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Lattice Classification of 8-Dimensional Chiral Heterotic Strings

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ABSTRACT

The eight-dimensional chiral rank 18 heterotic strings are classified using the covariant lattice approach.

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I. Introduction

A large class of four dimensional string theories with chiral fermions is provided by the covariant lattice construction [1]. In ref. [2] we have shown how to classify all four dimensional chiral string theories based on covariant lattices with a special world-sheet supercurrent (the triplet constraint of [1]). In this paper we wish to demonstrate how to carry out the complete classification in 8 dimensions and as a result we list all possible chiral models based on the triplet constraint. Our algorithm can be carried out without any change to 4 dimensional models. The number of inequivalent models is 444 (this should be compared to the number of 10 dimensional maximal rank strings, which is 8). Clearly in 4 dimensions the number of chiral models is quite large, though not an astronomical number (probably smaller than 10^6). However one would need some extra constraint to restrict the number of models to something manageable. It seems that space-time supersymmetry is a necessary consistency condition to avoid the cosmological constant problem and the associated tadpole divergences, as suggestions [3] that Atkin-Lehner symmetric models can provide examples of theories with zero cosmological constant without space-time supersymmetry are ruled out [4]. This however is still not restrictive enough so one has to find a more phenomenological input to define the interesting models. For the moment our primary goal is to implement the complete classification at least in the form of a database on a computer since there may be other definitions of interesting models. For example if one is interested in theories with gauge groups with rank smaller than 22 the rank reducing technique of [5] requires models where the gauge group contains some identical factors.

The plan of the paper is the following: In section two we give a short introduction to the necessary lattice methods. Then we present the main ingredients of the covariant lattice approach illustrated in the eight dimensional case, and also derive all supersymmetric models. In section four we give a detailed description of our algorithm which has been implemented on a personal computer. In section five we show that from the analysis of the partition functions there is a very substantial restricton on the tachyon free models. Section six contains a discussion of the results, while in the next one we present a small part of the result of the classification in four dimensions.

II. A lattice primer

In this chapter we give a brief compendium of the relevant definitions and theorems in lattice theory needed for our purposes. For a detailed introduction to lattices we refer to the book of Conway and Sloane [6] and to the reviews by Goddard and Olive [7] and Lerche, Schellekens and Warner [8].

Consider a basis, $\{e_i\}_{i=1}^{N}$ of an N-dimensional lattice , Λ_N, then all lattice vectors can be written by definition as integer linear combinations of the basis vectors:

$$v = \sum_{i=1}^{N} n_i e_i, \qquad n_i \in \mathbf{Z}, \quad (v \in \Lambda_N).$$

The matrix of scalar products, (Gram matrix), given as

$$g_{ab} = e_a \cdot e_b$$

contains all information about the lattice . The volume of the unit cell $vol(\Lambda_N)$ is simply

$$vol(\Lambda_N) = \sqrt{|\det g|}.$$

The dual of Λ_N is defined as:

$$\Lambda_N^* = \{w : w \cdot v \in \mathbf{Z}, \forall v \in \Lambda_N\}.$$

Clearly

$$vol(\Lambda_N^*) = (vol(\Lambda_N))^{-1}.$$

A lattice can be either Euclidean or Lorentzian, depending on the definitness of the Gram matrix. Of particular interest are the integral lattices, when all entries of the Gram matrix are integers. Equivalently a lattice Λ is integral if and only if

$$\Lambda \subseteq \Lambda^*.$$

An important (Abelian) group associated with an integral lattice is its *dual quotient group* Λ^*/Λ, which has order $|\det g(\Lambda)|$. In the case of the root lattices, Λ_R, of the simply laced Lie algebras (A-D-E) the corresponding dual lattices are just the weight lattices Λ_W. The dual quotient group is the center of the Lie group and its order is just the number of conjugacy classes of Λ_W, i.e. the number of inequivalent 'n-ality classes'. The lattice Λ is called unimodular if $vol(\Lambda) = 1$. An integral, unimodular lattice is self-dual, that is $\Lambda^* = \Lambda$, and vice versa. Note that the only indecomposable self-dual simply laced root lattice is that of E_8. If Λ is an integral lattice , and for all $x \in \Lambda$, $x \cdot x$ is an *even* integer, then Λ is called *even*; otherwise *odd*. The classification of odd and even self-dual lattices is of great importance in mathematics, here we just summarize the main known results.

1.a *Even, Lorentzian* lattices exist only if $|p - q| = 0 \bmod(8)$.

1.b *Odd, Lorentzian* lattices are all Lorentz transformations of the lattice $\mathbf{Z}^{p,q}$, which is simply given by the set of vectors $\{(\mathbf{n}, \mathbf{m}) : \mathbf{n} \in \mathbf{Z}^p, \mathbf{m} \in \mathbf{Z}^q\}$.

In the classification of Euclidean lattices the following theorems are important:

Theorem 1. *Any integral lattice Λ containing vectors of norm of 1 is decomposable:*

$$\Lambda = \mathbf{Z}^r \oplus \tilde{\Lambda}$$

where the minimum norm of vectors in $\tilde{\Lambda}$ is at least 2.

Theorem 2 (Witt). *For any integral lattice its minimal lattice (i.e. the sublattice generated by norm 2 vectors) is given by a direct sum of root lattices .*

2.a *Even, Euclidean* lattices exist only in $d = 8n$ dimensions and are completely known for d =8,16 and 24.

2.b *Odd, Euclidean* lattices can be obtained from the even self-dual ones by removing suitable D_n factors. Presently these lattices have been classified up to dimension 25 [6].

As for the enumeration of the eight dimensional heterotic strings we need all of the 18 dimensional odd, Euclidean lattices classified by Conway and Sloane we present briefly their results. In fact they classified these lattices up to dimension 23. According to Witt's theorem an integral lattice , containing vectors of minimum norm 2 has a sublattice of the form

$$\tilde{\Lambda} = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_k \oplus \Lambda_0$$

where the components Λ_i, $i = 1, \ldots, k$ are isomorphic to members of the $A - D - E$ series while the minimal norm of vectors in Λ_0 is at least 3. We shall write Λ_0 in terms of orthogonal $U(1)$ lattices as a 'root' lattice Λ_{0_R} of Λ_0 with suitable conjugacy classes from its 'weight' lattice $\Lambda_{0_W} \equiv \Lambda^*_{0_R}$. The notation $U_n, n \in Z_+$ refers to one dimensional $U(1)$ lattices with generating vector of norm n. So $U_1 = \mathbf{Z}$, $U_2 = A_1$ and the U_4 lattice is sometimes denoted by D_1 in this paper. Clearly the dual of these lattices (the 'weight' lattice) contains n conjugacy classes with generating vector of norm $1/n$ and these classes form the cyclic group $\mathbf{Z}_n$.

The lattices enumerated by Conway and Sloane are generated by $\tilde{\Lambda}$ together with certain glue vectors $g = (g_1, \ldots, g_k, g_0)$ where g_i is the corresponding glue vector component for Λ_i. Clearly $g_i \in \Lambda_i^*$, so it is a weight vector of the corresponding Lie algebra. The norm of the glue vectors is at least 3 and quite clearly their role is to decrease the volume of the root lattice to unity. The Abelian group generated by the glue vectors is a subgroup of the direct product of the dual quotient groups of corresponding Witt components:

$$G = G_1 \times G_2 \times \ldots G_k \times G_0.$$

The dual quotient groups for the A-D-E-U series are the following:

A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8	U_n
$\mathbf{Z}_{n+1}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}_3$	$\mathbf{Z}_2$	—	$\mathbf{Z}_n$

Using the above notation we shall specify a 'glue group' by giving the glue vectors, the generators of the corresponding subgroup of G. A glue vector will be given by specifying its i-th component as an element of the corresponding cyclic group or $\mathbf{Z}_2 \times \mathbf{Z}_2$. For example (1), (3), (2) denote the two inequivalent spinor and the vector conjugacy classes of the D_{2n+1} while for a D_{2n} the corresponding conjugacy classes are denoted by (01), (10) and (11). Conjugation of the Lie algebra representations corresponds to the reflection $k \longrightarrow -k$ mod(n) in the $\mathbf{Z}_n$ factors of the dual quotient group.

In 18 dimensions there are 4 'genuine' (without vectors of norm 1) and 9 decomposable lattices reproduced in the following table:

Components		Glue vectors
1.	$E_8\mathbf{Z}^{10}$	—
2.	$D_{12}\mathbf{Z}^6$	(01)
3.	$D_{16}\mathbf{Z}^2$	(01)
4.	$\mathbf{Z}^{18}$	—
5.	$E_8^2\mathbf{Z}^2$	—
6.	$D_8^2\mathbf{Z}^2$	(01,11) (11,01)
7.	$E_7^2\mathbf{Z}^4$	(1,1)
8.	$A_{15}\mathbf{Z}^3$	(4)
9.	$A_{11}E_6\mathbf{Z}$	(2,1)
10.	D_6^3	(00,01,10) + cycl. perms.
11.	A_9^2	(1,3)
12.	$A_{17}A_1$	(3,1)
13.	$D_{10}E_7A_1$	(01,1,0), (10,0,1)

In the above table the labelling of the glue vector representatives should be clear. For the A_n algebras (k) is a weight vector of the k-th antisymmetric tensor product, for the E_n algebras (i) (i =0,1,2) denote the three or two inequivalent conjugacy classes. E.g. for D_{12} (01) stands for $(\frac{1}{2})^{12}$. We remark here that the number of 22 dimensional odd self-dual lattices (which are relevant for the construction of 4 dimensional models) is 68, out of which 28 are 'genuine' 22 dimensional.

As in our construction of the eight dimensional, chiral heterotic strings we shall work with nonintegral lattices , namely the definite parts of a self-dual Lorentzian lattice, their classification can be done in terms of their dual lattices since in this case these are already integer lattices therefore one can use the whole machinery mentioned above.

Also it is of great importance to decide when two lattices are equivalent, that is assuming they have the same Witt components to establish an isomorphism between the glue vectors by the automorphism group of one the lattices. We note that the automorphism group in our notation generated by permutations of isomorphic Witt components and automorphisms of a single Witt component. In the case of a root lattice of the $A-D-E$ Lie algebras these are the automorphisms of the corresponding Dynkin diagrams, that can be realized by reflections or permutations in the dual quotient groups of A_n, D_{2n+1} and D_{2n}, respectively. Apart from the reflections there can be also rotations of finite order in the case of two or more U_n lattices.

III. 8–dimensional Heterotic Strings

First we recall the main ingredients of model building based on even self-dual lattices [1],[8]. The eight dimensional lattice compactified heterotic string contains the following degrees of freedom. The matter fields consist of 8 left and 8 right moving bosons, X^μ, corresponding to space-time, 18 left moving internal bosons, X_L^I, compactified on a torus, 8 right moving fermions, Ψ^μ and 3 right moving internal bosons X_R^I, also compactified on a torus. The momenta of the left and right moving compactified bosons lie on a momentum lattice, Λ_L and λ_R, respectively. In addition we have the conformal ghost system (b,c) on both sides and the superconfomal ghost system $(\beta\ ,\gamma)$ on the right.

Bosonization of the space-time fermions and the ghosts leads to the following lattices: the 8 fermions, Ψ^μ, correspond to 4 bosons, whose momenta lie on a D_4 lattice. The (β,γ) system corresponds to a boson (with the wrong sign in its propagator) quantized on a D_1 lattice. Because of the correlation between the Ramond and Neveu-Schwarz sectors of the fermions and the ghosts, this part of the theory is in fact described by a $D_{4,1}$ lattice. This is a five dimensional, Lorentzian lattice with metric $<(+)^4,(-)>$ and has four conjugacy classes similar to the Euclidean D_n lattices.

The bosonized theory is therefore charaterized by a

$$\Gamma_{18;7,1} = \Lambda_L \times \lambda_R \times D_{4,1} \tag{3.1}$$

lattice. In fact $\Gamma_{18;7,1}$ is an integer lattice with respect to the

$$<(+)^{18};(-)^7,(+)> \tag{3.2}$$

metric.

As is known from [1] there are two additional consistency conditions satisfied by the theory: modular invariance and world sheet supersymmetry. Modular invariance is guaranteed by imposing self duality on the lattice $\Gamma_{18;7,1}$. The second condition, world sheet supersymmetry, which is crucial for Lorentz invariance of the theory in the light-cone gauge (and is also used in the 'picture changing' operation) is nontrivial to satisfy (especially in a purely bosonic theory). Since this is the only not completely solved issue in the covariant lattice approach we shall discuss the general construction of the supercurrent (not restricted to 8 dimensions) in some detail. The right moving energy-momentum tensor and supercurrent are given by [9]:

$$\begin{aligned} T(z) = & -\frac{1}{2}\partial_z X^\mu \partial_z X^\mu + \frac{1}{2}\Psi^\mu \partial_z \Psi^\mu + T_{int}(z) \\ & + c\partial_z b + 2(\partial_z c)b - \frac{1}{2}\gamma\partial_z\beta - \frac{3}{2}(\partial_z\gamma)\beta \end{aligned} \tag{3.3a}$$

$$S(z) = -\Psi^\mu \partial_z X^\mu + S_{int}(z) - 2c\partial_z\beta - 3(\partial_z c)\beta + \gamma b \tag{3.3b}$$

where

$$T_{int}(z) = -\frac{1}{2}\partial_z X_R^I \partial_z X_R^I \tag{3.4}$$

so the only nontrivial problem is to find an S_{int} acting on the space of compactified bosons X_R^I so that $T(z)$ and $S(z)$ satisfy the following operator product algebra (equivalent to the super Virasoro algebra):

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \tag{3.5a}$$

$$T(z)S(w) = \frac{3/2S(w)}{(z-w)^2} + \frac{\partial_w S(w)}{z-w} + \cdots \tag{3.5b}$$

$$S(z)S(w) = \frac{2/3c}{(z-w)^3} + \frac{2T(w)}{z-w} + \cdots \tag{3.5c}$$

The most general ansatz for $S_{int}(z)$ in a bosonic theory is:

$$S_{int}(z) = \sum_{\mathrm{t}} A(\mathrm{t}) e^{i\mathrm{t}\cdot X_R(z)} + i\sum_{\mathrm{l}} \mathbf{B}(\mathrm{l}) \cdot \partial_z X_R e^{i\mathrm{l}\cdot X_R(z)} \tag{3.6}$$

where

$$\mathrm{t}, \mathrm{l} \in \lambda_R \quad \text{with} \quad \mathrm{t}^2 = 3,\ \mathrm{l}^2 = 1$$

Substituting ansatz (3.6) into (3.5c) we get a complicated set of quadratic equations for the coefficients $A(\mathrm{t})$ and $\mathbf{B}(\mathrm{l})$ (see Appendix A). The general solution of this system is not known for four dimensional strings. However a large number of solutions has been found using various techniques [10],[11].

Given a bosonic supercurrent, world sheet supersymmetry requires that all constraint vectors of the form (written according to (3.1)):

$$(0; \mathrm{t}, v) \qquad (0; 1, v)$$

give integer scalar products with all lattice vectors. The self duality of the lattice $\Gamma_{18;7,1}$ implies that the constraint vectors themselves are on the lattice.

After passing to the even formulation of ref. [1] the space-time part of the lattice $(D_{4,1})$ is mapped into a D_7 lattice. All scalar products and norms change in such a way that

$$\Gamma_{18;10} = \Lambda_L \times \lambda_R \times D_7 \tag{3.7}$$

is even self-dual. We shall mostly use this formalism in our paper. Physical states of the string are characterized by $(w_L; w_R)$ (neglecting oscillator excitations) where

$$w_L \in \Lambda_L \qquad w_R \in \lambda_R \times D_7$$

The mass formulae are especially simple in the even formalism:

$$\begin{aligned} M_L^2 &= w_L^2 - 2 \\ M_R^2 &= w_R^2 - 2 \end{aligned} \tag{3.8}$$

Physical states must satisfy the additional constraint $M_L^2 = M_R^2$. It is convenient to make connection with the light-cone formalism where it is easy to identify the physical particle content. (The use of the light cone formalism is also neccessary to compute the physical partition function.) Writing $w_R = (u_R, v_R)$ which corresponds to the $\lambda_R \times D_7$ decomposition, space-time properties of the states can be read off by mapping the D_7 classes to the corresponding D_3 ones.(Here D_3 plays the role of the transverse Lorentz group.) The light cone transition rules are as follows [1]:

D_7 class	D_3 class
(0)	(v)
(v)	(0)
(s)	(c)
(c)	(s)

These rules allow us to discuss the particle spectrum.

Charged gauge bosons correspond to vectors of the form $(w_g; 0, 0)$ that satisfy the condition $w_g^2 = 2$. These vectors generate a root system that defines our gauge group. (We ignore the possibility of additional gauge bosons coming from the right lattice.)

Massless fermions are described by vectors of the form $F = (w_f; u_f, s)$, together with

$$w_f^2 = 2 \qquad u_f^2 = \frac{1}{4}$$

The requirement of chirality excludes the presence of vectors of the form

$$k = (0; \mathbf{l}, v) \qquad \text{with} \quad \mathbf{l}^2 = 1$$

To see that k indeed spoils chirality we just add it to (or substract it from) F which yields

$$F' = (w_f; u'_f, c) \qquad \text{with} \quad u'^2_f = \frac{1}{4}$$

F' has opposite chirality but is in the same representation of the gauge group.

Apart from the gravity multiplet which is always present, the massless spectrum may contain scalars. They correspond to

$$(w_s; u_s, v) \qquad \text{with} \quad w_s^2 = 2 \quad u_s^2 = 1$$

Let us now return to the question of constructing the supercurrent. The necessary condition for chirality implies that the second term is absent in (3.6). This greatly simplifies eqs. (A8) but it is still not known whether in four dimensions there are additional solutions apart from the ones found in [11]. In the much simpler case of eight dimensional strings however there are only two different solutions.

The first solution is based on the eight (length square three) vectors

$$(\pm 1, \pm 1, \pm 1) \tag{3.9a}$$

All $A(\mathbf{t})$ coefficients are equal to 1/2 in this case.

The other solution is based on the vectors

$$(\pm\sqrt{3}, 0, 0) \qquad (0, \pm\sqrt{3}, 0) \qquad (0, 0, \pm\sqrt{3}) \tag{3.9b}$$

with

$$A(t) = \frac{1}{\sqrt{3}}$$

In the first case the lattice λ_R must then contain the $(0,0,0)$ and (v,v,v) classes of the D_1^3 weight lattice and corresponds to the well known 'triplet constraint' of refs. [10]. In the second case the lattice contains the U_{12}^3 root lattice together with the conjugacy classes $(6,0,0),(0,6,0),(0,0,6)$ corresponding to the constraint vectors given in (3.9b).

After this short summary of the necessary ingredients of the model building based on self-dual lattices we now start to actually construct the right lattices.

First we deal with the right lattice Γ_{R_1} based on the supercurrent generated by the constraint vectors given in (3.9a). Because of the correlation between the space-time and internal degrees of freedom, Γ_{R_1} must contain the (v,v,v,v) conjugacy class of the $D_1^3 \times D_7$ weight lattice. Since they have to give integer scalar product with all the vectors from Γ_{R_1} we define $\Gamma_{R_1}^*$, the $(\mathbf{0})$ conjugacy class of the right lattice Γ_{R_1} to be

$$(\mathbf{0}) = (0,0,0,0) + (v,v,v,v) \tag{3.10}$$

With respect to $(\mathbf{0})$ there are 64 conjugacy classes in Γ_{R_1} generated by

$$\begin{aligned} s_1 &= (s,0,0,s) \\ s_2 &= (0,s,0,s) \\ s_0 &= (s,s,s,s) \\ v_0 &= (0,0,0,v) \end{aligned} \tag{3.11}$$

where s_1 and s_2 are fourth order elements, whereas s_0 and v_0 are of second order. This is the 'maximal' right lattice in the sense that all other possible right lattices are sublattices of it. Since $\Gamma_{R_1}^*$ has to be an even integer lattice due to the even self duality of $\Gamma_{18,10}$ one can easily see that any admissible enlargement of $\Gamma_{R_1}^*$ by conjugacy classes in Γ_{R_1} spoils chirality. Therefore one has to stick to this maximal solution in order to have chirality. Since in this paper we concentrate on chiral theories we shall use Γ_{R_1} given in (3.11) in the case of the first supercurrent.

In the case of the other supercurrent $\Gamma_{R_2}^*$ is generated by the root lattice of $U_{12}^3 \times D_7$ together with the constraint vectors:

$$(\mathbf{0}) = \langle (0,0,0,0); (6,0,0,v); (0,6,0,v); (0,0,6,v) \rangle \tag{3.12}$$

that serves as the zero conjugacy class of Γ_{R_2}. There are 108 conjugacy classes in Γ_{R_2} with respect to $(\mathbf{0})$ generated by

$$\begin{aligned} s_1 &= (1,1,1,s) \\ s_2 &= (1,-1,1,s) \\ s_3 &= (1,1,-1,s) \\ v_0 &= (0,0,0,v). \end{aligned} \tag{3.13}$$

Apart from this 'maximal' lattice there is another right lattice, $\Gamma_{R_3} \subseteq \Gamma_{R_2}$, which can also generate chiral models. It is constructed by enlarging $\Gamma^*_{R_2}$ with the conjugacy class $(3,3,3,c)$:

$$(\mathbf{0})' = (\mathbf{0}) + (3,3,3,c). \tag{3.14}$$

Then the 27 conjugacy classes of Γ_{R_3} are generated by

$$\begin{aligned} s'_1 &= (1,1,1,s) \\ s'_2 &= (-1,-1,1,s) \\ s'_3 &= (-1,1,-1,s) \end{aligned} \tag{3.15}$$

where all these vectors are third order elements.

Before turning to the classification of all chiral theories, as an illustration of our techniques, first we show how to construct the space-time supersymmetric models based on the supercurrents discussed before. In order to have space time supersymmetry we enlarge the zero conjugacy classes by adding a zero mass fermion vector, e.g. s_1. The resulting $\tilde{\Gamma}^*_{R_i}$; $i = 1,2,3$ sublattices are easily seen to be $D_2 \oplus E_8$ in the first case and $A_2 \oplus E_8$ in the other two cases. $\tilde{\Gamma}_{R_1}$ has four conjugacy classes generated by $2s_2 + v_0$ and s_0–s_1 given in (3.11), that are the v and s conjugacy classes of D_2, respectively. The three conjugacy classes of $\tilde{\Gamma}_{R_2} \equiv \tilde{\Gamma}_{R_3}$ are generated by s_3–s_2 given in (3.13) or (3.15).

The construction of the possible models with $\tilde{\Gamma}_{R_1}$ as the right lattice is almost trivial. $\tilde{\Gamma}^*_{R_1}$ together with the v conjugacy class form a self-dual lattice, namely $\Gamma_{10} = Z_2 \oplus E_8$, therefore the corresponding conjugacy classes of the left lattice also give a self-dual lattice, Γ_{18}, as a consequence of the group structure and scalar product matchings among the two sides [2]. In this way we mapped the original $\Gamma_{18,10}$ lattice into a direct sum lattice: $\Gamma_{18} \oplus \Gamma_{10}$. By reversing this procedure one can show that the supersymmetric lattice models are in one to one correspondence with the 13 Conway lattices in 18 dimensions because the left lattices, Γ_{L_1}, are the duals of the even sublattices of these Conway lattices. Gluing diagonally the four conjugacy classes of Γ_{L_1} with the corresponding ones from the right, one gets all possible $\Gamma_{18,10}$ even self-dual Lorentzian lattices corresponding to the space-time supersymmetric models based on the first world-sheet supercurrent.

The construction is not so simple in the case of the other supercurrent because $\tilde{\Gamma}_{R_2}$ does not contain a self-dual sublattice. Therefore we first enlarge $\Gamma_{18,10}$ by an auxiliary (self-dual, even, Lorentzian) lattice , demanding that its right part contains as many conjugacy classes as the original $\tilde{\Gamma}_{R_2}$ and also that the enlarged right lattice should now contain a self-dual lattice. In our particular case the auxiliary lattice can be chosen as the diagonal Lorentzian sum of two E_6 weight lattice, $\Gamma_{6,6}$. Therefore the enlarged direct sum Lorentzian lattice, $\Lambda_{24,16}$, contains the following conjugacy classes:

$$\Lambda_{24,16} = \bigcup_{i,j=0}^{2} (\Lambda_L^{ij}, \Lambda_R^{ij}) \tag{3.16}$$

where i, j parametrize the three conjugacy classes of the original $\Gamma_{18,10}$ and the auxiliary $\Gamma_{6,6}$ lattices in the decomposition with respect to the dual of their right lattices. Since the lattice

$$\Lambda_{16} = \bigcup_{i=0}^{2} \Lambda_R^{ii} \tag{3.17}$$

is a self-dual even Euclidean lattice, namely $E_8 \oplus E_8$, the corresponding conjugacy classes on the left hand side must form a 24–dimensional even self-dual lattice, Λ_{24} (Niemeier lattice). If an E_6 lattice can be embedded into Λ_{24} so that

$$\Lambda_{24} = \bigcup_{i=0}^{2} (E_6^i, \Gamma_L^i) \tag{3.18}$$

then the lattices corresponding to possible space-time supersymmetric models based on the second supercurrent are

$$\Gamma_{18,10} = \bigcup_{i=0}^{2} (\Gamma_L^i; \Gamma_{R_2}^i) \; . \tag{3.19}$$

The models generated this way are given in the following table:

Niemeier lattice	Gauge group
$A_{17}E_7$	$A_{17}U_6$
$A_{11}D_7E_6$	$A_{11}D_7$
$D_{16}E_8$	$D_{16}A_2$
$D_{10}E_7^2$	$D_{10}E_7U_6$
E_8^3	$E_8^2A_2$
E_6^4	E_6^3

As we have concentrated on the first supercurrent (corresponding to the triplet constraint) we do not go into a more detailed discussion of the classification of models with Γ_{R_2} and Γ_{R_3} as the right lattices.

IV. Construction of the chiral models

In this section we describe in detail how to construct all lattices, Γ_{L_1}, such that the Lorentzian lattice $(\Gamma_{L_1};\Gamma_{R_1})$ is even, self-dual. This amounts to a complete solution of our classification problem, as it has been shown [2] that any even self-dual Lorentzian lattice, $\Lambda_{k,l}$, admits the following decomposition:

$$\Lambda_{k,l} = \bigcup_{i=0}^{N-1} (\Delta_i^k;\Delta_i^l)$$

where Δ_0^k (resp.Δ_0^l) denotes the dual of the 'cut' lattice Λ_k (resp.Λ_l) and the Δ_i are the conjugacy classes with respect to Δ_0, furthermore the group structure of the conjugacy classes under addition is the same for the left and right lattices, which is also true for the scalar products, mod(1), and norms, mod(2). That is we have reduced our problem to finding the left handed counterparts of the conjugacy classes s_1, s_2, s_0 and v_0 which we denote by σ_1, σ_2, σ_0 and β_0, respectively.

Since the ten dimensional Euclidean lattice generated from $\Gamma_{R_1}^*$ by the conjugacy classes $2s_1, 2s_2, v_0$ is an odd self-dual one

$$\Lambda_C = \langle\ \Gamma_{L_1}^*;\ 2\sigma_1,\ 2\sigma_2,\ \beta_0\ \rangle \tag{4.1}$$

is also an odd self-dual lattice (i.e.one of the 13 possible Conway lattices in 18 dimensions). In this way the 8–dimensional chiral lattice models can naturally be divided into 13 families. β_0 generates the odd length square conjugacy class from the even sublattice of Λ_C:

$$\Lambda_C = \Lambda_C^{even} \cup (\Lambda_C^{even} + \beta_0) \ , \tag{4.2}$$

while $2\sigma_1$and $2\sigma_2$ are in the even sublattice. This follows from the fact that the four generators σ_i, β_0 must have the same scalar products as the corresponding generators of the right lattice Γ_{R_1}:

$$\begin{array}{llllr} s_1^2 = s_2^2 = 0 & \mathrm{mod}(2) & s_1 \cdot s_2 = -\frac{1}{4} & \mathrm{mod}(1)\ , & (4.3a) \\ s_1 \cdot s_0 = s_2 \cdot s_0 = 0 & \mathrm{mod}(1) & s_1 \cdot v_0 = s_2 \cdot v_0 = \frac{1}{2} & \mathrm{mod}(1)\ , & (4.3b) \\ s_0^2 = \frac{1}{2},\ v_0^2 = 1 & \mathrm{mod}(2) & s_0 \cdot v_0 = \frac{1}{2} & \mathrm{mod}(1)\ . & (4.3c) \end{array}$$

in order that the diagonal sum of the conjugacy classes form an even self-dual Lorentzian lattice. If one chooses σ_1 and σ_2 as the halves of primitive vectors from Λ_C^{even} with the given scalar products, one can decompose the even sublattice into four conjugacy classes:

$$\Lambda_C^{even} = \Lambda_{00} \cup \Lambda_{0\frac{1}{2}} \cup \Lambda_{\frac{1}{2}0} \cup \Lambda_{\frac{1}{2}\frac{1}{2}} \tag{4.4}$$

where

$$\Lambda_{a_1,a_2} = \{w \in \Lambda_C^{even} : w \cdot \sigma_i = a_i \ \mathrm{mod}(1),\ i = 1,2\} \ . \tag{4.5}$$

One can easily see that they correspond to the $(0)_L$, $2\sigma_1$, $2\sigma_2$, $2(\sigma_1+\sigma_2)$ conjugacy classes of Γ_{L_1}, that is $\Lambda_{00} = \Gamma_{L_1}^*$. Since the self-dual lattice Λ_C has eight conjugacy classes with respect to Λ_{00}, the volume of this lattice is eight. Thus $\Lambda_{00}^* = \Gamma_{L_1}$ contains 64 conjugacy classes with respect to Λ_{00} which correspond to the ones generated by σ_i, $i = 0,1,2$ and β_0. Therefore finding σ_1 and σ_2 is sufficient to determine $\Gamma_{L_1}^*$ uniquely. One of the remaining generators, β_0, can be found in the odd conjugacy class of Λ_C, while σ_0 is in the conjugacy class with non–integer length square of $(\Lambda_C^{even})^*$.

We note that, since $\Gamma_{L_1}^*$, the zero conjugacy class of Γ_{L_1} is a sublattice of one of the Conway lattices and since the root lattice of $\Gamma_{L_1}^*$ determines the gauge group, the only possible gauge groups are the regular subgroups of the groups associated to the Conway lattices.

To actually construct Γ_{L_1}, first one has to choose a Λ_C, and then one has to find representatives of the four generators, σ_i, β_0. For pedagogical reasons let us start with the simplest family corresponding to $\mathbf{Z}_{18}$. We use the standard orthonormal lattice basis $\mathbf{e}_i$; $i = 1, \ldots, 18$, that is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \; . \tag{4.6}$$

As explained above, $(\mathbf{0})_L$ is given by the set of vectors from $\mathbf{Z}_{18}^{even}$ giving integer scalar products with σ_1 and σ_2. If we define:

$$\Sigma_i \equiv 2\sigma_i \qquad i = 1, 2$$

then the relation $w \cdot \Sigma_i \in 2\mathbf{Z}$; $i = 1, 2$ holds for every $w \in (\mathbf{0})_L$. Since $(\mathbf{0})_L$ determines Γ_{L_1} uniquely ($(\mathbf{0})_L^* = \Gamma_{L_1}$) we can add vectors of the form $2v$, $v \in \mathbf{Z}_{18}$ to Σ_i without changing the resulting lattice. Using this freedom one can reduce any pair of Σ_1 and Σ_2 to the 'standard' form:

$$\begin{aligned} \tilde{\Sigma}_1 &= (1, 1, \ldots, 1|0, 0, \ldots, 0|0, 0, \ldots, 0|1, 1, \ldots, 1) \\ \tilde{\Sigma}_2 &= (0, 0, \ldots, 0|1, 1, \ldots, 1|0, 0, \ldots, 0|1, 1, \ldots, 1) \end{aligned} \tag{4.7}$$

On the other hand since $\sigma_0 \in (\mathbf{Z}_{18}^{even})^*$, $\Sigma_0 \equiv 2\sigma_0$ gives even scalar products with even vectors of $\mathbf{Z}_{18}$, therefore the standard form of Σ_0 is necessarily

$$\tilde{\Sigma}_0 = (1, 1, \ldots, 1|1, 1, \ldots, 1|1, 1, \ldots, 1|1, 1, \ldots, 1) \tag{4.8}$$

Because of eq. (4.3), $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ contain an even number of 1 entries with an odd number of overlap between them. Therefore the vectors $\tilde{\Sigma}_i$, $i = 1, 2$ can be characterized by the distribution of the 18 basis vectors in four equivalent boxes, each containing an odd number of elements. These boxes are defined according to the distribution of the basis vectors between $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$, namely B_1 contains those which appear only in $\tilde{\Sigma}_1$, B_2 contains those appearing only in $\tilde{\Sigma}_2$. The third (resp. fourth) box consists of the basis vectors appearing in none (resp. both) of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$. They are equivalent because the roles of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ can be interchanged, moreover one can add $\tilde{\Sigma}_0$ to either of them without changing the resulting $(\mathbf{0})_L$. Therefore the members of the $\mathbf{Z}_{18}$ family correspond to such partitions of basis vectors. Conversely given an odd partition of the basis vectors into four boxes we can define B_α ($\alpha = 1, \ldots, 4$) as the sum of the $\mathbf{e}_i$'s contained in the α^{th} box. We define:

$$\begin{aligned}
\tilde{\sigma}_1 &= \frac{1}{2}(B_1 + B_4) \\
\tilde{\sigma}_2 &= \frac{1}{2}(B_2 + B_4) \\
\tilde{\sigma}_0 &= \frac{1}{2}(B_1 + B_2 + B_3 + B_4)
\end{aligned} \tag{4.9}$$

Finally $\tilde{\beta}_0$ can be chosen to be any basis vector belonging to B_4. These vectors satisfy (4.3c), however (4.3a–b) are only satisfied mod(1/2). It is not difficult to show that one can always modify them to get the vectors σ_i, β_0 satisfying (4.3). For example, if $\tilde{\sigma}_1^2$ is half–integer then $(\tilde{\sigma}_1 + \tilde{\sigma}_0)^2$ is integer. Furthermore if it is an odd integer then adding $2\tilde{\sigma}_2$ changes its norm to 0 mod(2). So in this case

$$\sigma_1 = \tilde{\sigma}_1 + \tilde{\sigma}_0 + 2\tilde{\sigma}_2$$

One can repeat this procedure for σ_2 and by changing its sign, if necessary, (4.3a) will be satisfied. Finally by adding $2\sigma_1$ and/or $2\sigma_2$ to $\tilde{\sigma}_0$ and/or $\tilde{\beta}_0$ one can satisfy (4.3b). Altogether there are 11 different odd partitions of 18 which in the case of the $\mathbf{Z}_{18}$ family give 11 different models.

As an example we consider the partition

$$18 = 11 + 3 + 3 + 1$$

leading to the root lattice $D_{11} \times D_3 \times D_3 \times D_1$. $(\mathbf{0})_L$ is given as this minimal lattice together with the single glue vector $\gamma = (v, v, v, v)$. In this notation the generating vectors of Γ_{L_1} are:

$$\begin{aligned}
\sigma_1 &= (c, v, 0, s) \\
\sigma_2 &= (0, c, v, s) \\
\sigma_0 &= (s, s, c, s) \\
\beta_0 &= (0, 0, 0, v)
\end{aligned} \tag{4.10}$$

The lowest lying states are determined by the left hand partners of the following conjugacy classes of Γ_{R1}:

$$
\begin{aligned}
\text{gauge bosons} &: (0,0,0,0) \\
\text{massless fermions} &: (s,0,0,s),(c,0,0,s),(0,s,0,s) \\
&\quad (0,c,0,s),(0,0,s,s),(0,0,c,s) \\
\text{massless scalars} &: (v,0,0,v),(0,v,0,v),(0,0,v,v) \\
\text{tachyons of mass}^2 - 1 &: (0,0,0,v) \\
\text{tachyons of mass}^2 - 1/2 &: (s,s,0,v),(s,c,0,v),(c,s,0,v),(c,c,0,v) \\
&\quad (s,o,s,v),(s,0,c,v),(c,0,s,v),(c,0,c,v) \\
&\quad (0,s,s,v),(0,s,c,v),(0,c,s,v),(0,c,c,v)
\end{aligned}
\tag{4.11}
$$

To find the left counterparts of these classes we first express them in terms of the generators s_i, v_0. For example, three of the 12 conjugacy classes that can potentially contain tachyons of mass square –1/2 are expressed as:

$$
\begin{aligned}
t_1 &= (s,s,0,v) = s_1 + s_2 \\
t_2 &= (s,0,c,v) = s_2 + s_0 + 2s_1 \\
t_3 &= (0,c,c,v) = v_0 + s_1 + s_0
\end{aligned}
\tag{4.12}
$$

Using (4.10) their left partners are:

$$
\begin{aligned}
\tau_1 &= \sigma_1 + \sigma_2 = (c,s,v,v) \\
\tau_2 &= \sigma_2 + \sigma_0 + 2\sigma_1 = (c,0,s,0) \\
\tau_3 &= \beta_0 + \sigma_1 + \sigma_0 = (0,c,c,0)
\end{aligned}
\tag{4.13}
$$

Among these three only the last one contains physical tachyons since the minimal length square is 7/2 for τ_1 and τ_2. Thus we see that the level matching conditions for the lowest lying states may not be satisfied for some classes. We remark that on the other hand there may be several inequivalent representations of the same mass in a given class due to the presence of glue vectors in $(\mathbf{0})_L$.

The construction of the $\mathbf{Z}_{18}$ family lends itself to a straightforward generalization. This is based on the possibility of finding an 'odd' basis for the other Conway lattices, Λ_C satisfying

$$
\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \bmod(2) \tag{4.14}
$$

In Appendix B we prove the existence of such an 'odd' basis for any odd self-dual lattice. Using such a basis one can go through the construction given on the $\mathbf{Z}_{18}$ example with only minor modifications.

To find the different Γ_{L_1} lattices it is sufficient to give all representatives of σ_1 and σ_2, which lead to different $\Gamma^*_{L_1} = \Lambda_{00} \subseteq \Lambda_C^{even}$ lattices. Since both of the representatives $\tilde{\sigma}_1, \tilde{\sigma}_2$ can be written as halves of primitive vectors in Λ_C^{even}:

$$\tilde{\sigma}_i = \frac{1}{2}\sum_{j=1}^{18} n_{ij}\mathbf{e}_j, \qquad i = 1,2;\ n_{ij} \in \mathbf{Z}\ , \tag{4.15}$$

and Λ_{00} is given by eq.(4.5), it follows that models differing in their n matrices as

$$n_{ij} = n'_{ij} \bmod(2), \qquad i = 1,2;\ j = 1,\ldots,18 \tag{4.16}$$

lead to the same Λ_{00} lattice. Therefore we can choose n_{ij} to be either zero or one, which means that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ correspond to certain subsets of the basis vectors. Since

$$\tilde{\sigma}_i \cdot \tilde{\sigma}_j = \frac{1}{4}\sum_{k,l=1}^{18} n_{ik}n_{jl}\mathbf{e}_k \cdot \mathbf{e}_l = \left[\frac{1}{4}\sum_{k=1}^{18} n_{ik}n_{jk}\right] \quad \bmod\left(\frac{1}{2}\right) \tag{4.17}$$

both $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ should be the sum of an even number of basis vectors with an odd number of common elements, to ensure that their norms and scalar products are integers and $\frac{1}{4} \bmod(\frac{1}{2})$, respectively, as indicated in (4.3).

These requirements lead to the same odd partition of the basis vectors introduced earlier in the case of the $\mathbf{Z}_{18}$ lattice. The only difference between that simple example and the general case is that in the first case the permutation group S_{18}, acting on the set of basis vectors was a symmetry group of the metric $g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j$, while in the latter case not all of the basis vectors are equivalent, only a permutation subgroup of S_{18} is a symmetry group of the metric. Therefore the distribution is not determined by the partition alone in general (whereas this was true for $\mathbf{Z}_{18}$). One needs to know in addition how the inequivalent basis vectors are distributed among the four boxes, $\mathrm{B}_1, \ldots, \mathrm{B}_4$.

If we define the representatives of the generating conjugacy classes to be

$$\begin{aligned}
\tilde{\sigma}_1 &= \frac{1}{2}\sum_{i=1}^{18}\left(\delta_{\mathrm{e}_i\in \mathrm{B}_1}+\delta_{\mathrm{e}_i\in \mathrm{B}_4}\right)\cdot \mathbf{e}_i \ , \\
\tilde{\sigma}_2 &= \frac{1}{2}\sum_{i=1}^{18}\left(\delta_{\mathrm{e}_i\in \mathrm{B}_2}+\delta_{\mathrm{e}_i\in \mathrm{B}_4}\right)\cdot \mathbf{e}_i \ , \\
\tilde{\sigma}_0 &= \frac{1}{2}\sum_{i=1}^{18}\mathbf{e}_i \ , \\
\tilde{\beta}_0 &= \sum_{i=1}^{18}\delta_{\mathrm{e}_i\in \mathrm{B}_4}\cdot \mathbf{e}_i
\end{aligned} \tag{4.18}$$

we find that the length squares are:

$$\begin{aligned}
\tilde{\beta}_0^2 &= 1 \bmod(2) & \tilde{\sigma}_1^2 &= \frac{1}{2} \bmod(\frac{1}{2}) \\
\tilde{\sigma}_0^2 &= \frac{1}{2} \bmod(1) & \tilde{\sigma}_2^2 &= \frac{1}{2} \bmod(\frac{1}{2})
\end{aligned} \tag{4.19}$$

which is a direct consequence of the odd basis. The only nontrivial case, the value of $\tilde{\sigma}_0^2$, follows from the equality $\det g = 1$, which implies that there is an even number of diagonal elements of the form $4n+3,\ n\in \mathbf{Z}$ in the metric g. Therefore

$$\tilde{\sigma}_0^2 = \frac{1}{4}\Big(\sum_i g_{ii} + 2\sum_{i<j} g_{ij}\Big) = \frac{1}{2} \bmod(1) \ . \tag{4.20}$$

Now $\Gamma_{L_1}^*$ is determined by $\tilde{\sigma}_1,\ \tilde{\sigma}_2$, namely $\Gamma_{L_1}^*$ is that even sublattice of Λ_C, whose vectors give integer scalar products with both $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. (Though the original scalar products among $\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_2$ and $\tilde{\beta}_0$ should be modified in general as illustrated for the case of $\mathbf{Z}_{18}$, this however does not change $\Gamma_{L_1}^*$.) To characterize the lattice $\Delta \equiv \Gamma_{L_1}^*$ we give its root lattice $\Delta_r \subseteq \Delta$, generated by the length square 2 vectors corresponding to the root system of semisimple Lie–algebras and possibly by longer vectors generating orthogonal $U(1)$ directions. Usually Δ_r is a proper sublattice of Δ, since Δ contains conjugacy classes from the weight lattice of Δ_r as well. The generators of these conjugacy classes are referred to as glue vectors, and they have at least length square 4.

Since Δ^* contains 64 conjugacy classes with respect to Δ, that is the volume of Δ is equal to 8, the order of the Abelian group generated by the glue vectors has to be

$$|G| = \frac{1}{8} vol(\Delta_r) \ . \tag{4.21}$$

To distinguish between the models one has to compare the various lattices Δ. This can be done in two steps: first comparing the root lattices, Δ_r, and if they are the same, then comparing the glue vectors. If those are also the same then the models are identical, otherwise they are different. Of course one has to write the lattices in a form independent of the choice of a particular coordinate basis, and one has to find a canonical form of the generating glue vectors in order to compare the lattices, which usually look different only because of their different embeddings in the Conway lattice, Λ_C. Fortunately, we need to compare those models only wich come from the same Conway lattice. Of course it can happen (it does actually) that the root lattices Δ_r of two models obtained from different Λ_C's are the same, but the whole lattices Δ are certainly different. This follows from the fact that Λ_C can be built up from Δ in the unique way described in eq. (4.1).

Having found the inequivalent models one can easily determine the zero mass fermions, scalars and the tachyons for each model. (Of course the gauge group is determined by the root lattice Δ_r.) For zero mass matter fields or tachyons one has to look for vectors of norm 2 or 1 and 3/2 respectively, in the sectors of the left lattice, Γ_{L_1}, corresponding to the right conjugacy classes given in (4.11). Since the glue vectors are elements of $\Gamma^*_{L_1}$ they do not change the conjugacy classes of Γ_{L_1}, but since they transform non–trivially under the gauge group one has to keep in mind that there are vectors with different representations of the gauge group within one conjugacy class of Γ_{L_1}.

We would like to note that one does not get chiral models automatically: it can happen that there are no zero mass fermions at all. But if there is at least one zero mass fermion multiplet, the model is chiral. This follows from the fact that none of the sums of two right conjugacy classes corresponding to zero mass fermions (given in (4.11)) is equal to the zero conjugacy class of the right lattice Γ_{R_1}.

We close this chapter by a short description of the algorithm for finding the inequivalent chiral heterotic models.

1. Choose an 18–dimensional self-dual euclidean lattice Λ_C.
2. Find an odd basis for Λ_C.
3. List the inequivalent odd partitions of the odd basis.
4. Build up the generators $\tilde{\sigma}_i$, $i = 0, 1, 2$ and β_0 from the given partitions.

5. Find the lattice Δ_r, that is those 'good' root vectors of Λ_C, which give integer scalar product with $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$.
6. Determine the Lie–algebra corresponding to Δ_r, that is find the simple roots in the orthogonal clusters of the 'good' root vectors.
7. Find the glue vectors from Λ_C^{even}, that is the generators of the Δ/Δ_r cosets. They are among the sums of 'bad' roots (elements of $\Lambda_{\frac{1}{2}0}, \Lambda_{0\frac{1}{2}}, \Lambda_{\frac{1}{2}\frac{1}{2}}$) which give integer scalar product with $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$.
8. Arrange the generating glue vectors in a basis independent canonical form.
9. Select the inequivalent $\Delta = \Gamma_{L_1}^*$ lattices.
10. Determine the matter representation of the given model.

V. Restrictions on tachyon free models

In this chapter we are going to demonstrate that demanding the absence of tachyons from the spectrum reduces the size of the classification problem substantially. Although the actual details of our analysis are specific to the 8-dimensional case, a similar analysis can also be performed for the case of 4-dimensional string models.

We start by recalling that the structure of the right lattice determines the conjugacy class structure of the left lattice together with their norms and mutual scalar products (modulo 2 and 1, respectively). This information is necessary and sufficient to determine the modular transformation properties of the left lattice partition functions and this is why the requirement of modular invariance is equivalent to the self-duality of the lattice $\Gamma_{18;10}$.

In the case of chiral 8-dimensional string models considered in this paper the right lattice Γ_{R_1} is uniquely fixed by its (**0**) conjugacy class (3.10) and generating vectors (3.11). It consists of 64 conjugacy classes with respect to (**0**), however, using the permutation and reflection symmetries of Γ_{R_1}, they can be grouped into 9 clusters. The elements of a given cluster are conjugacy classes that are related by permutation or reflection symmetries of Γ_{R_1} and therefore obviously have the same partition functions. These clusters are obtained by completing (4.11) and are listed below:

Cluster	Representative class	Elements	Norm	Physical states
E_1	$(000,0)$	1	0	gauge bosons
E_2	$(vv0,0)$	3	0	massless scalars
E_3	$(s00,s)$	12	0	massless fermions
Q_1	$(000,v)$	1	1	tachyons of mass^2 -1
Q_2	$(v00,0)$	3	1	massive bosons
Q_3	$(s0v,s)$	12	1	massive fermions
Y	$(ssv,0)$	12	3/2	tachyons of mass$^2 - \frac{1}{2}$
Z_1	$(ss0,0)$	12	1/2	massive bosons
Z_2	(sss,s)	8	1/2	massive fermions

If we denote by $l_A(\tau)$ and l_A $(A = 1,\ldots,64)$ the partition function and representative vector of the A^{th} conjugacy class of the left lattice respectively, then the modular transformations are summarized by [8]

$$\begin{aligned} l_A(\tau) &= \sum_B T_{AB} l_B(\tau+1) \qquad & T_{AB} &= e^{-i\pi l_A^2}\delta_{AB} \\ l_A(\tau) &= i^k \tau^{-k} \sum_B S_{AB} l_B(-\frac{1}{\tau}) \qquad & S_{AB} &= \frac{1}{v} e^{2\pi i (l_A \cdot l_B)} , \end{aligned} \tag{5.1}$$

where $k = 9$ and $v = 8$. Since all norms l_A^2 are quantized in units of $\frac{1}{2}$, the 64×64 matrix T in (5.1) satisfies $T^4 = 1$. The lattice partition functions $l_A(\tau)$ are modular forms of the modular subgroup $\Gamma(4)$, whereas the matrices generated by T and S form a representation of the quotient group $\Gamma/\Gamma(4) = SL(2,Z_4)/Z_2 \cong S_4$. The important observation is that the full, modular invariant partition function can be written as

$$Z(\tau,\bar{\tau}) = \sum_{A=1}^{64} l_A(\tau)\bar{r}(\bar{\tau}) = \sum_{\gamma=1}^{9} Z(\gamma)\bar{r}(\gamma)$$

where $r(\gamma)$ is the partition function of any representative conjugacy class of the γ cluster and $Z(\gamma)$ is the sum of partition functions of all conjugacy classes from the left lattice paired with the elements of the given cluster γ. Modular invariance implies that the 9 cluster functions $Z(\gamma)$ are closed under modular transformations. The corresponding 9×9 matrices T and S can be calculated using (5.1): T is diagonal with phases determined by the modulo 2 norm of the cluster, whereas $8S$ is given by

$$\begin{array}{c|ccc|ccc|c|cc}
 & E_1 & E_2 & E_3 & Q_1 & Q_2 & Q_3 & Y & Z_1 & Z_2 \\
E_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
E_2 & 3 & 3 & -1 & 3 & 3 & -1 & -1 & -1 & 3 \\
E_3 & 12 & -4 & 0 & -12 & 4 & 0 & -4 & 4 & 0 \\
Q_1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
Q_2 & 3 & 3 & 1 & 3 & 3 & 1 & -1 & -1 & -3 \\
Q_3 & 12 & -4 & 0 & -12 & 4 & 0 & 4 & -4 & 0 \\
Y & 12 & -4 & -4 & 12 & -4 & 4 & 0 & 0 & 0 \\
Z_1 & 12 & -4 & 4 & 12 & -4 & -4 & 0 & 0 & 0 \\
Z_2 & 8 & 8 & 0 & -8 & -8 & 0 & 0 & 0 & 0
\end{array} \tag{5.2}$$

(The matrix (5.2) is not symmetric in the cluster basis, but can be brought to a symmetric form by appropriately normalizing $Z(\gamma)$.)

Using S_4 characters one can immediately see that the 9×9 matrix (5.2) (together with T) is decomposed into the following irreducible S_4 representations: $2\times \mathbf{3}+\mathbf{3}'$. Unfortunately, in order to be able to see the restrictions on the possible models imposed by modular invariance, we have to carry out the decomposition explicitly.

For this purpose, it is convenient to expand the cluster partition functions in terms of θ-functions. If the left lattice is a sublattice of the D_1^{18} weight lattice, then the partition functions are already given in this form, since the θ-functions are nothing but D_1 partition functions. However, any left lattice partition function can be written as some 18^{th} order polynomial in even powers of θ-functions, simply because even powers of θ-functions form a complete basis for the ring of modular forms of $\Gamma(4)$ [4].

Using this basis, we can form the following triplets of S_4:

$$L_k^{\pm} = \left\{ \begin{aligned} Z_k^{\pm} &= [\theta_3^{18-2k} \pm \theta_4^{18-2k}]\theta_2^{2k} \\ N_k^{\pm} &= [\theta_3^{18-2k} \pm \theta_2^{18-2k}]\theta_4^{2k} \\ -P_k^{\pm} &= [\theta_4^{18-2k} \pm (-1)^{\frac{9-k}{2}} \theta_2^{18-2k}]\theta_3^{2k} \end{aligned} \right\} \qquad k = 1,3,5,7,9$$

These are the generalizations of the obvious triplet $\{\theta_2^{18}, \theta_3^{18}, \theta_4^{18}\}$, but because of the Riemann identity $\theta_3^4 - \theta_4^4 = \theta_2^4$, only 5 of them are linearly independent. We choose our basis as:

$$\begin{array}{cccc} & L_1^- \text{ and } L_3^+ & (\mathbf{3}') \\ & L_1^+,\ L_3^- \text{ and } L_5^+ & (\mathbf{3}) \end{array} \tag{5.3}$$

Altogether there are 19 linearly independent combinations of θ_i^2 of order 18. 15 of them are occuring in (5.3), while the rest of them form the multiplets $\mathbf{1}, \mathbf{1}'$ and $\mathbf{2}$ and therefore play no role in our analysis.

Introducing the T-diagonal combinations

$$U_k^\pm = N_k^\pm + P_k^\pm \qquad \text{and} \qquad V_k^\pm = N_k^\pm - P_k^\pm$$

the cluster partition functions are expanded as

$$\begin{aligned} Z(E_i) &= \bar{a}_i V_1^- + \bar{b}_i V_3^+ + \bar{c}_i V_1^+ + \bar{d}_i V_3^- + \bar{e}_i V_5^+ \\ Z(Q_i) &= a_i U_1^- + b_i U_3^+ + c_i U_1^+ + d_i U_3^- + e_i U_5^+ \qquad i = 1,2,3 \\ Z(Y) &= A Z_1^- + B Z_3^+ \\ Z(Z_\alpha) &= C_\alpha Z_1^+ + D_\alpha Z_3^- + E_\alpha Z_5^+ \qquad \alpha = 1,2 \end{aligned}$$

where the coefficients are arbitrary constants. Comparing the transformation properties of the triplets under S with (5.2) we find the following constraints among these coefficients:

$$\begin{aligned} \bar{x}_1 = -\bar{x}_2 = x_1 = -x_2 &= \frac{1}{8} X \\ -\bar{x}_3 = x_3 &= \frac{1}{2} X \end{aligned} \tag{5.4a}$$

for $x = a, b$ and $X = A, B$ ($\mathbf{3}'$ representations) and

$$\begin{aligned} \bar{x}_1 &= \frac{1}{8}X_1 + \frac{1}{8}X_2 & x_1 &= \frac{1}{8}X_1 - \frac{1}{8}X_2 \\ \bar{x}_2 &= -\frac{1}{8}X_1 + \frac{3}{8}X_2 & x_2 &= -\frac{1}{8}X_1 - \frac{3}{8}X_2 \\ \bar{x}_3 &= \frac{1}{2}X_1 & x_3 &= -\frac{1}{2}X_1 \end{aligned} \tag{5.4b}$$

for $x = c, d, e$ and $X = C, D, E$ ($\mathbf{3}$ representations). Using (5.4) we see that the 9 cluster functions can be characterized by 8 independent coefficients: A, B; $C_1, D_1, E_1, C_2, D_2, E_2$.

The next step is to compare the q-expansion of the lattice functions with that of the θ-function triplets. Defining

$$\begin{aligned}
Z(E_i) &= N_0(E_i) + qN(E_i) + \cdots \\
Z(Q_i) &= q^{1/2}N(Q_i) + \cdots \\
Z(Y) &= q^{3/4}N(Y) + \cdots \\
Z(Z_\alpha) &= q^{1/4}N(Z_\alpha) + \cdots
\end{aligned} \tag{5.5}$$

we find

$$\begin{aligned}
N_0(E_i) &= 2(\bar{a}_i + \bar{b}_i + \bar{c}_i + \bar{d}_i + \bar{e}_i) \\
N(E_i) &= 712\bar{a}_i + 72\bar{b}_i + 712\bar{c}_i + 72\bar{d}_i + 456\bar{e}_i \\
N(Q_i) &= 56a_i + 24b_i + 56c_i + 24d_i - 8e_i \\
N(Y) &= 256A + 128B \\
N(Z_\alpha) &= 8C_\alpha
\end{aligned} \tag{5.6}$$

Now, taking into account that the origin of the lattice appears in the E_1 cluster only,

$$N_0(E_i) = \delta_{i1}, \qquad i = 1,2,3 \tag{5.7}$$

the number of independent parameters is finally reduced to 5. These we choose to be the following:

$n_1 = N(Q_1) + N(Q_2)$: number of unit length vectors in the Conway lattice Λ_C which the model is based on; $\frac{1}{2}n_1$ is the number of Z-factors in Λ_C

$n_2 = N(E_1) + N(E_2)$: number of roots in Λ_C

$N(-1/2) = N(Y)$: number of tachyons of mass^2 $-\frac{1}{2}$

$N(-1) = N(Q_1)$: number of tachyons of mass^2 -1

$N(G) = N(E_1)$: number of roots of the gauge group

Putting everything together, we find that the multiplicities of the smallest norm vectors satisfy

$$N(Z_1) = \frac{1}{64}\{4N(G) - 12n_1 - n_2 + 228 + 48N(-1) - 4N(-1/2)\} \tag{5.8a}$$

$$N(Z_2) = \frac{1}{64}\{n_2 - 12n_1 - 180\} \tag{5.8b}$$

$$N(Q_3) = n_1 - 12 - 4N(-1) + \frac{1}{4}N(-1/2) \tag{5.8c}$$

$$N(F) = N(E_3) = 852 - n_2 + 4N(G) - 5N(-1/2) \tag{5.8d}$$

These constraints, apart from the last one, which gives the number of massless fermions, are uninteresting from the point of view of low-energy physics, since they give the number of massive particles as a function of massless and tachyonic ones. Note however, that all $N(\gamma)$-s are non-negative integers, so from (5.8c), assuming the absence of tachyons from the physical spectrum of the model we can derive the important inequality

$$n_1 \geq 12. \tag{5.9}$$

This implies that tachyon-free models can only be based on Conway lattices containing at least 6 Z-factors! Since models based on Z_{18} will always contain tachyons of mass^2 -1, this leaves us with the first two Conway lattices only, namely E_8Z_{10} and $D_{12}Z_6$. The restriction on tachyon-free models therefore reduces the classification problem significantly.

Finally we remark that there are only two tachyon-free models, one for each of the admissable Conway lattices. Although the lattice partition functions of these models are clearly different, the physical (light-cone) partition functions turn out to be the same. The integral of this partition function over the fundamental domain of the modular group yields a non-vanishing one-loop cosmological constant for these models.

VI. Discussion of the results

The result of the classification of the 8–dimensional chiral heterotic strings based on the world sheet supersymmetry given by (3.9a) can be seen in Table I. There are 275 different gauge groups and 444 different models. To cut this paper to a manageable size, we list only the gauge groups and the number of different models with the given gauge group. Complete lists containing the matter representations and some other data necessary to reproduce the results are also available. In the following table we present for convenience the total number of gauge groups and chiral models for each of the 13 Conway-Sloane lattices .

Lattice	Number of gauge groups	Number of chiral models
1.	25	30
2.	43	58
3.	14	16
4.	8	8
5.	10	13
6.	40	64
7.	43	55
8.	33	40
9.	43	60
10.	26	32
11.	18	23
12.	11	11
13.	30	34

As we argued earlier, models coming from different Conway-Sloane lattices are necessarily inequivalent,therefore the sum of the entries of the last column simply gives the total number of different models, i.e. 444. However, since a given gauge group can be generated from different parent lattices the sum of the entries in the second column is greater than 275.

There are only two tachyon free models among the chiral ones having gauge groups $D_4^2 A_7 A_1^2 U_8$ and $A_3^4 A_1^6$. They are derived from the Conway lattices $E_8 Z_{10}$ and $D_{12} Z_6$ respectively, in agreement with the results of the previous section. The details of these models are given in Table II.

Since the vectors of the left lattice corresponding to the zero mass matter multiplets have length square two, the possible representations of the gauge group are very restricted: only those representations are allowed where the maximal length square of the weight vectors is less than or equal to two. These can be the adjoint representations of all simply laced algebras, the **78** and the **27**, **27*** in the case of E_7 and E_6, respectively, the vector and the two spinor representation (if $n \leq 8$) in the case of D_n and the k–fold antisymmetrized tensors in the case of A_n

if $k \cdot (n+1-k)/(n+1) \leq 2$.

One gets further restrictions from the structure of the right lattice Γ_R. In the non–supersymmetric case of Γ_{R_1} there can be only scalars and vectors in the $[(\mathbf{0})_L; (\mathbf{0})_R]$ conjugacy class, in the corresponding supersymmetric case it contains fermions as well. These and only these states are in the adjoint representation of the gauge group. But the existence of zero mass adjoint scalars and the level matching conditions imply that the right inner part of the corresponding lattice vectors should have length square one, as the scalars are in the (v) conjugacy class of the space–time lattice in the covariant formulation. But this is in contradiction with the assumption of chirality, therefore in chiral models we never get zero mass adjoint scalars favoured in the Higgs sectors of grand unified models.

Since in the non–supersymmetric case no fermions are contained in the zero conjugacy class, the existence of adjoint zero mass fermions is excluded. Therefore the constraints mentioned above are very close to the ones given in [12] to avoid exotic fermion representations with respect to the colour group $SU(3)$ in unified gauge models.

VII. Perspectives in four dimensions

In order to get a feeling what will happen in the case of four space–time dimensions, we have carried out a small part of the classification of 4–dimensional chiral rank 22 heterotic strings. The main differences from the 8–dimensional case are:

— there are not 13 but 68 odd self-dual lattices in 22 dimensions which the models can be derived from,

— the possible world sheet supersymmetries, therefore the possible right lattices have not been classified yet,

— there are chiral, space-time supersymmetric ($N = 1$) models,

— one accepts only the tachyon free models.

Apart from these points the classification can be done along the same lines as in the 8–dimensional case.

It is clearly important to see how the number of the different models grows as compared to the 8–dimensional case, so we have chosen a particular lattice, namely the $E_8 Z_{14}$ one, and a particular right lattice based on the 'triplet constraints', which leads to chiral models and contains 64 conjugacy classes similarly to the case of Γ_{R_1} in 8 dimensions.

The right lattice Γ_R is generated by the conjugation classes

$$\begin{aligned} s_1 &= (s00\ s00\ s00\ s) \\ s_2 &= (0s0\ 0s0\ 0s0\ s) \\ s_0 &= (sss\ sss\ sss\ s) \\ v_0 &= (000\ 000\ 000\ v) \end{aligned} \tag{7.1}$$

while the zero conjugacy class Γ_R^* is given by the root lattice of $D_1^9 D_5$ and the following weight vectors:

$$\begin{aligned} &(vvv\ 000\ 000\ v), (000\ vvv\ 000\ v), (000\ 000\ vvv\ v) \\ &(v00\ v00\ 000\ 0), (000\ v00\ v00\ 0) \\ &(0v0\ 0v0\ 000\ 0), (000\ 0v0\ 0v0\ 0) \end{aligned} \tag{7.2}$$

The final results can be found in tables III. and IV. Table III. contains all the Γ_L^* lattices leading to chiral models, the zero mass matter multiplets corresponding to the 8 tachyon free models can be read off from table IV. We make the following remarks concerning the main differences between the four and the eight dimensional case. The number of odd partitions is 18, not 11. In 4 dimensions there are 44 different gauge groups, while from the corresponding $E_8 Z_{10}$ case one gets 25. The corresponding numbers of the inequivalent (tachyon free) models are 54 (8) and 30 (1), respectively. As the increase in the number of models coming from similar lattices is not so substantial, we think that mainly the larger number of self-dual lattices and even more importantly the right lattices will increase the number of chiral models in 4 dimensions.

An interesting feature of the right lattice given in (7.1–7.2) that it does not lead to mass square -1/2 tachyons, since the corresponding sectors of Γ_R contain vectors with length square greater than 2 only. Thus the indecomposable 22–dimensional self-dual Euclidean lattices that do not contain Z factors, therefore vectors with length square 1, give rise to tachyon free models automatically.

Another interesting property of this right lattice is that due to the presence of the length square two vectors in Γ_R^*, there is a global $SU(4)$ symmetry in these models. As a consequence, every zero mass fermion (scalar) multiplet comes with a multiplicity of four (six), as is easily seen by examining the corresponding conjugacy classes of the right lattice.

VIII. Conclusions

We succeeded to give a complete classification of all 8 dimensional chiral heterotic strings which can be obtained from the covariant lattice approach based on the triplet constraint. The extension of our method to obtain all such lattice based models in 4 dimensions is entirely straightforward. Our algorithm has been implemented on an IBM Personal Computer in compiled BASIC (Microsoft QUICK BASIC 4.0). To carry out the classification in 4 dimensions one has to port the program to a mainfraim machine as the running time on a PC would be prohibitively long. However as the total number of four dimensional chiral, supersymmetric models would certainly exceed several thousands, the results can be only stored in a database. However for the sake of completeness one should first classify all possible world-sheet supercurrents, which seems to be rather difficult. It is clear that one can play with these models but some more fundamental understanding is needed in string theory as how to make contact with reality. As a side remark we mention that one could also try to apply our approach for the classification of higher dimensional (> 25) self-dual lattices , where the number of lattices is still manageable.

Appendix A

In this Appendix we will discuss the construction of the internal part of the world sheet supercurrent in some detail. Our starting point is Ansatz (3.6) which is the most general form the internal supercurrent can take in the lattice compactified bosonic formulation. Actually (3.6) can be written in the more accurate form

$$S_{\text{int}}(z) = \sum_{a^2=3} A(a)\Omega(a) \colon e^{ia\cdot\phi(z)} \colon + i \sum_{u^2=1} B^j(u)\Omega(u) \colon \partial_z \phi^j(z) e^{iu\cdot\phi(z)} \colon \qquad (A.1)$$

taking into account the following two points not mentioned in Section 3. First, the exponentials are normal ordered with respect to the usual decomposition of the internal bosonic fields $\phi^i(z)$ into their positive and negative frequency parts

$$\phi^i(z) = \phi^i_-(z) + \phi^i_+(z), \qquad i = 1, 2, \ldots, N. \qquad (A.2)$$

Here

$$\phi_{-}^{i}(z) = -ip^{i} \ln z + i \sum_{n>0} \frac{1}{n} \alpha_{n}^{i} z^{-n} \tag{A.3a}$$

and

$$\phi_{+}^{i}(z) = x^{i} + i \sum_{n<0} \frac{1}{n} \alpha_{n}^{i} z^{-n}. \tag{A.3b}$$

In the expansion $(A.3)$ the $\{\alpha_n^i\}$ are the usual string oscillators, x^i is the center-of-mass coordinate and p^i is the momentum-operator taking its value in the momentum lattice λ_R. N is the dimension of the internal space which is 3 and 9 for 8 and 4 dimensional strings respectively. Second, the vertex operators in $(A.1)$ are multiplied by the cocycle generating Klein factors Ω, the role of which is to ensure that the different pieces of the supercurrent anticommute rather than commute. Such factors can be constructed for any integer lattice [13]. If we assume that Ω is of the form

$$\Omega(\alpha) = e^{-ip\wedge\alpha} \qquad \alpha \in \lambda_R \tag{A.4}$$

where

$$p \wedge \alpha = p^i \Lambda_{ij} \alpha^j \quad , \qquad \Lambda_{ij} = -\Lambda_{ji} \tag{A.5}$$

then the antisymmetric matrix Λ has to be chosen in such a way that for any pair of lattice vectors α, β

$$e^{2i\alpha\wedge\beta} = (-1)^{\alpha^2\beta^2}(-1)^{\alpha\cdot\beta}. \tag{A.6}$$

The coefficients $A(a)$ and $B^i(u)$ in $(A.1)$ have to satisfy the following constraints:

$$A^*(a) = A(-a) \quad , \qquad B^{*i}(u) = B^i(-u) \quad , \tag{A.7a}$$

$$u^i B^i(u) = 0. \tag{A.7b}$$

$(A.7a)$ and $(A.7b)$ follow from the hermicity of S_{int} and the requirement that it is a conformal spin 3/2 object with respect to the internal energy-momentum operator T_{int}, respectively.

Now by substituting $(A.1)$ into $(3.5c)$ we obtain the following set of quadratic equations for the coefficients $A(a)$ and $B^i(u)$:

$$\sum_a |A(a)|^2 a^i a^j + \sum_u \{2B^{*i}(u)B^j(u) + u^i u^j B^{*k}(u)B^k(u)\} = 2\delta^{ij} \tag{A.8a}$$

$$\sum_{a+b=\mu} e^{ia\wedge b} A(a)A(b)a^i$$

$$+ \sum_{a+u=\mu} e^{ia\wedge u} A(a)\{a^j B^j(u)(u-a)^i + 2B^i(u)\} \tag{A.8b}$$

$$+ \sum_{u+v=\mu} e^{iu\wedge v}\Big\{2u^j B^j(v)B^i(u) + v^i[B^j(u)B^j(v) - v^j u^k B^j(u)B^k(v)]\Big\} = 0$$

$$\sum_{a+b=\nu} e^{ia\wedge b} A(a)A(b) + \sum_{2u=\nu} B^i(u)B^i(u)$$
$$-2 \sum_{a+u=\nu} e^{ia\wedge u} A(a)a^i B^i(u) = 0. \tag{A.8c}$$

Here $(A.8b)$ has to be satisfied for all $\mu \in \lambda_R$ with $\mu^2 = 2$. Similarly $(A.8c)$ must hold for any $\nu \in \lambda_R$ for which $\nu^2 = 4$.

As discussed in Section 3., if we want the spectrum of the string model to be chiral, λ_R must not contain any vectors of unit norm. In this case the second term on the right-hand-side of $(A.1)$ is absent and the equations $(A.8)$ simplify accordingly:

$$\sum_a |A(a)|^2 a^i a^j = 2\delta^{ij} \tag{A.9a}$$

$$\sum_{a+b=\mu} e^{ia\wedge b} A(a)A(b)a^i = 0 \tag{A.9b}$$

$$\sum_{a+b=\nu} e^{ia\wedge b} A(a)A(b) = 0. \tag{A.9c}$$

From $(A.9a)$ it is seen that for chiral theories the lattice λ_R must be generated by vectors of norm 3. In 1 dimension , the solution of $(A.9)$ is

$$a = \pm\sqrt{3} \quad ; \quad A(a) = \frac{1}{\sqrt{3}}. \tag{A.10}$$

In 2 dimensions there are two indecomposable lattices generated by norm 3 vectors, however, neither of them allows for a solution of $(A.9)$. Hence the only possible supercurrent in two dimensions is based on the direct sum of two copies of $(A.10)$.

In 3 dimensions, which is the relevant one for 8 dimensional strings, the number of such indecomposable lattices is 9, but a solution of $(A.9)$ exists for only one of them. This solution correspons to the 'triplet constraint' and this is the one we have used in our construction of 8 dimensional strings.

In higher dimensions the number of lattices generated by norm 3 vectors increases rapidly. In [10] and [11] a large number of supercurrents have been constructed using different methods, but whether these solutions exhaust all possibilities is not known.

Appendix B

Our construction of 8-dimensional string models was based on the existence of an 'odd' basis for odd self-dual lattices. In this Appendix we will show that it is indeed always possible to find such a basis for any odd self-dual lattice.

Let us first recall that the set of vectors $\{e_i\}_{i=1}^N$ is a basis for the N-dimensional lattice Λ_N if all lattice vectors can be written as integer linear conbinations of the basis vectors:

$$w = \sum_{i=1}^{N} n_i e_i \qquad n_i \in \mathbf{Z} \quad (w \in \Lambda_N).$$

Clearly the choice of basis is not unique for a given lattice and a change of basis is characterized by the transition matrix Q:

$$\tilde{e}_i = \sum_{j=1}^{N} Q_{ij} e_j \qquad e_i = \sum_{j=1}^{N} Q_{ij}^{-1} \tilde{e}_j. \tag{B.1}$$

Since the elements of both the 'old' and the 'new' bases themselves are lattice vectors, both Q and its inverse Q^{-1} must be integer matrices. This is possible if $\det(Q) = \pm 1$ or in other words

$$Q \in SL(N, \mathbf{Z}).$$

The Gram matrix

$$M_{ij} = e_i \cdot e_j$$

plays the role of the metric of the lattice and it transforms as a symmetric tensor under (B.1):

$$\widetilde{M} = Q M Q^T.$$

M is an integer matrix for integer lattices, furthermore $|\det(M)| = 1$ for self-dual lattices. (These are basis independent properties of M.)

Now we wish to show that there always exists an 'odd' basis in which

$$M_{ij} \equiv \delta_{ij}. \tag{B.2}$$

(In this Appendix $\equiv$ means congruent modulo 2, unless otherwise stated.) It is rather easy to show the existence of such a basis for those self-dual lattices that contain at least one $\mathbf{Z}$ factor. In this case the $N+1$-dimensional lattice Λ_{N+1} is the direct sum of an N-dimensional self-dual lattice λ_N and $\mathbf{Z}$:

$$\Lambda_{N+1} = \lambda_N \oplus \mathbf{Z}. \tag{B.3}$$

If $\{e_i\}_{i=1}^N$ denotes a λ_N-basis with Gram matrix m, then the set $\{x, e_i\}$ where

$$x^2 = 1 \qquad \text{and} \qquad x \cdot e_i = 0 \quad i = 1, 2, \ldots, N$$

is a basis for Λ_{N+1} and the Gram matrix of Λ_{N+1} takes the form:

$$M = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.4}$$

The transition matrix Q that transforms M to the desired form

$$QMQ^T \equiv E \tag{B.5}$$

(where E is the $(N+1) \times (N+1)$ dimensional unit matrix) can be found by the following trick. Consider the $N+1$-dimensional Lorentzian lattice

$$\Lambda'_{N;1} = \lambda_N \oplus \mathbf{Z}_- \tag{B.6}$$

where $\mathbf{Z}_-$ is a timelike direction. $\Lambda'_{N;1}$ has basis $\{x', e_i\}$ with

$$(x')^2 = -1 \qquad \text{and} \qquad x' \cdot e_i = 0 \quad i = 1, 2, \ldots, N$$

and in this basis its Gram matrix is given by

$$M' = \begin{pmatrix} m & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.7}$$

Clearly $\Lambda'_{N;1}$ is self-dual since

$$\det(M') = -\det(m) = -\det(M) = -1.$$

The fundamental theorem of Lorentzian self-dual lattices [14] mentioned in section two states that $\Lambda'_{N;1}$ is isomorphic to $\mathbf{Z}_N \oplus \mathbf{Z}_-$, the $N+1$ dimensional hypercubic Lorentzian lattice :

$$\Lambda'_{N;1} \cong \mathbf{Z}_N \oplus \mathbf{Z}_-. \tag{B.8}$$

This means that it is possible to find an $(N+1)\times(N+1)$ Q such that

$$QM'Q^T = \begin{pmatrix} \varepsilon & 0 \\ 0 & -1 \end{pmatrix}, \tag{B.9}$$

where ε is the $N \times N$ unit matrix. Now (B.5) follows from (B.9) (with the same Q transition matrix) taking into account that

$$M \equiv M' \qquad \text{and} \qquad \begin{pmatrix} \varepsilon & 0 \\ 0 & -1 \end{pmatrix} \equiv E.$$

The proof of the existence of an odd basis for those lattices that do not contain $\mathbf{Z}$ factors is more involved. Let us assume that λ_N itself is such a lattice . We can still use the enlarged lattice Λ_{N+1} and assume that for Λ_{N+1} an odd basis $\{f_i\}_{i=1}^{N+1}$ has been found. To complete the construction for λ_N we will have to perform an additional basis transformation so that x becomes one of the basis vectors and the rest are orthogonal to it and at the same time the 'oddness' of the basis is preserved. To show that this is always possible we use the following

Lemma

Any primitive vector of the lattice can be extended to a complete lattice basis .
(A vector is called primitive if it is not a multiple of any other lattice vector.)

We will give an inductive proof of this Lemma so we start at 2 dimensions. In this case we are given

$$f_1 = a_1 e_1 + a_2 e_2$$

where (a_1, a_2) are relative primes and we have to find

$$f_2 = u e_1 + v e_2$$

so that $\{f_1, f_2\}$ is a new lattice basis . In other words

$$\begin{pmatrix} a_1 & a_2 \\ u & v \end{pmatrix} \in SL(2, \mathbf{Z}). \tag{B.10}$$

If $a_1 = 0$ then $a_2 = \pm 1$ and a solution is

$$u = 1 \qquad v = 0.$$

If $a_1 \neq 0$ then (B.10) is equivalent to finding a solution of the congruence

$$a_1 v \equiv 1 \pmod{a_2} \tag{B.11}$$

which is always possible since (a_1, a_2) are relative primes. Note that if a_2 is odd then v can always be chosen to be even (by changing u and v to $u' = u + a_1$ and $v' = v + a_2$ if necessary). If v is chosen to be even then u becomes odd.

Now we proceed to the general case. f_1 is given by its components

$$(\, a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_n \,) \tag{B.12}$$

and we are looking for an $SL(n, \mathbf{Z})$ matrix Q the first row of which is (B.12). (B.12) can be rewritten as

$$(\, a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_n \,) = (\, a_1 \quad m\alpha_2 \quad m\alpha_3 \quad \ldots \quad m\alpha_n \,)$$

where (a_1, m) are relative primes and the set $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ are relative primes. Now using the induction hypothesis the $(n-1)$ dimensional row $(\alpha_2, \alpha_3, \ldots, \alpha_n)$ can be extended to an $SL(n-1, \mathbf{Z})$ matrix

$$\begin{pmatrix} \alpha_2 & \alpha_3 & \ldots & \alpha_n \\ \beta_2 & \beta_3 & \ldots & \beta_n \\ \vdots & & \ddots & \vdots \\ \delta_2 & \delta_3 & \ldots & \delta_n \end{pmatrix}. \tag{B.13}$$

With the help of (B.13) we now build the $n \times n$ matrix

$$Q = \begin{pmatrix} a_1 & m\alpha_2 & m\alpha_3 & \ldots & m\alpha_n \\ u & v\alpha_2 & v\alpha_3 & \ldots & v\alpha_n \\ 0 & \beta_2 & \beta_3 & \ldots & \beta_n \\ \vdots & & & & \vdots \\ 0 & \delta_2 & \delta_3 & \ldots & \delta_n \end{pmatrix}. \tag{B.14}$$

The condition for Q being an element of $SL(n, \mathbf{Z})$ is

$$a_1 v - mu = 1,$$

but this is the same as the two-dimensional problem we solved already. This completes the proof of the Lemma.

Actually we have proven a slightly stronger statement, namely that Q can be chosen to be almost triangular:

$$Q = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ u_2 & \frac{v_2}{m_2}a_2 & \frac{v_2}{m_2}a_3 & \frac{v_2}{m_2}a_4 & \dots & \frac{v_2}{m_2}a_{n-1} & \frac{v_2}{m_2}a_n \\ 0 & u_3 & \frac{v_3}{m_3}a_3 & \frac{v_3}{m_3}a_4 & \dots & \frac{v_3}{m_3}a_{n-1} & \frac{v_3}{m_3}a_n \\ 0 & 0 & u_4 & \frac{v_4}{m_4}a_4 & \dots & \frac{v_4}{m_4}a_{n-1} & \frac{v_4}{m_4}a_n \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & u_n & v_n \end{pmatrix}. \qquad (B.15)$$

Furthermore, if a_n is odd (which can be assumed without loss of generality) then $m = m_2, m_3, \dots$ are all odd and $v = v_2, v_3, \dots$ can all be chosen to be even so $u_2, u_3, \dots$ are all odd. In this case Q has the following structure:

$$Q \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let us now return to the problem of separating the x direction from the lattice Λ_{N+1} without destroying the oddness of the basis . We will achieve this in three steps.

In terms of the $\{f_i\}_{i=1}^{N+1}$ basis x can be written as

$$x = \sum_{i=1}^{N+1} a_i f_i$$

where we will assume that

$$a_1 \equiv a_2 \equiv \ldots \equiv a_k \equiv 0$$

and

$$a_{k+1} \equiv a_{k+2} \equiv \ldots \equiv a_{N+1} \equiv 1.$$

(Clearly the total number of odd components must be odd since x is odd.)

Let us first perform the basis transformation corresponding to the matrix (B.15). The new basis vectors are

$$\begin{aligned} x = g_0 &\equiv (0,0,\ldots,0;1,\ldots,1,1) \\ g_1 &\equiv (1,0,\ldots,0;0,\ldots,0,0) \\ g_2 &\equiv (0,1,\ldots,0;0,\ldots,0,0) \\ &\vdots \\ g_k &\equiv (0,0,\ldots,1;0,\ldots,0,0) \\ g_{k+1} &\equiv (0,0,\ldots,0;1,\ldots,0,0) \\ &\vdots \\ g_N &\equiv (0,0,\ldots,0;0,\ldots,1,0) \end{aligned} \tag{B.16}$$

Next we define the set of vectors $\{e_i\}_{i=1}^N$ where

$$e_i = g_i - (g_i{\cdot}x)\, x \qquad i = 1,2,\ldots,N.$$

The e_i vectors are orthogonal to x and form a basis for λ_N. Finally the odd basis for λ_N is obtained by defining

$$\begin{aligned} \tilde{e}_1 = e_1 + y \quad &\equiv (1,0,\ldots,0;1,1,\ldots,1,0) \\ \tilde{e}_2 = e_2 \quad &\equiv (0,1,\ldots,0;0,0,\ldots,0,0) \\ &\vdots \\ \tilde{e}_k = e_k \quad &\equiv (0,0,\ldots,1;0,0,\ldots,0,0) \\ \tilde{e}_{k+1} = e_{k+1} + e_1 &\equiv (1,0,\ldots,0;0,1,\ldots,1,1) \\ \tilde{e}_{k+2} = e_{k+2} - e_1 &\equiv (1,0,\ldots,0;1,0,\ldots,1,1) \\ &\vdots \\ \tilde{e}_{N-1} = e_{N-1} + e_1 &\equiv (1,0,\ldots,0;1,1,\ldots,1,1) \\ \tilde{e}_N = e_N - e_1 \quad &\equiv (1,0,\ldots,0;1,1,\ldots,0,1) \end{aligned} \tag{B.17}$$

where

$$y = \sum_{i=k+1}^{N} e_i.$$

Using (B.17) it is not difficult to see that $\{\tilde{e}_i\}_{i=1}^{N}$ is indeed an odd basis for λ_N, i.e. it satisfies (B.2).

This concludes the proof of the existence of an odd basis for any odd self-dual λ_N. Unfortunately the proof was based on an other existence theorem (the uniqueness theorem of Lorentzian self-dual lattices [14]) so it does not provide us with an algorithm for actually constructing the odd basis for a given λ_N. Although we were able to construct by trial an error the odd basis for the 13 Conway lattices that are relevant for 8 dimensional strings, it would be interesting to find such an algorithm since the number and complexity of the Conway lattices relevant for 4 dimensional strings is much greater.

Table I. The 275 possible gauge groups of 8–dimensional chiral heterotic string models corresponding to self-dual lattices with world sheet supersymmetry connected to the 'triplet constraint'. The semisimple groups and those of corresponding to tachyon-free models are marked. The numbers in brackets are the numbers of inequivalent models with the given gauge group.

$E_8E_6A_1^2U_1^2$	(1)	$E_8D_7U_1^3$	(1)	$E_8D_5A_3U_1^2$	(1)
$E_8A_7A_1^2U_1$	(1)	$E_8A_3^3U_1$	(1)	$E_7^2U_1^4$	(1)
$E_7E_6A_1^3U_1^2$	(1)	$E_7D_7A_1U_1^3$	(2)	$E_7D_6A_1U_1^4$	(1)
$E_7D_5A_3U_1^3$	(3)	$E_7D_4A_5U_1^2$	(2)	$E_7D_4A_1^3U_1^4$	(1)
$E_7A_7A_1^3U_1$	(2)	$E_7A_5A_1^4U_1^2$	(2)	$E_7A_3^3A_1U_1$	(2)
$E_6^2A_3U_1^3$	(1)	$E_6^2A_1^2U_1^4$	(1)	$E_6D_8A_1^2U_1^2$	(2)
$E_6D_6D_4U_1^2$	(1)	$E_6D_6A_1^4U_1^2$	(2)	$E_6D_5A_3U_1^4$	(1)
$E_6D_5A_1^2U_1^5$	(1)	$E_6D_4^2A_1^2U_1^2$	(2)	$E_6D_4A_1^6U_1^2$	(1)
$E_6A_9A_1U_1^2$	(1)	$E_6A_8U_1^4$	(1)	$E_6A_7A_3U_1^2$	(1)
$E_6A_7A_1^2U_1^3$	(2)	$E_6A_6A_2U_1^4$	(1)	$E_6A_5A_3A_1U_1^3$	(2)
$E_6A_5A_1^3U_1^4$	(1)	$E_6A_4^2U_1^4$	(1)	$E_6A_4A_2^2U_1^4$	(1)
$E_6A_3^3U_1^3$	(1)	$E_6A_3^2A_1^2U_1^4$	(1)	$E_6A_2^4U_1^4$	(1)
$D_{13}A_3U_1^2$	(1)	$D_{13}A_1^2U_1^3$	(1)	$D_{12}A_3U_1^3$	(2)
$D_{11}A_3^2U_1$	(1)	$D_{11}A_3A_1^2U_1^2$	(1)	$D_{10}A_7U_1$	(1)
$D_{10}A_5A_1U_1^2$	(2)	$D_{10}A_3A_1^2U_1^3$	(2)	$D_9D_7U_1^2$	(1)
$D_9D_6U_1^3$	(1)	$D_9D_5A_3U_1$	(1)	$D_9D_5A_1^2U_1^2$	(1)
$D_9D_4A_1^2U_1^3$	(1)	$D_9A_3^2A_1^2U_1$	(1)	$D_9A_1^6U_1^3$	(1)
$D_8D_7U_1^3$	(1)	$D_8D_5A_3U_1^2$	(1)	$D_8D_5A_1^2U_1^3$	(2)
$D_8D_4A_3U_1^3$	(2)	$D_8A_7A_1^2U_1$	(3)	$D_8A_5A_1^3U_1^2$	(2)
$D_8A_3^3U_1$	(1)	$D_8A_3^2A_1^2U_1^2$	(2)	$D_8A_3A_1^4U_1^3$	(1)
$D_7^2A_3U_1$	(1)	$D_7^2A_1^2U_1^2$	(1)	$D_7D_6A_3U_1^2$	(1)
$D_7D_6A_1^2U_1^3$	(2)	$D_7D_5^2U_1$	(1)	$D_7D_5A_3^2$	ss (1)
$D_7D_5A_3A_1^2U_1$	(1)	$D_7D_4A^3A_1^2U_1^2$	(2)	$D_7D_4A_1^4U_1^3$	(1)
$D_7A_7U_1^4$	(2)	$D_7A_5A_1U_1^5$	(2)	$D_7A_3^3A_1^2$	ss (1)
$D_7A_3^2U_1^5$	(1)	$D_7A_3A_1^6U_1^2$	(1)	$D_6^2A_3U_1^3$	(4)
$D_6^2A_1^2U_1^4$	(2)	$D_6D_5^2U_1^2$	(1)	$D_6D_5A_3^2U_1$	(1)
$D_6D_5A_3A_1^2U_1^2$	(4)	$D_6D_5A_1^4U_1^3$	(2)	$D_6D_4A_7U_1$	(2)
$D_6D_4A_5A_1U_1^2$	(5)	$D_6D_4A_3A_1^2U_1^3$	(3)	$D_6D_4A_1^4U_1^4$	(1)
$D_6A_7A_1^4U_1$	(4)	$D_6A_7U_1^5$	(1)	$D_6A_5A_1^5U_1^2$	(5)
$D_6A_5A_1U_1^6$	(1)	$D_6A_3^3A_1^2U_1$	(2)	$D_6A_3^2A_1^4U_1^2$	(3)
$D_6A_3^2U_1^6$	(1)	$D_6A_3A_1^6U_1^3$	(2)	$D_5^3A_3$	ss (1)
$D_5^3A_1^2U_1$	(1)	$D_5^2D_4A_1^2U_1^2$	(2)	$D_5^2A_3^2A_1^2$	ss (1)
$D_5^2A_3U_1^5$	(1)	$D_5^2A_1^6U_1^2$	(1)	$D_5^2A_1^2U_1^6$	(1)
$D_5D_4^2A_3U_1^2$	(1)	$D_5D_4^2A_1^2U_1^3$	(2)	$D_5D_4A_3^2A_1^2U_1$	(2)
$D_5D_4A_3A_1^4U_1^2$	(1)	$D_5D_4A_1^6U_1^3$	(1)	$D_5A_{11}U_1^2$	(1)
$D_5A_9A_1U_1^3$	(3)	$D_5A_8U_1^5$	(1)	$D_5A_7A_3U_1^3$	(4)
$D_5A_7A_1^2U_1^4$	(3)	$D_5A_6A_2U_1^5$	(2)	$D_5^3U_1^3$	(3)

(Table I. continued)

$D_5^2A_3A_1U_1^4$	(6)	$D_5A_5A_1^3U_1^5$	(2)	$D_5A_4^2U_1^5$	(2)
$D_5A_4A_2^2U_1^5$	(2)	$D_5A_3^3U_1^4$	(2)	$D_5A_3^2A_1^6U_1$	(1)
$D_5A_3^2A_1^2U_1^5$	(3)	$D_5A_2^4U_1^5$	(1)	$D_5A_1^{10}U_1^3$	(1)
$D_4^3A_3U_1^3$	(1)	$D_4^2A_7A_1^2U_1$	tf (3)	$D_4^2A_5A_1^3U_1^2$	(3)
$D_4^2A_3^3U_1$	(1)	$D_4^2A_3^2A_1^2U_1^2$	(3)	$D_4^2A_3A_1^4U_1^3$	(4)
$D_4^2A_1^6U_1^4$	(1)	$D_4A_{11}U_1^3$	(2)	$D_4A_9A_1U_1^4$	(1)
$D_4A_8U_1^6$	(1)	$D_4A_7A_3U_1^4$	(3)	$D_4A_7A_1^6U_1$	(1)
$D_4A_7A_1^2U_1^5$	(1)	$D_4A_6A_2U_1^6$	(1)	$D_4A_5^2U_1^4$	(2)
$D_4A_5A_3A_1U_1^5$	(2)	$D_4A_5A_1^7U_1^2$	(3)	$D_4A_5A_1^3U_1^6$	(1)
$D_4A_4^2U_1^6$	(1)	$D_4A_4A_2^2U_1^6$	(1)	$D_4A_3^4A_1^2$	ss (1)
$D_4A_3^3A_1^4U_1$	(1)	$D_4A_3^3U_1^5$	(1)	$D_4A_3^2A_1^6U_1^2$	(2)
$D_4A_3^2A_1^2U_1^6$	(2)	$D_4A_3A_1^8U_1^3$	(1)	$D_4A_2^4U_1^6$	(1)
$A_{15}A_1^2U_1$	(1)	$A_{15}U_1^3$	(1)	$A_{14}A_1U_1^3$	(1)
$A_{13}A_1^3U_1^2$	(2)	$A_{13}A_1U_1^4$	(1)	$A_{12}A_3U_1^3$	(1)
$A_{12}A_2A_1U_1^3$	(1)	$A_{12}A_1^2U_1^4$	(1)	$A_{11}A_4U_1^3$	(1)
$A_{11}A_3^2U_1$	(1)	$A_{11}A_3A_1^2U_1^2$	(3)	$A_{11}A_3U_1^4$	(1)
$A_{11}A_2^2U_1^3$	(1)	$A_{11}A_1^4U_1^3$	(1)	$A_{11}A_1^2U_1^5$	(1)
$A_{10}A_4A_1U_1^3$	(1)	$A_{10}A_3A_2U_1^3$	(1)	$A_{10}A_2^2A_1U_1^3$	(1)
$A_{10}A_2A_1^2U_1^4$	(2)	$A_9A_7A_1U_1$	(1)	$A_9A_6U_1^3$	(1)
$A_9A_5A_1^2U_1^2$	(3)	$A_9A_5U_1^4$	(1)	$A_9A_4A_2U_1^3$	(1)
$A_9A_4A_1U_1^4$	(2)	$A_9A_3^2A_1U_1^2$	(2)	$A_9A_3A_1^3U_1^3$	(2)
$A_9A_3A_1U_1^5$	(1)	$A_9A_2^3U_1^3$	(1)	$A_9A_2^2A_1U_1^4$	(1)
$A_9A_1^5U_1^4$	(1)	$A_9A_1^3U_1^6$	(1)	$A_8A_6A_1U_1^3$	(1)
$A_8A_5A_1U_1^4$	(1)	$A_8A_4A_3U_1^3$	(1)	$A_8A_4A_2A_1U_1^3$	(1)
$A_8A_4A_1^2U_1^4$	(2)	$A_8A_3A_2^2U_1^3$	(1)	$A_8A_3A_1^2U_1^5$	(1)
$A_8A_2^3A_1U_1^3$	(1)	$A_8A_2^2A_1^2U_1^4$	(2)	$A_7^2A_3U_1$	(1)
$A_7^2A_1^2U_1^2$	(5)	$A_7^2U_1^4$	(1)	$A_7A_6A_1U_1^4$	(1)
$A_7A_5A_3A_1U_1^2$	(2)	$A_7A_5A_1^3U_1^3$	(6)	$A_7A_5A_1U_1^5$	(1)
$A_7A_4A_3U_1^4$	(2)	$A_7A_4A_2A_1U_1^4$	(2)	$A_7A_4A_1^2U_1^5$	(2)
$A_7A_3^3U_1^2$	(2)	$A_7A_3^2A_1^2U_1^3$	(6)	$A_7A_3^2U_1^5$	(1)
$A_7A_3A_2^2U_1^4$	(1)	$A_7A_3A_1^4U_1^4$	(2)	$A_7A_3A_1^2U_1^6$	(1)
$A_7A_2^3A_1U_1^4$	(1)	$A_7A_2^2A_1^2U_1^5$	(1)	$A_6^2A_3U_1^3$	(1)
$A_6^2A_2A_1U_1^3$	(1)	$A_6^2A_1^2U_1^4$	(2)	$A_6^2U_1^6$	(2)
$A_6A_5A_3U_1^4$	(1)	$A_6A_5A_2A_1U_1^4$	(2)	$A_6A_5A_1^2U_1^5$	(1)
$A_6A_4^2A_1U_1^3$	(1)	$A_6A_4A_3A_2U_1^3$	(1)	$A_6A_4A_2^2A_1U_1^3$	(1)
$A_6A_4A_2A_1^2U_1^4$	(3)	$A_6A_4A_2U_1^6$	(2)	$A_6A_3^2A_1U_1^5$	(1)
$A_6A_3A_2^3U_1^3$	(1)	$A_6A_3A_2A_1^2U_1^5$	(2)	$A_6A_3A_1^3U_1^6$	(1)
$A_6A_2^3A_1^2U_1^4$	(1)	$A_5^2A_4U_1^4$	(2)	$A_5^2A_3^2U_1^2$	(2)
$A_5^2A_3A_1^2U_1^3$	(6)	$A_5^2A_3U_1^5$	(1)	$A_5^2A_2^2U_1^4$	(1)
$A_5^2A_1^4U_1^4$	(5)	$A_5^2A_1^2U_1^6$	(1)	$A_5^2U_1^8$	(1)
$A_5A_4^2A_1U_1^4$	(2)	$A_5A_4A_3A_2U_1^4$	(2)	$A_5A_4A_3A_1U_1^5$	(3)

(Table I. continued)

$A_5A_4A_2^2A_1U_1^4$		(2)	$A_5A_4A_2A_1^2U_1^5$	(2)	$A_5A_4A_1^3U_1^6$	(2)
$A_5A_3^3A_1U_1^3$		(4)	$A_5A_3^2A_1^3U_1^4$	(5)	$A_5A_3^2A_1U_1^6$	(1)
$A_5A_3A_2^3U_1^4$		(1)	$A_5A_3A_2^2A_1U_1^5$	(1)	$A_5A_3A_1^5U_1^5$	(2)
$A_5A_2^4A_1U_1^4$		(1)	$A_5A_2^3A_1^2U_1^5$	(1)	$A_5A_2^2A_1^3U_1^6$	(1)
$A_4^3A_3U_1^3$		(1)	$A_4^3A_2A_1U_1^3$	(1)	$A_4^3A_1^2U_1^4$	(1)
$A_4^3U_1^6$		(1)	$A_4^2A_3A_2^2U_1^3$	(1)	$A_4^2A_3A_1^2U_1^5$	(2)
$A_4^2A_2^2A_1^2U_1^4$		(2)	$A_4^2A_2^2U_1^6$	(3)	$A_4A_3^3U_1^5$	(1)
$A_4A_3^2A_2A_1U_1^5$		(2)	$A_4A_3^2A_1^2U_1^6$	(2)	$A_4A_3A_2^2A_1^2U_1^5$	(2)
$A_4A_3A_2A_1^3U_1^6$		(2)	$A_4A_2^4U_1^6$	(1)	$A_3^5U_1^3$	(1)
$A_3^4A_1^6$	tf,ss	(1)	$A_3^4A_1^2U_1^4$	(4)	$A_3^4U_1^6$	(1)
$A_3^3A_2^2U_1^5$		(1)	$A_3^3A_1^4U_1^5$	(2)	$A_3^3U_1^9$	(1)
$A_3^2A_2^3A_1U_1^5$		(1)	$A_3^2A_2^2A_1^2U_1^6$	(1)	$A_3^2A_1^{10}U_1^2$	(1)
$A_3^2A_1^6U_1^6$		(1)	$A_3A_2^4A_1^2U_1^5$	(1)	$A_3A_2^3A_1^3U_1^6$	(1)
$A_3A_1^{12}U_1^3$		(1)	$A_2^6U_1^6$	(1)		

Table II. The left lattices and the matter representations corresponding to the two 8-dimensional chiral tachyon-free models.

Γ_L^* root lattice:		$U_8A_7A_1^2D_4^2$	$A_3^4A_1^6$
glue vectors:		22110001	1111000011
		04000101	0202001111
			0022110011
Γ_L generating conjugacy classes:			
	s_0:	66101111	0200010101
	s_1:	42100000	2112000001
	s_2:	75110011	3122000100
	v_0:	40110000	3113000000
zero mass fermions:		−17001100	1001000001
		06010000	0031000100
		17000010	1030010000
		20100001	0031001000
		−20010100	1300001000
		06100000	0330000001
		17000011	1300000100
		20010001	0301010000
		−20100100	1001000010
		−17001000	0330000010
			1030100000
			0301100000
zero mass scalars:		04000000	0220000000
		00000101	0000111100
		00110001	2002000000
		00110100	0022000000
		22000000	0000001111
		26000000	0000110011
		−22000000	2200000000
		−26000000	2020000000
		40000000	0202000000
		−40000000	

Table III. The 44 possible gauge groups of 4–dimensional chiral heterotic string models (derived from the $E_8 Z_{14}$ Conway–Sloane lattice) that correspond to self-dual lattices with world sheet supersymmetry connected to the "triplet constraint". The semisimple groups and those of corresponding to tachyon free models are marked. The numbers in brackets are the numbers of inequivalent models with the given gauge group.

$E_8D_9A_3U_1^2$		(1)	$E_8D_7D_5U_1^2$		(1)	$E_8D_7A_3^2U_1$		(1)
$E_8D_5^2A_3U_1$		(1)	$E_8D_5A_3^3$	ss	(1)	$E_7D_{11}A_1U_1^3$		(1)
$E_7D_9A_3A_1U_1^2$		(1)	$E_7D_7D_5A_1U_1^2$		(2)	$E_7D_7A_3^2A_1U_1$		(2)
$E_7D_5^2A_3A_1U_1$		(1)	$E_7D_5A_3^3A_1$	ss	(1)	$E_6D_{12}A_1^2U_1^2$		(1)
$E_6D_{10}A_1^4U_1^2$	tf	(1)	$E_6D_8D_6U_1^2$		(1)	$E_6D_8D_4A_1^2U_1^2$		(1)
$E_6D_6^2A_1^2U_1^2$	tf	(1)	$E_6D_6D_4^2U_1^2$		(1)	$E_6D_6D_4A_1^4U_1^2$		(1)
$E_6D_4^3A_1^2U_1^2$		(1)	$D_{14}A_7U_1$	tf	(1)	$D_{12}A_7A_1^2U_1$	tf	(1)
$D_{11}D_6A_1^2U_1^3$		(1)	$D_{10}D_4A_7U_1$	tf	(2)	$D_{10}A_7A_1^4U_1$		(1)
$D_9D_8A_3U_1^2$		(2)	$D_9D_6A_3A_1^2U_1^2$		(1)	$D_9D_4^2A_3U_1^2$		(1)
$D_8D_7D_5U_1^2$		(1)	$D_8D_7A_3^2U_1$		(2)	$D_8D_6A_7U_1$	tf	(1)
$D_8D_5^2A_3U_1$		(1)	$D_8D_5A_3^3$	ss	(1)	$D_8D_4A_7A_1^2U_1$	tf	(2)
$D_8A_7A_1^6U_1$		(1)	$D_7D_6D_5A_1^2U_1^2$		(2)	$D_7D_6A_3^2A_1^2U_1$		(1)
$D_7D_4^2A_3^2U_1$		(1)	$D_6^2A_7A_1^2U_1$		(2)	$D_6D_5^2A_3A_1^2U_1$		(2)
$D_6D_5A_3^3A_1^2$	ss	(1)	$D_6D_4^2A_7U_1$	tf	(2)	$D_6D_4A_7A_1^4U_1$		(1)
$D_5^2D_4^2A_3U_1$		(1)	$D_4^3A_7A_1^2U_1$		(1)			

Table IV. The left lattices and the matter representations corresponding to the eight 4-dimensional tachyon-free models.

Γ_L^* root lattice:	$U_{24}U_8E_6A_1^4D_{10}$	$U_{24}U_8E_6A_1^2D_6^2$
glue vectors:	2 21001111	2 21001111
	0 40110011	0 40110011
Γ_L generating conjugacy classes:		
s_0:	2 61010110	6 20101010
s_1:	18 00000100	9 10100000
s_2:	19 32010000	3 50110010
v_0:	12 00001100	6 60110000
zero mass fermions:	−6 00000100	0 20001000
	−6 00001000	−3−30010000
	−3 10101100	3 10000010
	−1−11010000	3 10000001
	−3 10011100	−3 10100011
	0−20001011	1 12010000
	0−20000111	3−10101100
	0 20111000	−3−30100000
	0 20110100	−3 10010011
	−1−11100000	1 12100000
	2 01000100	3−10011100
	3 30010000	0 20000100
	2 01001000	
	3−10100011	
	3−10010011	
	3 30100000	
zero mass scalars:	2−21000000	−6 20000000
	−6 20000000	6 20000000
	6 20000000	0 40000000
	−4 01000000	0 00110011
	−2−22000000	−2−22000000
	0 00001111	4 02000000
	0 40000000	0 00001111
	2 21000000	2 21000000
	0 00110011	6−20000000
	4 02000000	−6−20000000
	6−20000000	−2 22000000
	−6−20000000	0 00111100
	−2 22000000	2−21000000
	0 00111100	−4 01000000

(Table IV. continued)

Γ_L^* root lattice:	$U_8A_7D_{14}$	$U_8A_7A_1^2D_{12}$
glue vectors:	4400	221111
Γ_L generating conjugacy classes:		
s_0:	2201	660110
s_1:	6001	370000
s_2:	7300	751110
v_0:	4011	340110
zero mass fermions:	−1300	370000
	1711	020100
	3100	111100
	3700	−200111
	1111	−130000
	−1500	021000
		−201011
		−170011
zero mass scalars:	4000	−260000
	−2600	−220000
	2600	400000
	0400	001111
	2200	220000
	−2200	260000
		040000

(Table IV. continued)

Γ_L^* root lattice:	$U_8A_7D_4D_{10}$	$U_8A_7D_6D_8$
glue vectors:	220111	221111
Γ_L generating conjugacy classes:		
s_0:	661010	440110
s_1:	370000	370000
s_2:	531100	750010
v_0:	620100	401100
zero mass fermions:	370000	370000
	171000	−201000
	−130000	111100
	171100	−200100
	−170100	−130000
	110011	−170011
zero mass scalars:	−260000	−260000
	260000	−220000
	040000	400000
	000111	001111
	220000	220000
	−220000	260000
	400000	040000

(Table IV. continued)

Γ_L^* root lattice:	$U_8A_7A_1^2D_4D_8$	$U_8A_7D_4^2D_6$
glue vectors:	22110011	22000111
	04000111	04010100
Γ_L generating conjugacy classes:		
s_0:	22101110	04101010
s_1:	46100000	33110000
s_2:	77001100	75011000
v_0:	04110000	22010000
zero mass fermions:	−17001100	−17100000
	−20010011	−11001100
	−20100011	−17110000
	02100000	−11001000
	02010000	−20000010
	20010100	−20000001
	20100100	
	−17001000	
zero mass scalars:	04000000	−26000000
	−26000000	−22000000
	−22000000	40000000
	00000111	00000111
	00110011	00010011
	00110100	22000000
	40000000	26000000
	22000000	04000000
	26000000	00010100

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