambda}} \mathcal{V}(\omega,k)\, z(k), \tag{2.14}$$

where

$$N(\omega) \text{ is the number of jumps of a path } \omega, \tag{2.15}$$

$$\begin{aligned} \chi_\Lambda(\omega) &= 1, \text{ if } \omega(u)\in\Lambda \text{ for any } u\in[0,\tau],\\ &= 0, \text{ otherwise,} \end{aligned} \tag{2.16}$$

$$\mathcal{V}(\omega,\omega') = \int_0^\tau du\, \Phi(|\omega(u)-\omega'(u)|), \tag{2.17a}$$

$$\mathcal{V}(\omega,k) = \int_0^\tau du\, \Phi(|\omega(u)-k|). \tag{2.17b}$$

Finally, given a pair, $\Omega=\{\omega\}, \Omega'=\{\omega'\}$, of families of paths we set

$$\begin{aligned} \mathcal{V}(\Omega\cup\Omega'|z) = &\sum_{\substack{\omega,\omega'\in\Omega:\\ \omega\neq\omega'}} \mathcal{V}(\omega,\omega') + \sum_{\substack{\omega,\omega'\in\Omega':\\ \omega\neq\omega'}} \mathcal{V}(\omega,\omega') +\\ &+ \sum_{\omega\in\Omega,\omega'\in\Omega'} \mathcal{V}(\omega,\omega') + \sum_{\omega\in\Omega\cup\Omega', k\in\Lambda} \mathcal{V}(\omega,k). \end{aligned} \tag{2.18}$$

For later use, it is convenient to consider Hamiltonians of a slightly more general form-those which include a term representing an external field generated by a fixed occupation number configuration. Precisely, let $H_{\Lambda,z}$ denote the operator in $\mathcal{H}$ given by

$$\begin{aligned} H_{\Lambda,z} = &-\tfrac{1}{2}\sum_{j\in\Lambda} a_j^+(\Delta a)_j + \sum_{j,j'\in\Lambda} \Phi(|j-j'|)\, n_j n_{j'} +\\ &+ \sum_{j,k\in\Lambda} \Phi(|j-k|)\, n_j\, z(k), \end{aligned} \tag{2.19}$$

and $\tilde{H}_{\Lambda,z}$ be the operator in $\mathcal{H}_\Lambda\otimes\mathcal{H}_\Lambda$ of the form

$$\tilde{H}_{\Lambda,z} = H_{\Lambda,z}\otimes \mathbb{1} + \mathbb{1}\otimes H_{\Lambda,z} + V, \tag{2.20}$$

where V is the cross interaction energy

$$V = \sum_{k,k' \in \Lambda} \Phi(|k-k'|)\, n_k \otimes n_{k'} . \tag{2.21}$$

Lemma 2.1 Given a bounded interval $\Lambda \subset \mathbb{Z}^1$, a quadruple of occupation number configurations y, y', y_1, y_1' in Λ with $|y| = |y'|$, $|y_1| = |y_1'|$ and $t \geq 0$, we have:

a) the matrix elements of the operators $e^{\pm itH_{\Lambda,z}}$ are given by

$$(e^{\pm itH_{\Lambda,z}})_{y,y'} = e^{t(1\pm i)|y|} \sum_{\Gamma \in \mathcal{O}_{y,y'}} \int_{W^t_\Gamma} P^t_y(d\Omega)\chi_\Lambda(\Omega) \times$$
$$\times\, i^{\mp N(\Omega)} \exp(\pm i U(\Omega|z)), \tag{2.22}$$

b) the matrix elements of the operators $e^{\pm it\tilde{H}_{\Lambda,z}}$ are given by

$$(e^{\pm it\tilde{H}_{\Lambda,z}})_{(y,y_1),(y',y_1')} = e^{t(1\pm i)(|y|+|y_1|)} \sum_{\Gamma \in \mathcal{O}_{y,y'},\, \Gamma_1 \in \mathcal{O}_{y_1,y_1'}} \times$$
$$\int_{W^t_\Gamma} P^t_y(d\Omega) \int_{W^t_{y_1}} P^t_y(d\Omega_1)\chi_\Lambda(\Omega)\chi_\Lambda(\Omega_1)\, i^{\mp(N(\Omega)+N(\Omega_1))} \exp(\pm i U(\Omega\cup\Omega_1|z)) . \triangleleft \tag{2.23}$$

Notice that if $|y| \neq |y'|$ or $|y_1| \neq |y_1'|$, the matrix elements of $e^{\pm itH_{\Lambda,z}}$ and $e^{\pm it\tilde{H}_{\Lambda,z}}$ vanish due to the gauge invariance of the Hamiltonians $H_{\Lambda,z}$ and $\tilde{H}_{\Lambda,z}$.

The proof of Lemma 2.1 is standard: Observe that for each choice of sign the RHS's of (2.22) and (2.23) obey the operator semigroup rule and then calculate the infinitesimal generators.

3. Proof of Theorem 1

For definiteness, assume that $t > 0$. Notice that the RHS of (2.8) vanishes if $|x| \neq |x'|$. Hence, we can assume that $|x| = |x'|$. Denote

$$\varphi^0_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) = \varphi(\Pi^{(\Lambda)}(m) e^{itH_\Lambda} E^{(\Lambda^0)}_{x,x'} e^{-itH_\Lambda} \Pi^{(\Lambda)}(m)), \tag{3.1}$$

where $\Pi^{(\Lambda)}(m)$ is the orthogonal projector in $\mathcal{H}_\Lambda$ onto the subspace generated by those basis vectors $e_y^{(\Lambda)}$ for which the occupation number configuration y belong to $\mathcal{M}(m)$ (see (2.5)).

Lemma 3.1. The following equality holds true:

$$\lim_{m\to\infty} \sup_{\Lambda\supset I_m} |\varphi_{\Lambda,t}(E_{x,x'}^{(\Lambda^0)}) - \varphi^0_{\Lambda,t,m}(E_{x,x'}^{(\Lambda^0)})| = 0, \tag{3.2}$$

where I_m is defined in (2.2). ◁

Proof. Let $\Lambda\supset I_m$ (this will be always assumed in the proof of intermediate assertions which follow). We have

$$\begin{aligned}|\varphi_{\Lambda,t}(E_{x,x'}^{(\Lambda^0)}) - \varphi^0_{\Lambda,t,m}(E_{x,x'}^{(\Lambda^0)})| &\leq |\varphi((\mathbb{1}-\Pi^{(\Lambda)}(m))e^{itH_\Lambda}\times \\ E_{x,x'}^{(\Lambda^0)}e^{-itH_\Lambda}(\mathbb{1}-\Pi^{(\Lambda)}(m)))| &+ |\varphi((\mathbb{1}-\Pi^{(\Lambda)}(m))e^{itH_\Lambda}\times \\ E_{x,x'}^{(\Lambda^0)}e^{-itH_\Lambda}\Pi^{(\Lambda)}(m))| &+ |\varphi(\Pi^{(\Lambda)}(m)e^{itH_\Lambda}E_{x,x'}^{(\Lambda^0)}e^{-itH_\Lambda}(\mathbb{1}-\Pi^{(\Lambda)}(m)))|\end{aligned} \tag{3.3}$$

The first term in the RHS of (3.3) is upper-bounded by $\varphi(\mathbb{1}-\Pi^{(\Lambda)}(m))$ whereas both the second and the third ones do not exceed $(\varphi(\mathbb{1}-\Pi^{(\Lambda)}(m))\varphi(\Pi^{(\Lambda)}(m)))^{1/2}$. Finally, we get that the LHS of (3.3) is less than or equal to

$$\varphi(\complement\mathcal{M}(m)) + 2\varphi(\complement\mathcal{M}(m))^{1/2}, \tag{3.4}$$

where $\complement$ denotes the set-theoretical complement. Due to (2.4), this yields the proof. □

The next approximation to $\varphi_{\Lambda,t}(E_{x,x'}^{(\Lambda^0)})$ is provided by the following quantity:

$$\varphi^1_{\Lambda,t,m}(E_{x,x'}^{(\Lambda^0)}) = \varphi(\Pi^{(\Lambda)}(m)D^L_{t,\Lambda,m}E_{x,x'}^{(\Lambda^0)}D^R_{-t,\Lambda,m}\Pi^{(\Lambda)}(m)), \tag{3.5}$$

where (cf. (2.22) for the particular case $z=0$)

$$(D^{L/R}_{\pm t,\Lambda,m})_{y,y'} = e^{t(1\pm i)|y|} \sum_{\Gamma\in\mathcal{O}_{y,y'}} \int_{W^t_\Gamma} P^t_y(d\Omega) \times \tag{3.6}$$

$$\times \chi_\Lambda(\Omega) i^{\mp N(\Omega)} \exp(\pm i U(\Omega))\, \chi^{L/R}_{(m)}(\Omega).$$

New indicators $\chi^L_{(m)}$ and $\chi^R_{(m)}$ are relating to free intervals $J^\pm$ (see (2.5)) . We remind that, because of presence of the projections $\Pi^{(\Lambda)}(m)$ in (3.5), we have $y\in\mathcal{M}(m)$ when deal with D^L and $y'\in\mathcal{M}(m)$ when deal with D^R . So,

$$\chi^{L/R}_{(m)}(\Omega) = \prod_{\omega\in\Omega} \chi^{L/R}_{(m)}(\omega). \tag{3.7}$$

Here, for a given path ω from a family $\Omega\in W^t_\Gamma$, we have set

$$\chi^L_{(m)}(\omega) = 1, \text{ if } \omega(u)\notin(\hat{J}^-\cup\hat{J}^+) \text{ for any } u\in[0,t], \tag{3.8}$$

$$= 0, \text{ otherwise},$$

where the intervals $\hat{J}^\pm \subset J^\pm(y,m)$ (see (2.5)) are defined as follows: if $J^\pm(y,m)=[v_1^\pm, v_2^\pm]$, then

$$\hat{J}^\pm = \left[v_1^\pm + [\tfrac{1}{3}m^\delta],\ v_2^\pm - [\tfrac{1}{3}m^\delta] \right] \tag{3.9}$$

(distinguish parenthesis for the integer part of a positive number from that for an interval of the lattice $\mathbb{Z}^1$). Recall that the lengths $v_2^\pm - v_1^\pm$ of $J^\pm$'s are $\geqslant [m^\delta]$. Likewise,

$$\chi^{R}_{(m)}(\omega) = 1, \text{ if } \omega(u) \notin (\hat{J}^{-} \cup \hat{J}^{+}) \text{ for any } u \in [0, t], \qquad (3.10)$$

$$= 0, \text{ otherwise},$$

where the intervals $\hat{J}^{\pm} \subset J^{\pm}(y,m)$ (see (2.5) again) are defined in the similar way.

Physically speaking, in the integral in the RHS of (3.6) we forbid the paths to go "too far" into free intervals $J^{\pm}$. As a result, the paths which are on one hand side of $J^{\pm}$ will not interact with those which are on the other hand side. Such an interaction breakdown means, as we shall see, that $\varphi^{1}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'})$ depends "very weakly" on Λ.

We now want to estimate the diffeence between $\varphi^{0}_{\Lambda,t,m}$ and $\varphi^{1}_{\Lambda,t,m}$:

Lemma 3.2 The following equality takes place:

$$\lim_{m\to\infty} \sup_{\Lambda \supset I_m} \left| \varphi^{0}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) - \varphi^{1}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) \right| = 0. \triangleleft \qquad (3.11)$$

Proof. Comparing (2.22) (where z is taken to be zero) and (3.6), we shall write the difference between $\varphi_{\Lambda,t,m}$'s in a form where the integration over the trajectories which obey the taboo imposed in (3.6) is separated from that over the trajectories which violate it. Denote by $\mathcal{M}_{\Lambda}$ the set of occupation number configurations in Λ:

$$\mathcal{M}_{\Lambda} = \{ X \in \mathcal{M} : X(j) = 0 \ \ \forall\, j \notin \Lambda \}$$

and set $\mathcal{M}_{\Lambda}(m) = \mathcal{M}_{\Lambda} \cap \mathcal{M}(m)$. To simplify the notations, we set $\varphi(E^{(\Lambda)}_{y,y}) = \varphi^{(\Lambda)}(y)$ and omit the indicator χ_{Λ} in all the integrands. Denoting by $w \restriction \tilde{\Lambda}$ the restriction of a function w to a set $\tilde{\Lambda}$, we have

$$\varphi^{0}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) - \varphi^{1}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) = \sum_{\substack{y^0,y^1,y^2 \in \mathcal{M}_{\Lambda}(m): \\ |y^0|+|y^1|+|y^2| \geq 1}} e^{t(2|y^0|+|y^1|+|y^2|)} \times$$

$$\sum_{\substack{z^1,z^2\in\mathcal{M}_\Lambda:\\ |y^o|+|y^1|=|z^1|,\,|z^2|=|y^2|+|y^o|}} \sum_{\substack{\Gamma^1\in\mathcal{O}_{y^o+y^1,z^1},\\ \Gamma^2\in\mathcal{O}_{z^2,y^2+y^o}}} \sum_{\substack{y_1,y_2\in\mathcal{M}_\Lambda:\\ y^o+y^1+y_1=y_2+y^2+y^o\in\mathcal{M}_\Lambda(m)}} e^{t(|y_1|+|y_2|)}$$

$$\sum_{\substack{z_1,z_2\in\mathcal{M}_\Lambda:\\ (z^1+z_1)\restriction\Lambda^o=x,\,x'=(z^2+z_2)\restriction\Lambda^o,\\ (z^1+z_1)\restriction\Lambda\setminus\Lambda^o=(z^2+z_2)\restriction\Lambda\setminus\Lambda^o,\\ |z_1|=|z^1|,\,|z^2|=|z_2|}} \sum_{\substack{\Gamma_1\in\mathcal{O}_{y_1,z_1},\\ \Gamma_2\in\mathcal{O}_{z_2,y_2}}} \int_{W^t_{\Gamma_1}} P^t_{y^o+y^1}(d\Omega^1)\, i^{-N(\Omega^1)}\bar{\chi}_{(m)}(\Omega^1)\times$$

$$\int_{W^t_{\Gamma_2}} P^t_{z^2}(d\Omega^2)\, i^{N(\Omega_2)}\bar{\chi}_{(m)}(\Omega^2) \int_{W^t_{\Gamma_1}} P^t_{y_1}(d\Omega_1)\, i^{-N(\Omega_1)}\chi_{(m)}(\Omega_1) \int_{W^t_{\Gamma_2}} P^t_{z_2}(d\Omega_2)\, i^{N(\Omega_2)}\chi_{(m)}(\Omega_2)$$

$$\times \exp\left(i\mathcal{U}(\Omega^1\cup\Omega_1)-i\mathcal{U}(\Omega^2\cup\Omega_2)\right) \quad \varphi^{(\Lambda)}(y^o+y^1+y^2). \quad (3.12)$$

Here Ω^1 is the collection of paths from the (e^{itH_Λ})-integral which ignore the taboo, and Ω^2 is that of those from the (e^{-itH_Λ})-integral. On the other hand, Ω_1 is the collection of taboo-obeying paths from e^{itH_Λ} and Ω_2 is that of those from e^{-itH_Λ}. Correspondingly, the indicator $\bar{\chi}_{(m)}$ is totally disjoint from χ_m :

$$\bar{\chi}_{(m)}(\Omega^\alpha)=\prod_{\omega\in\Omega^\alpha}\left(1-\chi_{(m)}(\omega)\right), \quad \alpha=1,2. \qquad (3.13)$$

Finally, notice that the taboo under question is established relating to the occupation number configuration $y^o+y^1+y_1=y^o+y^2+y_2$ which, by assumption, belongs to $\mathcal{M}_\Lambda(m)$.

We shall now rewrite the RHS of (3.12) in a sligtly more convenient form. For further simplifications, we omit from the notations (but of course, keep in mind) regulations concerning occupation number comfigurations which are in the RHS of (3.12). We also omit, whenever possible, the references to z^1, z^2, z_1, z_2, because these occupation number configurations may be recovered from matchings $\Gamma^1, \Gamma^2, \Gamma_1$ and Γ_2. So, the RHS of (3.12) equals

$$\sum_{\substack{J^+_*\subset I^+_m,\,J^-_*\subset I^-_m:\\ |J^\pm_*|\geq[m^\delta]}} \sum_{\substack{y^o,y^1,y^2:\\ y^o+y^1+y^2\restriction J^+_*\cup J^-_*=0}} e^{t(2|y^o|+|y^1|+|y^2|)} \sum_{\Gamma^1,\Gamma^2}$$

$$\int_{W^t_{\Gamma^1}} P^t_{y^o+y^1}(d\Omega^1)\; i^{-N(\Omega^1)}\;\bar{\chi}_{(\hat{J}^\pm_*)}(\Omega^1) \quad \int_{W^t_{\Gamma^2}} P^t_{z^2}(d\Omega^2)\; i^{N(\Omega^2)}\bar{\chi}_{(\hat{J}^\pm_*)}(\Omega^2$$

$$\sum_{\substack{y_1, y_2: \\ J^{\pm}(y^o+y^1+y_1,m)=J^{\pm}_*}} e^{t(|y_1|+|y_2|)} \sum_{\Gamma_1,\Gamma_2} \int_{W^t_{\Gamma_1}} P^t_{y_1}(d\Omega_1)\, i^{-N(\Omega_1)} \chi_{(\hat{J}^{\pm}_*)}(\Omega_1) \times$$

$$\times \int_{W^t_{\Gamma_2}} P^t_{z^2}(d\Omega_2)\, i^{N(\Omega_2)} \chi_{(\hat{J}^{\pm}_*)}(\Omega_2) \times$$

$$\times \ \exp\left(iU(\Omega^1 \cup \Omega_1) - iU(\Omega^2 \cup \Omega_2) \right) \varphi^{(\Lambda)}(y^o + y^1 + y_1). \quad (3.14)$$

Here $|J^{\pm}_*|$ stands for the length $(v^{\pm}_{2*} - v^{\pm}_{1*})$ of an interval $J^{\pm}_* = [v^{\pm}_{1*}, v^{\pm}_{2*}]$ and $\hat{J}^{\pm}_*$ denotes, as before (cf. (3.10)), the corresponding taboo subinterval,

$$\hat{J}^{\pm}_* = [v^{\pm}_{1*} + [\tfrac{1}{3} m^{\delta}] \ , \ v^{\pm}_{2*} - [\tfrac{1}{3} m^{\delta}]\,]. \quad (3.15)$$

Correspondingly, $\bar{\chi}_{(\hat{J}^{\pm}_*)}$ is the indicator which vanishes provided at least one trajectory ω from $\Omega^{\alpha}, \alpha=1,2$, obeys the taboo and $\chi_{(\hat{J}^{\pm}_*)}$ is the totally disjoint indicator which equals 1 provided all the trajectories ω from $\Omega_{\alpha}, \alpha=1,2$, obey the taboo. The sum $\sum_{y_1, y_2}$ is now restricted to those pairs of occupation number configurations y_1, y_2 for which the reference intervals $J^{\pm}_*$ are just the free intervals for the full configuration $y^o + y^1 + y_1$.

Now we estimate (3.15) from above by

$$\sum_{J^{\pm}_*} \sum_{y^o, y^1, y^2} e^{t(2|y^o|+|y^1|+|y^2|)} \sum_{\Gamma^1, \Gamma^2} \int_{W^t_{\Gamma^1}} P^t_{y^o+y^1}(d\Omega^1)\, \bar{\chi}_{(\hat{J}^{\pm}_*)}(\Omega^1) \times$$

$$\times \int_{W^t_{\Gamma^2}} P^t_{z^2}(d\Omega^2)\, \bar{\chi}_{(\hat{J}^{\pm}_*)}(\Omega^2) \Big| \sum_{y_1, y_2} e^{t(|y_1|+|y_2|)} \sum_{\Gamma_1, \Gamma_2} \int_{W^t_{\Gamma_1}} P^t_{y_1}(d\Omega_1)\, i^{-N(\Omega_1)},$$

$$\chi_{(\hat{J}^{\pm}_*)}(\Omega_1) \int_{W^t_{\Gamma_2}} P^t_{z^2}(d\Omega_2)\, i^{N(\Omega_2)} \chi_{(\hat{J}^{\pm}_*)}(\Omega_2) \times$$

$$\times \exp\left(i\bar{U}(\Omega_1 | \Omega^1) - i\bar{U}(\Omega_2 | \Omega^2) \right) \varphi^{(\Lambda)}(y^o + y^1 + y_1) \Big| . \quad (3.16)$$

Here we continued the policy to omit from the notations varius sum conditions. Notice where the absolute value appears: this will play the crucial role just below. The potential energy exponent is defined by

$$U(\Omega_\alpha|\Omega^\alpha)\;(=U(\Omega_\alpha\cup\Omega^\alpha)-U(\Omega^\alpha))=$$

$$=\sum_{\substack{\omega,\omega'\in\Omega_\alpha:\\ \omega\neq\omega'}} U(\omega,\omega')+\sum_{\substack{\omega\in\Omega_\alpha,\\ \omega'\in\Omega^\alpha}} U(\omega,\omega'); \qquad (3.17)$$

this is related to a "conditional" energy of a "domestic" path family Ω_α in the environment created by a "wild" path family Ω^α.

Proposition 3.3. The quantity under the absolute value sign in (3.16) is equal to the following trace-like sum

$$\sum_{\substack{y\in\mathcal{M}_\Lambda:\\ y^0+y^1+y\in\mathcal{M}_\Lambda(m),\\ J^\pm(y^0+y^1+y,m)=J^\pm_*}} \varphi^{(\Lambda)}(y+y^1+y^0)\big(\mathcal{E}_t(\Omega^1;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))\,E^{(\Lambda^0)}_{x-z^1\upharpoonright\Lambda^0,\,x'-z^2\upharpoonright\Lambda^0}\times$$

$$\times\,\Pi^{(\Lambda\setminus\Lambda^0)}_{\cdot+z^1\upharpoonright\Lambda\setminus\Lambda^0=\cdot+z^2\upharpoonright\Lambda\setminus\Lambda^0}\,\mathcal{E}_{-t}(\Omega^2;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))\,\Pi^{(\Lambda)}_{\cdot+y^2+y^0=\cdot+y^0+y^1}\big)_{y,y},\qquad (3.18)$$

where:

(a) $\mathcal{E}_t(\Omega^1;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))$ and $\mathcal{E}_{-t}(\Omega^2;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))$ are the products of unitary operators in $\mathcal{H}_\Lambda$:

$$\mathcal{E}_t(\Omega^1;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))=\prod_{j:0\to N(\Omega^1)} e^{iT_j(\Omega^1)H_{\Lambda\setminus(\hat{J}^-\cup\hat{J}^+),j,\Omega^1}}, \qquad (3.19)$$

$$\mathcal{E}_{-t}(\Omega^2;\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))=\prod_{j:0\to N(\Omega^2)} e^{-iT_j(\Omega^2)H_{\Lambda\setminus(\hat{J}^-\cup\hat{J}^+),j,\Omega^2}}, \qquad (3.20)$$

(b) $\Pi^{(\Lambda\setminus\Lambda^0)}_{\cdot+z^1\upharpoonright\Lambda\setminus\Lambda^0=\cdot+z^2\upharpoonright\Lambda\setminus\Lambda^0}$ is the partial isometry, operator in $\mathcal{H}_{\Lambda\setminus\Lambda^0}=\bigotimes_{k\in\Lambda\setminus\Lambda^0}\mathcal{H}_k$ with the entries

$$\big(\Pi^{(\Lambda\setminus\Lambda^0)}_{\cdot+z^1\upharpoonright\Lambda\setminus\Lambda^0=\cdot+z^2\upharpoonright\Lambda\setminus\Lambda^0}\big)_{z_1,z_2}=1,\ \text{if } z_1+z^1\upharpoonright\Lambda\setminus\Lambda^0=z_2+z^2\upharpoonright\Lambda\setminus\Lambda^0,$$

$$=0,\ \text{otherwise}, \qquad (3.21)$$

(c) likewise, $\Pi^{(\Lambda)}_{\cdot+y^2+y^0=\cdot+y^0+y^1}$ is the partial isometry operator in $\mathcal{H}_\Lambda$ with

$$\left(\Pi^{(\Lambda)}_{\cdot+y^2+y^\circ=\cdot+y^\circ+y^1}\right)_{y_1,y_2} = 1, \text{ if } y_1+y^\circ+y^1 = y_2+y^2+y^\circ,$$
$$= 0, \text{ otherwise.} \tag{3.22}$$

Furthermore, $T_j(\Omega^\alpha)=\tau_{j+1}(\Omega^\alpha)-\tau_j(\Omega^\alpha)$ is the time between the epochs of the (j + 1)th and jth jump for the path family Ω^α (we set $\tau_0(\Omega^\alpha)=0, \tau_{N(\Omega^\alpha)+1}(\Omega^\alpha)=t$), $\alpha=1,2$, and the self-adjoint operators $H_{\Lambda\setminus(\hat{J}^-\cup\hat{J}^+),j,\Omega^\alpha}$ are the Hamiltonians of the motion in the external potential field created by the occupation number configuration $\Omega^\alpha(u)$ (i.e., by the time section of this wild path family Ω^α at a moment u), $\tau_j(\Omega^\alpha)<u<\tau_{j+1}(\Omega^\alpha)$, $0\le j\le N(\Omega^\alpha)$, $\alpha=1,2$:

$$H_{\Lambda\setminus(\hat{J}^-\cup\hat{J}^+),j,\Omega^\alpha} = -\tfrac{1}{2}\sum_{k\in\Lambda\setminus(\hat{J}^-\cup\hat{J}^+)} a_k^+(\Delta a)_k + \sum_{k,k'\in\Lambda\setminus(\hat{J}^-\cup\hat{J}^+)} \Phi(|k-k'|)\, n_k n_{k'} + \sum_{\substack{k\in\Lambda\setminus(\hat{J}^-\cup\hat{J}^+),\\ \omega\in\Omega^\alpha}} \Phi(|k-\omega(u)|)\, n_k , \tag{3.23}$$

with the Dirichlet boundary condition on $\partial(\Lambda\setminus(\hat{J}^-\cup\hat{J}^+))$. ◁

Proof of proposition 3.3 - by inspection, using Lemma 2.1 a).

Since all the operators in (3.18) are of norm ≤ 1, we conclude from Proposition 3.3 that the absolute value in (3.16) is upper-bounded by

$$\sum_{\substack{\tilde{y}\in\mathcal{M}_\Lambda(m):\\ \tilde{y}\ge\max[y^1,y^2]+y^\circ,\\ J^\pm(\tilde{y},m)=J^\pm_*}} \varphi^{(\Lambda)}(\tilde{y}) \le \sum_{\substack{y\in\mathcal{M}_\Lambda(m):\\ J^\pm(y,m)=J^\pm}} \varphi^{(\Lambda)}(y). \tag{3.24}$$

Now we pass to estimate the internal sum $\sum_{\Gamma^1,\Gamma^2}$ in (3.16) :

$$\sum_{\Gamma^1,\Gamma^2} \int_{W^t_{\Gamma^1}} P^t_{y^1}(d\Omega^1)\bar{\chi}_{(\hat{J}^{\pm}_*)}(\Omega^1) \int_{W^t_{\Gamma^2}} P^t_{z^2}(d\Omega^2)\bar{\chi}_{(\hat{J}^{\pm}_*)}(\Omega^2) \tag{3.25}$$

for fixed intervals $J^{\pm}_*$ and occupation number configurations y^0, y^1, y^2. This is obviously less than the product

$$\prod_{j\in\Lambda}\Big(P^t_0\big(\sup_{u\in[0,t]}|\omega(u)| \geq [\tfrac{1}{3}m^{\delta}] + \mathrm{dist}(j,J^-)\big) + P^t_0\big(\sup_{u\in[0,t]}|\omega(u)| \geq [\tfrac{1}{3}m^{\delta}] + \mathrm{dist}(j,J^+)\big)\Big)^{2y^0(j)+y^1(j)+y^2(j)}. \tag{3.26}$$

Making the summation, over y^0, y^1 and y^2 yields the upper bound

$$\sum_{n=1}^{\infty} 3^n e^{2tn}\Big(4\sum_{r\geq[\frac{1}{3}m^{\delta}]} P^t_0(\sup|\omega(u)|\geq r)\Big)^n, \tag{3.27}$$

whereas the sum over $J^{\pm}_*$ gives a value ≤ 1 due to (3.23). Assuming that m is chosen so large that

$$24e^t \sum_{r\geq[\frac{1}{3}m^{\delta}]} P^t_0(\sup|\omega(u)|\geq r) < 1, \tag{3.28}$$

we get that (3.16) is less than LHS of (3.28). This goes to zero because

$$P^t_0(\sup|\omega(u)|\geq r) \leq \sum_{\ell\geq r}\frac{t^{\ell}}{\ell!}e^{-t} \leq \frac{t^r}{r!}. \tag{3.29}$$

Lemma 3.2 is proved. □

Now write

$$\varphi^{(1}_{\Lambda,t,m}(E^{(\Lambda^0)}_{x,x'}) = \sum_{\substack{J^{\pm}_*\subset I^{\pm}_m:\\ |J^{\pm}_*|\geq[m^{\delta}]}} \sum_{\substack{y\in\mathcal{M}_\Lambda(m):\\ J^{\pm}(y,m)=J^{\pm}_*}} \varphi^{(\Lambda)}(y)\,(D^L_{t,\Lambda,m} \times E^{(\Lambda^0)}_{x,x'} D^R_{-t,\Lambda,m})_{y,y} \tag{3.30}$$

and notice that for $y\in\mathcal{M}_\Lambda(m)$ and m large enough

$$\left(D^{L}_{t,\Lambda,m}\; E^{(\Lambda^0)}_{x,x'}\; D^{R}_{-t,\Lambda,m}\right)_{y,y} = \left(D^{L}_{t,I_m,m}\; \times\right.$$

$$\left.\times\, E^{(\Lambda^0)}_{x,x'}\; D^{R}_{-t,I_m,m}\right)_{y\restriction I_m,\, y\restriction I_m}, \qquad (3.31)$$

since both LHS and RHS coinside with the diagonal entry of the operator

$$\exp\left(itH_{(I_m\setminus(\hat{J}^-\cup\hat{J}^+))_{int}}\right)\; E^{(\Lambda^0)}_{x,x'}\; \times$$

$$\times \exp\left(-itH_{(I_m\setminus(\hat{J}^-\cup\hat{J}^+))_{int}}\right), \qquad (3.32)$$

where $I_m\setminus(\hat{J}^-\cup\hat{J}^+)_{int}$ stands for the "internal connected component" of the set-theoretical difference $I_m\setminus(\hat{J}^-\cup\hat{J}^+)$.

Having this in mind and combining the arguments from the proof of Lemmas 3.1 and 3.2, one can get

Lemma 3.4. The following relation is valid:

$$\lim_{m\to\infty}\ \sup_{\Lambda\supset I_m} \left|\varphi^{1}_{\Lambda,t,m}\left(E^{(\Lambda^0)}_{x,x'}\right) - \right.$$

$$\left. -\varphi_{I_m,t}\left(E^{(\Lambda^0)}_{x,x'}\right)\right| = 0. \lhd \qquad (3.33)$$

This finishes the proof of Theorem 1. $\square$

4. Proof of Theorem 2.

Assume again that $t>0$. The proof of Theorem 2 proceeds along a similar way. First of all, we write

$$\varphi_{\Lambda,t}\left(\Pi^{(\Lambda^0)}_{\geqslant s}\right) = \varphi\left(\Pi^{(\bar{\Lambda}^0([s^{1/2}])}_{\geqslant s}\, e^{itH_\Lambda}\, \Pi^{(\Lambda^0)}_{\geqslant s}\, e^{-itH_\Lambda}\, \Pi^{(\bar{\Lambda}^0([s^{1/2}]))}_{\geqslant s}\right) +$$

$$+\,\varphi\left(\Pi^{(\bar{\Lambda}^0([s^{1/2}])}_{<s}\, e^{itH_\Lambda}\, \Pi^{(\Lambda^0)}_{\geqslant s}\, e^{-itH_\Lambda}\, \Pi^{(\bar{\Lambda}^0([s^{1/2}])}_{<s}\right), \quad (4.1)$$

where $\Pi_{<s}^{(\bar{\Lambda}^{o}([s^{\frac{1}{2}}]))}$ denotes the complementary projector:

$$\Pi_{<s}^{(\bar{\Lambda}^{o}([s^{1/2}]))} = \mathbb{1} - \Pi_{\geqslant s}^{(\bar{\Lambda}^{o}([s^{1/2}]))}$$

and $\bar{\Lambda}^{o}([s^{\frac{1}{2}}])$ is obtained by stretching the interval $\Lambda^{o} = [v_1^o, v_2^o]$:

$$\bar{\Lambda}^{o}([s^{1/2}]) = [v_1^o - [s^{1/2}],\ v_2^o + [s^{1/2}]].$$

Due to condition (d_1^{**}), the first term in the RHS of (4.1) does not exceed, in the absolute value,

$$\varphi(\Pi_{\geqslant s}^{(\bar{\Lambda}^{o}([s^{1/2}]))}) \leq c \int_{\bar{s}}^{\infty} du \exp(-u^2/2), \tag{4.2}$$

where

$$\bar{s} = (\sigma(|\Lambda^o| + 2s^{1/2})^{1/2})^{-1}(s - (|\Lambda^o| + 2s^{1/2})b)_{+}\,. \tag{4.3}$$

We are now going to estimate the second term in the RHS of (4.1). For the sake of simplicity we write $\bar{\Lambda}^{o}$ instead of $\bar{\Lambda}^{o}([s^{1/2}])$.

Lemma 4.1 The following bound holds true

$$|\varphi(\Pi_{<s}^{(\bar{\Lambda}^{o})} e^{itH_{\Lambda}} \Pi_{\geqslant s}^{(\Lambda^{o})} e^{-itH_{\Lambda}} \Pi_{<s}^{(\bar{\Lambda}^{o})}) \leq \psi_{\Lambda^o, t}(s),$$

where $\lim_{s\to\infty} \psi_{\Lambda^o, t}(s) = 0$ for any Λ^o and t. ◁

Proof. The proof of Lemma 4.1 will remind that of Lemma 3.2. We have (cf. (3.13)):

$$\varphi(\Pi_{<s}^{(\bar{\Lambda}^{o})} e^{itH_{\Lambda}} \Pi_{\geqslant s}^{(\Lambda^{o})} e^{-itH_{\Lambda}} \Pi_{<s}^{(\bar{\Lambda}^{o})}) =$$

$$= \sum_{\substack{z^0, z^1, z^2 \in \mathcal{M}_{\Lambda^o}: \\ |z^0| + |z^\alpha| \geqslant 1,\ \alpha = 1,2}} e^{t(2|z^0| + |z^1| + |z^2|)} \sum_{\substack{y^1, y^2 \in \mathcal{M}_{\Lambda \setminus \Lambda^o}: \\ |y^1| = |z^0| + |z^1|, \\ |y^2| = |z^0| + |z^2|}}$$

$$\sum_{\substack{\Gamma^1\in\mho_{z^0+z^1,y^1}\\ \Gamma^2\in\mho_{y^2,z^0+z^2}}} \int_{W^t_{\Gamma^1}} P^t_{z^0+z^1}(d\Omega^1)\chi_\Lambda(\Omega^1)i^{N(\Omega^1)}\int_{\bar W^t_{\Gamma^2}} P^t_{y^2}(d\Omega^2)\chi_\Lambda(\Omega^2)i^{-N(\Omega^2)}$$

$$\sum_{\substack{z_1,z_2\in\mathcal{M}_{\Lambda^0}:\\ z^0+z^1+z_1=z_2+z^2+z^0,\\ |z^0|+|z^1|+|z_1|\geq s}} e^{t(|z_1|+|z_2|)} \sum_{\substack{y_1,y_2\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_\alpha|=|z_\alpha|,\alpha=1,2}} \sum_{\substack{\Gamma_1\in\mho_{z_1,y_1},\\ \Gamma_2\in\mho_{y_2,z_2}}}$$

$$\int_{W^t_{\Gamma_1}} P^t_{z_1}(d\Omega_1)\chi_\Lambda(\Omega_1)i^{N(\Omega_1)}\int_{\bar W^t_{\Gamma_2}} P^t_{y_2}(d\Omega_2)\chi_\Lambda(\Omega_2)i^{-N(\Omega_2)} \sum_{z\in\mathcal{M}_{\Lambda\setminus\Lambda^0}} e^{2t|z|}$$

$$\sum_{\substack{y,y'\in\mathcal{M}_\Lambda:\\ y^1+y_1+y=y'+y_2+y^2,\\ |y|=|z|=|y'|}} \sum_{\substack{\Gamma\in\mho_{z,y},\\ \Gamma'\in\mho_{y',z}}} \int_{W^t_z} P^t_z(d\Omega)\chi_\Lambda(\Omega)i^{N(\Omega)}\int_{\bar W^t_{y'}} P^t_{y'}(d\Omega')\chi_\Lambda(\Omega')i^{-N(\Omega')}\times$$

$$\times\exp(-iU(\Omega^1\cup\Omega_1\cup\Omega)+iU(\Omega^2\cup\Omega_2\cup\Omega'))\,\varphi^{(\Lambda)}(y^1+y_1+y). \tag{4.4}$$

As before, we omit henceforth the standard indicator χ_Λ and do not write regulations for summations. We estimate the RHS of (4.4) from above by

$$\sum_{z^0,z^1,z^2} e^{t(2|z^0|+|z^1|+|z^2|)} \sum_{y^1,y^2}\sum_{\Gamma^1,\Gamma^2}\int_{W^t_{\Gamma^1}} P^t_{z^0+z^1}(d\Omega^1)\int_{W^t_{\Gamma^2}} P^t_{y^2}(d\Omega^2)\times$$

$$\Big|\sum_{z_1,z_2} e^{t(|z_1|+|z_2|)}\sum_{y_1,y_2}\sum_{\Gamma_1,\Gamma_2}\int_{W^t_{\Gamma_1}} P^t_{z_1}(d\Omega_1)i^{N(\Omega_1)}\int_{W^t_{\Gamma_2}} P^t_{y_2}(d\Omega_2)i^{-N(\Omega_2)}\times$$

$$\sum_{z,z'} e^{2t|z|}\sum_{y,y'}\sum_{\Gamma,\Gamma'}\int_{W^t_\Gamma} P^t_z(d\Omega)i^{N(\Omega)}\int_{W^t_{\Gamma'}} P^t_{y'}(d\Omega')i^{-N(\Omega)}\times$$

$$\times\exp(-iU(\Omega_1\cup\Omega\,|\,\Omega^1)+iU(\Omega_2\cup\Omega'\,|\,\Omega^2))\times \tag{4.5}$$

$$\times\,\varphi^{(\Lambda)}(y^1+y_1+y)\Big|.$$

It is again important to notice where the absolute value sign appears in (4.5). The conditional energies $U(\Omega_2 \cup \Omega' | \Omega^2)$ and $U(\Omega_1 \cup \Omega | \Omega^1)$ are defined in analogy with (3.17).

Now an analog of proposition 3.3 comes:

Proposition 4.2 The quantity under the absolute value sign in (4.5) is equal to the following trace-like sum

$$\sum_{\substack{y_1 \in \mathcal{M}_{\bar{\Lambda}^0},\\ y \in \mathcal{M}_\Lambda}} \varphi^{(\Lambda)}(y^1+y_1+y)\Big(\tilde{\Pi}^{(\Lambda)}_{\cdot+y^1+\cdot=\cdot+y^2+\cdot}\,(\Pi^{(\Lambda)}_{\bar{\Lambda}^0}\otimes \mathbb{1})\ \tilde{\mathcal{E}}_t(\Omega^2;\Lambda)\times$$
$$\times(\Pi^{(\Lambda)}_{\cdot+z^0+z^2=\cdot+z^0+z^1}\otimes \Pi^{(\Lambda)}_{\Lambda\setminus\Lambda^0})\times((\Pi^{(\Lambda)}_{\Lambda^0}\,\Pi^{(\Lambda)}_{|\cdot|+|z^0|+|z^1|\geq s})\otimes \mathbb{1})\times$$
$$\times\,\tilde{\mathcal{E}}_{-t}(\Omega^1;\Lambda)\,(\Pi^{(\Lambda)}_{\Lambda^0}\otimes \mathbb{1})\Big)_{y_1,y}\ , \tag{4.6}$$

where

(a) $\tilde{\Pi}^{(\Lambda)}_{\cdot+y^1+\cdot=\cdot+y^2+\cdot}$ is the partial isometry operator in the tensor square $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$ with the entries

$$\begin{aligned}(\tilde{\Pi}^{(\Lambda)}_{\cdot+y^1+\cdot=\cdot+y^2+\cdot})_{(y_1,y),(y_2,y)} &= 1, \text{ if } y_1+y^1+y=y_2+y^2+y',\\ &= 0, \text{ otherwise,}\end{aligned} \tag{4.7}$$

(b) $\Pi^{(\Lambda)}_{\tilde{\Lambda}}$ is the orthogonal projector in $\mathcal{H}_\Lambda$ confining an occupation number configuration to a subset $\tilde{\Lambda} \subset \Lambda$,

(c) $\Pi^{(\Lambda)}_{\cdot+z^0+z^2=\cdot+z^0+z^1}$ is the partial isometry operator in $\mathcal{H}_\Lambda$ with the entries

$$\begin{aligned}(\Pi^{(\Lambda)}_{\cdot+z^0+z^2=\cdot+z^0+z^1})_{z_2,z_1} &= 1, \text{ if } z_2+z^0+z^2=z_1+z^0+z^1,\\ &= 0, \text{ otherwise,}\end{aligned} \tag{4.8}$$

(d) $\Pi^{(\Lambda)}_{|\cdot|+|z^0|+|z^1|\geq s}$ is the orthogonal projector in $\mathcal{H}_\Lambda$ onto the space corresponding to occupation number configurations z_1 with $|z_1|+|z^0|+|z^1|\geq s$,

(e) $\tilde{\mathcal{E}}_{-t}(\Omega^1;\Lambda)$ and $\tilde{\mathcal{E}}_t(\Omega^2;\Lambda)$ are the products of unitory operators in $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$:

$$\tilde{\mathcal{E}}_{-t}(\Omega^1;\Lambda) = \prod_{j:0\to N(\Omega^1)} e^{-iT_j(\Omega^1)\tilde{H}_{\Lambda,j,\Omega^1}}, \qquad (4.9)$$

$$\tilde{\mathcal{E}}_{t}(\Omega^2;\Lambda) = \prod_{j:0\to N(\Omega^2)} e^{iT_j(\Omega^2)\tilde{H}_{\Lambda,j,\Omega^2}}. \qquad (4.10)$$

Here, as in Proposition 3.3, $T_j(\Omega^\alpha)$ is the time between the epochs $\tau_j(\Omega^\alpha)$ and $\tau_{j+1}(\Omega^\alpha)$ of subsequent jumps for the path family Ω^α, $j=0,\dots,N(\Omega^\alpha)$, $\alpha=1,2$. The self-adjoint operators $\tilde{H}_{\Lambda,j,\Omega^\alpha}$ in $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$ are of the form

$$\tilde{H}_{\Lambda,j,\Omega^\alpha} = H_{\Lambda,j,\Omega^\alpha} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\Lambda,j,\Omega^\alpha} + V, \qquad (4.11)$$

where

$$H_{\Lambda,j,\Omega^\alpha} = -\tfrac{1}{2}\sum_{k\in\Lambda} a_k^+(\Delta a)_k + \sum_{k,k'\in\Lambda} \Phi(|k-k'|)n_k n_{k'} +$$

$$+ \sum_{k\in\Lambda} \Phi(|k-\omega(u)|)n_k, \qquad (4.12)$$

where $\tau_j(\Omega^\alpha) < u < \tau_{j+1}(\Omega^\alpha)$, $\alpha = 1,2$ (cf. (3.23)), and V is the cross interaction energy (2.21). ◁

Proof of Proposition 4.2, like that of Proposition 3.3, is done by inspection, using Lemma 2.1 b).

We use Proposition 4.2 and condition (d_2^{**}) to conclude that the absolute value in (4.5) is less than

$$\sum_{\substack{y_1\in\mathcal{M}_{\Lambda^0}:\\ |y_1|+|y^1|\geq s}} \sum_{y\in\mathcal{M}_\Lambda} \varphi^{(\Lambda)}(y_1+y^1+y) = \sum_{\substack{y_1\in\mathcal{M}_{\bar{\Lambda}^0}:\\ |y_1|+|y^1|\geq s}} \bar{\varphi}(y_1+y^1) \leq$$

$$\leq c_1^{2[s^{1/2}]+|\Lambda^0|}\, \rho^{|y^1|} \sum_{\substack{y_1\in\mathcal{M}_{\bar{\Lambda}^0}:\\ |y_1|\geq(s-|y^1|)_+}} \bar{\varphi}(y_1), \qquad (4.13)$$

where we have used the notation

$$\bar{\varphi}(y^\sim) = \varphi(\bar{B}_{y^\sim}). \qquad (4.14)$$

In the same way one can check that the absolute value in question is less than

$$c_1^{2[s^{1/2}]+|\Lambda^0|}\rho^{|y^2|}\sum_{\substack{y_2\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_2|\geq(s-|y^2|)_+}}\bar\varphi(y_2).$$

Hence, this absolute value is less than

$$c_1^{2[s^{1/2}]+|\Lambda^0|}\rho^{(|y^1|+|y^2|)}\Big(\sum_{\substack{y_1\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_1|\geq(s-|y^1|)_+}}\bar\varphi(y^1)\cdot\sum_{\substack{y_2\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_2|\geq(s-|y^2|)_+}}\bar\varphi(y^2)\Big)^{1/2}.$$

Now we estimate from above the sum $\sum_{y^1,y^2}\sum_{\Gamma^1,\Gamma^2}$ in (4.5) by

$$\prod_{j\in\Lambda^0}\left(P_0^t(|\omega(t)|\geq[s^{1/2}]+j-v_1^0)+P_0^t(|\omega(t)|\geq[s^{1/2}]+v_2^0-j)\right)^{2|z^0|+|z^1|+|z^2|}\times$$

$$c_1^{2[s^{1/2}]+|\Lambda^0|}\rho^{(|z^1|+|z^2|)/2+|z^0|}\Big(\sum_{\substack{y_1\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_1|\geq(s-|z^0|-|z^1|)_+}}\bar\varphi(y^1)\cdot$$

$$\sum_{\substack{y_2\in\mathcal{M}_{\bar\Lambda^0}:\\ |y_2|\geq(s-|z^0|-|z^2|)_+}}\bar\varphi(y_2)\Big)^{1/2}$$

Making the summation over z^0, z^1, z^2 yields, by virtue of (d_2^{**}), the following upper bound for (4.5):

$$\Big(\sum_{n=1}^{\infty}3^n e^{2tn}\big(4\sum_{r\geq[s^{1/2}]}P_0^t(|\omega(t)\geq r)\big)^n\rho^{2n}\big(c_1^{|\Lambda^0|+2s^{1/2}}\big)^2\bar\rho^{\,|\Lambda^0|+2s^{1/2}}\Big)^2,$$

where

$$\bar\rho=(1-\rho)^{-1}=\sum_{m\geq 0}\rho^m.$$

The final remark is similar to that in the end of the proof of Lemma 3.2. This finishes the proof of Lemma 4.1 and hence, that of Theorem 2. □

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