DIAS-STP-90-29

NON-STANDARD QUANTUM GROUP IN TODA AND WZNW THEORIES *

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The basic Poisson brackets in the chiral sectors of the WZNW theory and its Toda reduction are described in terms of a monodromy dependent r-matrix. In the case of the $sl(n)$ Lie algebras, and only then, this monodromy dependence can be 'gauged away'. The resulting non-trivial solution of the classical Yang-Baxter equation is the classical limit of the quantum R-matrix of the $SL(n)$ Toda theory found recently by Cremmer and Gervais. The deformations of $SL(n)$ and $\mathcal{U}(sl(n))$ defined by this R-matrix are studied in the simplest non-trivial case of $n = 3$. The multiplicative structure of this deformation of $\mathcal{U}(sl(3))$ can be transformed into that of the standard $\mathcal{U}_q(sl(3))$, but the coproduct is different. Possible generalizations for arbitrary n and applications in conformal field theory and in non-commutative differential geometry are briefly indicated. The Cremmer-Gervais R-matrix is 'Yang-Baxterized'. The resulting spectral parameter dependent R-matrix may give rise to a new series of integrable models.

* To appear in the proceedings of the XIV$^{\text{th}}$ John Hopkins Workshop on Current Problems in Particle Theory, Hungary, 27-31 August, 1990.

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1. Classical exchange algebra

The algebraic structures associated to the Yang-Baxter equation play a central role in present day 'integrable physics' [1-4]. In particular, the role played by quantum groups in conformal field theory is a subject of intense study [5-16].

There is a universal method, developed by the Leningrad school [3], by which one can associate a Hopf algebra to any solution of the Yang-Baxter equation (without spectral parameter), that is to any R-matrix satisfying

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ . \tag{1}$$

For example, when applied to the following R-matrix in the defining representation of $SL(n)$

$$R_{DJ}(q) = q^{-\frac{1}{n}}\left\{q\sum_i e_{ii}\otimes e_{ii} + \sum_{i\neq j} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{i>j} e_{ij}\otimes e_{ji}\right\} , \tag{2}$$

this method yields the quantum group $SL(n)_q$, whose dual is the quantized universal enveloping algebra $\mathcal{U}_q(sl(n))$ discovered by Drinfeld and Jimbo [1, 2].

In principle, one can associate a Hopf algebra to any conformal field theory by applying the 'Leningrad construction' to the solution of the Yang-Baxter equation provided by the braiding matrix of the conformal field theory.

If $R(q)$ has a classical limit,

$$R(q) = 1 - ihr + o(h) \qquad \text{for} \qquad q = e^{-ih} \ , \tag{3}$$

such that $r \in \mathcal{G}\otimes\mathcal{G}$ for some Lie algebra $\mathcal{G}$, then the quantum group associated to R can be interpreted as a quantization of the Poisson-Lie group structure defined by r on the corresponding Lie group G. On the other hand, the dual $\mathcal{U}(R)$ provides a deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ [1, 3, 4].

In a recent paper [13] we have shown that the basic Poisson brackets in the chiral sectors of the WZNW theory can be given as follows:

$$\{u_L(\xi_1) \overset{\otimes}{,} u_L(\xi_2)\} = -\frac{\pi}{k}[u_L(\xi_1)\otimes u_L(\xi_2)]\left[\mathrm{sign}(\xi_2-\xi_1)C + r(\omega)\right] , \tag{4}$$

where

$$r(\omega) = \sum_{\alpha\in\Phi^+} \frac{|\alpha|^2}{2}\coth\{\tfrac{1}{2}\alpha^\mu\omega_\mu\} e_\alpha \wedge e_{-\alpha} \ , \tag{5}$$

with Φ^+ denoting the set of positive roots and $\wedge$ standing for antisymmetric tensor product. These formulae constitute the r-matrix description of the chiral sectors of the WZNW theory. The reader is referred to [9] for related (but different) results on the 'classical exchange algebra'. In equation (4) $C = \lambda^a \otimes \lambda_a$ is the 'Casimir-operator' and $u_L(\xi)$ is the left-moving field appearing in the factorization of the WZNW solution

$$g(\tau,\sigma) = u_L(\tau+\sigma) \cdot u_R(\tau-\sigma). \tag{6}$$

It is a chiral primary field with respect to the Kac-Moody current $I(\xi)$ (with level k), which solves the differential equation

$$-\frac{k}{2\pi} u'_L(\xi) = I(\xi) u_L(\xi) \tag{7}$$

with diagonal monodromy:

$$u_L(\xi + 2\pi) = u_L(\xi) D \ , \qquad D = \exp[\omega^\mu h_\mu] \ . \tag{8}$$

These properties determine $u_L(\xi)$ up to a monodromy dependent normalization, which allows to consider a transformation of the form

$$u_L(\xi) \longrightarrow \hat{u}_L(\xi) = u_L(\xi) \cdot \Omega(\omega). \tag{9}$$

We have shown in [13] that by such a transformation it is possible to remove the monodromy dependence of the exchange algebra in the case of $sl(n)$ and that this is impossible for the other simple Lie algebras. In the $sl(n)$ case the monodromy independent, antisymmetric r-matrix turns out to be given as

$$\hat{r} = -2 \sum_{i<j} \sum_{k<l} e_{ij} \wedge e_{lk} \delta_{j-i,l-k} \theta(i-k) + \sum_{\mu=1}^{n-2} h^\mu \wedge h^{\mu+1} \ , \tag{10}$$

where θ is the usual step function, whose value is $\frac{1}{2}$ at 0, and the h_μ are the standard Cartan generators of $sl(n)$. By construction, $\hat{r}$ is an antisymmetric solution of the modified classical Yang-Baxter equation, and thus $\hat{r} \pm C$ are solutions of the classical Yang-Baxter equation (without spectral parameter). The transformed $SL(n)$ valued field $\hat{u}_L(\xi)$, which satisfies an exchange algebra of the form (4) with $\hat{r}$ replacing $r(\omega)$, can simply be written as

$$\hat{u}_L(\xi) = \big[u(\xi) \quad u(\xi+2\pi) \quad u(\xi+4\pi) \quad \ldots \quad u(\xi + 2(n-1)\pi) \big]. \tag{11}$$

In other words, $\hat{u}_L$ is built out of an arbitrary (generic) column vector solution of the differential equation (7) by means of successive translations by 2π.

Our results for the exchange algebra for the WZNW field $u_L(\xi)$ ($\hat{u}_L(\xi)$) are actually also valid for the corresponding chiral fields in the Toda theory, obtained from the WZNW theory by imposing two chiral sets of first-class constraints [14]. For the left-movers, the gauge transformations generated by the constraints are upper triangular KM transformations and therefore leave the last row of $u_L(\xi)$ ($\hat{u}_L(\xi)$) invariant. On the other hand, the exchange algebra (4) is given in terms of matrix multiplication from the right, mixing the columns of $u_L(\xi)$ ($\hat{u}_L(\xi)$). These two observations imply that the chiral Toda fields, that is the matrix elements in the last row of $u_L(\xi)$ ($\hat{u}_L(\xi)$) in the constrained WZNW (i.e. Toda) theory, indeed satisfy the same exchange algebra as the chiral fields of the the unconstrained WZNW theory.

We also mention that, for exactly the same reason as above, (4) also yields the exchange algebra for the gauge invariant fields of the generalized Toda theories recently discovered by O'Raifeartaigh and Wipf [15].

It seems an interesting question to ask what is the quantum group structure obtained by quantizing the Poisson-Lie group defined by the constant r-matrix (10). In this respect it is worth pointing out that our classical r-matrix (10) is qualitatively different from the standard Drinfeld-Jimbo r-matrix

$$r_{DJ} = -\sum_{i<j} e_{ij} \wedge e_{ji} \ , \tag{12}$$

which underlies the standard quantization of $SL(n)$. To see this qualitative difference we now explain how to identify (10) and (12) as special cases in the Belavin-Drinfeld classification of r-matrices of simple Lie algebras [17, 4].

Belavin and Drinfeld proved that every solution of the classical Yang-Baxter equation can be transformed into a 'standard solution' by a Lie algebra automorphism and classified the standard solutions in terms of certain discrete and continuous parameters. The discrete parameter consists of two subsets Δ_+ and Δ_- of the set of simple roots and a one-to-one map: $\tau\,:\,\Delta_+ \longrightarrow \Delta_-$, which preserves the scalar product and pushes every root out of Δ_+ if applied enough times (Δ_+ and Δ_- may intersect). The continuous parameter is the purely Cartan piece of the r-matrix. Up to multiplication by a constant, the standard solution is of the form $r = (\pm C + r_{\mathcal{H}} + r_{\mathcal{N}})$, where C is the Casimir operator, $r_{\mathcal{H}}$ is the purely Cartan piece, and $r_{\mathcal{N}}$ is the purely root piece. One extends τ to a map $\bar{\tau}\,:\,\bar{\Delta}_+ \longrightarrow \bar{\Delta}_-$, where $\bar{\Delta}_\pm$ are the subsets of the positive roots generated by $\Delta_\pm$, respectively, and one writes for $\beta \in \bar{\Delta}_-$ and $\alpha \in \bar{\Delta}_+$ that $\beta \succ \alpha$ if $\beta = \bar{\tau}^k(\alpha)$ for some natural number k. The root piece $r_{\mathcal{N}}$ is completely fixed by the discrete parameter as

$$r_{\mathcal{N}} = \sum_{\alpha\in\Phi+} \frac{|\alpha|^2}{2} e_\alpha \wedge e_{-\alpha} + 2\sum_{\alpha\in\bar{\Delta}_+}\sum_{\beta\succ\alpha} \frac{|\alpha|^2}{2} e_\beta \wedge e_{-\alpha} \ . \tag{13}$$

The Cartan piece $r_{\mathcal{H}} = \sum_{\mu,\nu} r^{\mu\nu}_{\mathcal{H}} h_\mu \otimes h_\nu$ is antisymmetric and satisfies the condition

$$\sum_{\mu\nu} r^{\mu\nu}_{\mathcal{H}} h_\mu (h_\nu, h_\alpha) - h_\alpha = \sum_{\mu\nu} r^{\mu\nu}_{\mathcal{H}} h_\mu (h_\nu, h_{\tau(\alpha)}) + h_{\tau(\alpha)} \quad \text{for} \quad \forall \alpha \in \Delta_+ \ . \tag{14}$$

In the above equations $e_{\pm\alpha}$, h_α are the step operators and the Cartan element associated to a positive root $\alpha \in \Phi^+$.

The **r-matrix** (12) corresponds to the simplest special case of the Belavin-Drinfeld standard r-matrices, namely $\Delta_\pm$ is the empty set and $r_{\mathcal{H}} = 0$. Interestingly, (10) can be identified as the other extreme special case of the $sl(n)$ standard solutions, in the sense that $\Delta_\pm$ are the largest possible sets:

$$\begin{aligned} \Delta_+ &= \{(\lambda_1 - \lambda_2), (\lambda_2 - \lambda_3), \ldots, (\lambda_{n-2} - \lambda_{n-1})\} , \\ \Delta_- &= \{(\lambda_2 - \lambda_3), (\lambda_3 - \lambda_4), \ldots, (\lambda_{n-1} - \lambda_n)\} , \end{aligned} \tag{15}$$

where the λ_i's are the weights of the defining representation of $sl(n)$. The map τ is given as follows, $\tau : (\lambda_{i-1} - \lambda_i) \longrightarrow (\lambda_i - \lambda_{i+1})$, and r in (10) is indeed of the form $r = -r_{\mathcal{N}} - r_{\mathcal{H}}$.

2. Non-standard quantum group and quantized enveloping algebra

Before our work on the classical exchange algebra [13] of the WZNW theory, Cremmer and Gervais [11] performed a thorough analysis of the quantum exchange algebra of the chiral primary fiels in $SL(n)$ Toda field theory (see also [10, 12] for related papers). The result which is most important to us is that they constructed certain chiral primary fields ψ_i whose braiding is given as

$$\psi_j(\xi)\psi_k(\xi') = \sum_{l,m=1}^{n} R_{jk,ml}\psi_l(\xi')\psi_m(\xi) , \tag{16}$$

where, for $(\xi - \xi') > 0$, one has

$$\begin{aligned} R = q^{-\frac{1}{n}} \Bigg\{ & q\sum_i^n e_{ii} \otimes e_{ii} + q\sum_{i>j} q^{-2(i-j)/n} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i<j} q^{2(j-i)/n} e_{ii} \otimes e_{jj} \\ & + (q - q^{-1}) \Bigg(\sum_{i>j} \sum_{k=0}^{i-j-1} q^{-2k/n} e_{i,j+k} \otimes e_{j,i-k} - \sum_{i<j} \sum_{k=1}^{j-i-1} q^{2k/n} e_{i,j-k} \otimes e_{j,i+k} \Bigg) \Bigg\} . \end{aligned} \tag{17}$$

In Ref. [11] the authors initiated the study of the quantum group defined by (17). They have also pointed out that the two distinct solutions of the Yang-Baxter equation given by (2) and (17) are related by a similarity transformation $Y \in$ Mat$(C^n \otimes C^n)$. However, Y does not have the factorized form $X \otimes X$, $X \in$ Mat(C^n), and therefore the quantum groups defined by (2) and (17) are not isomorphic [11].

It can be shown, by identifying the components of the last row of the chiral WZNW field $\hat{u}_L$ with the chiral Toda fields ψ_i, that the classical limit of (16) is the classical exchange algebra we found (independently). It also follows from the relationship of Toda and WZNW theories that it is (17), rather than the standard R-matrix (2), that describes the quantum exchange algebra of the chiral primary fields also in $SL(n)$ WZNW theory. For this reason we think that the quantum group structure defined by the Cremmer-Gervais R-matrix deserves further attention.

The aim of our latest work [16], which we are going to review now, is in fact to describe this quantum group and its dual, denoted as $\widehat{\mathcal{U}}_q(sl(n))$. The main question we are interested in is how different the Hopf algebra $\widehat{\mathcal{U}}_q(sl(n))$ is from the standard quantized universal enveloping algebra $\mathcal{U}_q(sl(n))$. We consider in detail the simplest nontrivial case of $sl(3)$ ((2) and (17) coincide for $sl(2)$). Our main result is that the multiplicative structures of $\mathcal{U}_q(sl(3))$ and that of $\widehat{\mathcal{U}}_q(sl(3))$ are isomorphic, but the complete Hopf algebra structures are qualitatively different.

For easier reference, first we need to recall the main points of the 'Leningrad construction' [3]. Let us consider a solution of the Yang-Baxter equation $R \in \mathrm{Mat}(V \otimes V)$, $V \simeq C^n$. The Hopf algebra $\mathcal{A}(R)$ ('formal quantum group') associated to R is the associative algebra generated by the unit element 1 and the set of generators t_{ij} $(i,j = 1 \dots n)$, arranged as a matrix $T \in \mathrm{Mat}(V, \mathcal{A})$, subject to the relations

$$RT_1T_2 = T_2T_1R \qquad (T_1 = T \otimes I \ , \ T_2 = I \otimes T) \ . \tag{18}$$

The coproduct and the counit of $\mathcal{A}$ are given by the simple formulae

$$\Delta(t_{ij}) = t_{ik} \otimes t_{kj} \quad \text{and} \quad \varepsilon(t_{ij}) = \delta_{ij} \ . \tag{19}$$

Depending on R, one could add further relations, like $\det_q T = 1$ for (2), corresponding to the center of the exchange algebra (18). The quantum determinant plays an important role in the construction of an antipode S, which requires inverting the matrix of generators T.

We note that by 'forgetting the coordinate dependence' (16) reduces to the defining relations of a 'quantum plane' [3, 11]. Similarly, apart from the trivial coordinate dependence, the classical exchange algebra satisfied by the rows of $\hat{u}_L$ is nothing but the classical version of the relations defining a quantum plane.

The dual space $\mathcal{A}'(R)$ is endowed with the bialgebra (Hopf algebra) structure naturally induced from that of $\mathcal{A}(R)$, e. g., the multiplication in $\mathcal{A}'(R)$ is defined as

$$< l_1 l_2, t >=< l_1 \otimes l_2, \Delta t > \ . \tag{20}$$

If R is invertible, then one can introduce the 'algebra of regular functionals' on $\mathcal{A}$, which is the sublagebra $\mathcal{U}(R) \subset \mathcal{A}'(R)$ generated by the functionals 1 and $l_{ij}^{(\pm)}$ defined by the following formulae:

$$< 1 \overset{\otimes}{,} T_1 \cdots T_k >= I^{\otimes k} \quad \text{and} \quad < L^{(\pm)} \overset{\otimes}{,} T_1 \cdots T_k >= R_1^{(\pm)} \cdots R_k^{(\pm)} \ . \tag{21}$$

Here $L^{(\pm)}$ denotes the matrix $(l_{ij}^{(\pm)})_{i,j=1}^{n}$, $T_i \in \mathrm{Mat}(V^{\otimes k}, \mathcal{A})$ contains the matrix T as its i^{th} factor and is the unit matrix in the other factors. The numerical matrices $R_i^{(\pm)} \in \mathrm{Mat}(V^{\otimes(k+1)})$ are non-trivial in those two factors of the multiple tensor product, which correspond to the positions of $L^{(\pm)}$ and T_i on the left hand side of the above formula, and coincide there with

$$R^{(+)} = PRP \qquad \text{and} \qquad R^{(-)} = R^{-1} \; , \tag{22}$$

respectively, where $P \in \mathrm{Mat}(V \otimes V)$ is the permutation operator. In the Hopf algebra $\mathcal{U}(R)$ the multiplication satisfies the relations

$$R^{(+)} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{(+)} \; , \quad R^{(+)} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{(+)} \; . \tag{23}$$

For the comultiplication and counit we have

$$\Delta(l_{ij}^{(\pm)}) = l_{ik}^{(\pm)} \otimes l_{kj}^{(\pm)} \; , \quad \Delta(1) = 1 \otimes 1 \; , \tag{24}$$

and

$$\varepsilon(l_{ij}^{(\pm)}) = \delta_{ij} \; , \quad \varepsilon(1) = 1 \; , \tag{25}$$

respectively. We note that the formula for antipode in $\mathcal{U}(R)$ is known for a class of R-matrices [3], but the matrix (17) is not in this class and thus the existence of the antipode has to be investigated separately in our case.

The functionals $l_{ij}^{(\pm)}$ are not independent, some combinations of the matrix elements of $L^{(\pm)}$ can be zero. In conclusion, one can regard $\mathcal{U}(R)$ as the bialgebra generated by the $l_{ij}^{(\pm)}$ and 1 subject to (23-25), modulo the identifications coming from (21), which depend on the form of R. For example, for the standard R-matrix (2) giving rise to $\mathcal{U}(R) = \mathcal{U}_q(sl(n))$, the matrices $L^{(\pm)}$ are upper and lower triangular, respectively, and their diagonal entries are inverses of each other [3].

Now we are in the position to investigate $\mathcal{A}(R)$ and its dual for the R-matrix (17), for $n = 3$. First we consider $\mathcal{A}(R)$ briefly.

We would like to impose on $\mathcal{A}(R)$ the quantum analogue of the unit determinant constraint of $SL(3)$. To do this, and also for the construction of the antipode, we need some analogue of the quantum determinant, which generates the center of the algebra (18) in the standard case. In our case, the appropriate deformation of the classical determinant is provided by the expression

$$\widehat{\det}_q T = t_{11}t_{22}t_{33} - \epsilon^2 t_{11}t_{23}t_{32} - \epsilon^2 t_{12}t_{21}t_{33} + \epsilon^6 t_{13}t_{21}t_{32} + \epsilon^6 t_{12}t_{23}t_{31} - \epsilon^8 t_{13}t_{22}t_{31} \; , \tag{26}$$

which can be verified to commute with all the generators t_{ij}. Here and below we use the notation

$$\epsilon = q^{1/3} \; . \tag{27}$$

Like the standard quantum determinant [3], $\widehat{\det}_q$ (26) has the multiplicative property

$$\widehat{\det}_q(T \cdot T') = \widehat{\det}_q(T)\widehat{\det}_q(T') \ . \tag{28}$$

for any two matrices T and T' containing two commuting copies of generators satisfying the same exchange algebra (18). Therefore we can consistently impose the constraint

$$\widehat{\det}_q(T) = 1 \ . \tag{29}$$

We designate the quantum deformation of $SL(3)$ obtained in this way as $\widehat{SL}_q(3)$.

By using (29), the antipode S can now be computed from the equation

$$S(T_{ij}) \cdot T_{jk} = \delta_{jk} \ . \tag{30}$$

One can verify that the solution is given by the following q-deformation of the standard inverse matrix formula:

$$S(T) = \begin{pmatrix} (t_{22}t_{33} - \epsilon^{-2}t_{23}t_{32}) & (-\epsilon^{-2}t_{12}t_{33} + t_{13}t_{32}) & (\epsilon^{-6}t_{12}t_{23} - \epsilon^{-4}t_{13}t_{22}) \\ (-\epsilon^{2}t_{21}t_{33} + \epsilon^{6}t_{23}t_{31}) & (t_{11}t_{33} - \epsilon^{4}t_{13}t_{31}) & (-\epsilon^{-4}t_{11}t_{23} + t_{13}t_{21}) \\ (\epsilon^{6}t_{21}t_{32} - \epsilon^{8}t_{22}t_{31}) & (-\epsilon^{4}t_{11}t_{32} + \epsilon^{6}t_{12}t_{31}) & (t_{11}t_{22} - \epsilon^{2}t_{12}t_{21}) \end{pmatrix} . \tag{31}$$

Having clarified the Hopf algebra structure of $\widehat{SL}_q(3)$, now we describe the corresponding dual object $\mathcal{U}(R)$, denoted as $\widehat{\mathcal{U}}_q(sl(3))$. First we introduce a notation for the components of $L^{(\pm)}$, which takes into account the identifications forced by (21), namely:

$$L^{(+)} = \begin{pmatrix} C & \phi & \rho \\ 0 & A & \omega \\ 0 & \bar{\phi} & D \end{pmatrix} \quad \text{and} \quad L^{(-)} = \begin{pmatrix} A & \omega & 0 \\ \bar{\phi} & D & 0 \\ \bar{\rho} & \bar{\omega} & C \end{pmatrix} . \tag{32}$$

The list of relations obtained by substituting (32) into (23) has been displayed in [16]. The main feature is that, roughly speaking, the diagonal components behave like exponentiated Cartan generators. In particular, the operator C measures the height of the off-diagonal components, which are the analogues of the step operators. However, there appear also significant differences from this general pattern, e.g., not all the diagonal entries commute.

It can be proven that the 'quantum determinant'

$$\mathcal{D} = \widehat{\det}_q L^{(\pm)} = C(AD - \epsilon^{-2}\omega\bar{\phi}) \tag{33}$$

commutes with the whole algebra, therefore we can impose the constraint

$$\mathcal{D} = 1 \ . \tag{34}$$

This relation allows for inverting the generator matrices $L^{\pm}$. The result turns out to be consistent with the special form of $L^{\pm}$ given in (32) and provides the formula for the antipode of $\widehat{\mathcal{U}}_q(sl(3))$ [16].

In the standard case, that is for R in (2), one finds the Chevalley generators of $\mathcal{U}(R) = \mathcal{U}_q(sl(n))$ with the aid of the Gauss decomposition of the triangular matrices $L^{(\pm)}$ [3]. It turned out [16] that a Chevalley basis can be introduced for $\widehat{\mathcal{U}}_q(sl(3))$ too, with the aid of the Gauss decomposition adapted to this case.

In our case the matrices $L^{(\pm)}$ are not triangular, but, in a formal sense, they still belong to certain subgroups of $SL(n)$, consisting of matrices of the same 'block structure' as $L^{(\pm)}$ in (32). To account for this block structure and for the identifications between the various components of $L^{(+)}$ and $L^{(-)}$ in (32), we consider the following special Gauss decomposition of the generator matrices:

$$L^{(+)} = \exp(\bar{a}e_{32}) \cdot [Ce_{11} + Ae_{22} + Be_{33}] \cdot \exp(be_{23}) \cdot \exp(be_{12}) \cdot \exp(ae_{12}) \cdot \exp(ce_{13}) \ , \tag{35.a}$$

and

$$L^{(-)} = \exp(\bar{c}e_{31}) \cdot \exp(\bar{a}e_{32}) \cdot \exp(\bar{a}e_{21}) \cdot \exp(\bar{b}e_{32}) \cdot [Ae_{11} + Be_{22} + Ce_{33}] \cdot \exp(be_{12}) \ . \tag{35.b}$$

Here, the various e_{ij} are numerical matrices, and the a, b etc. are our new variables. By using these new variables, the relations given by (23) take a simple form. First of all, A, B and C turn out to be mutually commuting Cartan variables, in terms of which we have

$$\mathcal{D} = ABC = 1 \ . \tag{36}$$

By using this relation, and assuming that A^{-1} and B^{-1} are also in the algebra (or, more precisely, adding these as new generators) we can eliminate C and work with the 'exponentiated Cartan generators' A and B. These Cartan generators act on the generators of height one a, b, $\bar{a}$, $\bar{b}$ as follows:

$$\begin{array}{llll} Aa = \epsilon^4 aA, & A\bar{a} = \epsilon^{-4}\bar{a}A, & Ba = \epsilon^{-2}aB, & B\bar{a} = \epsilon^2\bar{a}B, \\ Ab = \epsilon^{-2}bA, & A\bar{b} = \epsilon^2\bar{b}A, & Bb = \epsilon^4 bB, & B\bar{b} = \epsilon^{-4}\bar{b}B. \end{array} \tag{37}$$

The commutation relations of the height one generators are

$$[a,\bar{b}] = [b,\bar{a}] = 0, \qquad [a,\bar{a}] = z\epsilon^{-1}B(A^{-1} - A^2), \qquad [b,\bar{b}] = z\epsilon^{-1}A^{-1}(B^{-2} - B), \tag{38}$$

where $z = (q - q^{-1})$. By combining (23) and (35) we can express the height two off-diagonal generator c in terms of the height one generators as

$$zc = qba - q^{-1}ab, \quad \text{and analogously} \quad z\bar{c} = q(\bar{b}\bar{a} - \bar{a}\bar{b}). \tag{39}$$

The above equations tell us that a, b, $\bar{a}$ and $\bar{b}$ are the analogues of step operators associated to simple roots, and we can take them, together with the Cartan generators, as the generators of our algebra. These step operators and the 'composite objects' c and $\bar{c}$ satisfy

$$ac = ca, \qquad bc = cb, \qquad \bar{a}\bar{c} = q^{-2}\bar{c}\bar{a}, \qquad \bar{b}\bar{c} = q^{2}\bar{c}\bar{b}, \tag{40}$$

which are clearly the analogues of the Serre relations.

The above relations are strongly reminescent of the defining relations of the standard quantized universal enveloping algebra $\mathcal{U}_q(sl(3))$ [1, 2]. More than that, we can indeed introduce new variables in which these relations describing the multiplicative structure of $\widehat{\mathcal{U}}_q(sl(3))$ become *identical* to the standard ones. This is achieved by the change of variables

$$\begin{aligned} a &= x_1 E_1 A^k B^l\,, & b &= x_2 E_2 A^m B^n\,, & A &= q^{2H_1/3}\,,\\ \bar{a} &= y_1 A^{\frac{1}{2}-k} B^{1-l} F_1\,, & \bar{b} &= y_2 A^{-1-m} B^{-\frac{1}{2}-n} F_2\,, & B &= q^{2H_2/3}\,, \end{aligned} \tag{41}$$

where the k, l, etc., are arbitrary numerical coefficients subject to the conditions

$$4l + 2n - 2k - 4m = 3\,, \qquad x_1 y_1 = x_2 y_2 = -\epsilon z^{-2}\,. \tag{42}$$

It is easy to check that this transformation converts the multiplication table of our algebra into the standard one, that is we have

$$\begin{aligned} [H_i, H_j] &= 0, & E_1^2 E_2 - [2]_q E_1 E_2 E_1 + E_2 E_1^2 &= 0,\\ [H_i, E_j] &= K_{ji} E_j, & E_2^2 E_1 - [2]_q E_2 E_1 E_2 + E_1 E_2^2 &= 0,\\ [H_i, F_j] &= -K_{ji} F_j, & F_1^2 F_2 - [2]_q F_1 F_2 F_1 + F_2 F_1^2 &= 0,\\ [E_i, F_j] &= \delta_{ij} [H_i]_q, & F_2^2 F_1 - [2]_q F_2 F_1 F_2 + F_1 F_2^2 &= 0, \end{aligned} \tag{43}$$

where K_{ij} is the Cartan matrix of $sl(3)$ and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{44}$$

It should be noted that, strictly speaking, H_1 and H_2 do not belong to the original algebra, which consists of polynomials in the components of $L^{(\pm)}$. To consider transformation (41), one has to enlarge the original algebra in such a way to allow for power series in the H_i's. However, this issue is not specific to our case, it also arises for the standard R-matrix (2).

The formulae describing the coproduct and the counit of $\widehat{\mathcal{U}}_q(sl(3))$ in terms of the variables A, B, C, a, b, c, $\bar{a}$, $\bar{b}$ and $\bar{c}$ can be read off easily from (24) and (25).

However, due to fractional powers appearing in (41), we have not been able to write them down in terms of the Chevalley generators. Nevertheless, we can give the coproduct of the invertible elements A, B and C as follows. First, we obtain from (24) immediately that

$$\Delta(C) = C \otimes C \; . \tag{45}$$

and

$$\Delta(A) = A \otimes A\,[1 \otimes 1 + \epsilon^4 b \otimes \bar{a}] = [1 \otimes 1 + \epsilon^{-2} b \otimes \bar{a}]\, A \otimes A \; . \tag{46}$$

Combining this with

$$\Delta(ABC) = \Delta(1) = 1 \otimes 1 \; , \tag{47}$$

we obtain

$$\Delta(B) = B \otimes B\,[1 \otimes 1 + \epsilon^{-2} b \otimes \bar{a}]^{-1} \; . \tag{48}$$

By using (41) and expressing the inverse in (48) as a formal power series, we arrive at

$$\Delta(A) = A \otimes A\,[1 \otimes 1 + \epsilon^4 x_2 y_1 E_2 A^m B^n \otimes A^{\frac{1}{2}-k} B^{1-l} F_1] \; , \tag{49.a}$$

and

$$\Delta(B) = B \otimes B\Big[1 \otimes 1 + \sum\nolimits_{j=1}^{\infty} (-1)^j \left(\epsilon^{-2} x_2 y_1 E_2 A^m B^n \otimes A^{\frac{1}{2}-k} B^{1-l} F_1\right)^j\Big] \; . \tag{49.b}$$

This power series terminates in any finite dimensional representation because of the nilpotent character of the E_i's and F_i's.

We see from the above formulae that, in contrast to the standard case, the coproduct of $\widehat{\mathcal{U}}_q(sl(3))$ is *not cocommutative on the Cartan subalgebra*. For this reason $\widehat{\mathcal{U}}_q(sl(3))$ escapes the uniqueness theorem of Drinfeld [1] for quantum deformations of the enveloping algebra of a simple Lie algebra.

We have seen that $\mathcal{U}_q(sl(3))$ and $\widehat{\mathcal{U}}_q(sl(3))$ are isomorphic as associative algebras, but not as Hopf algebras, since their coproduct operations are different. This means that $\mathcal{U}_q(sl(3))$ and $\widehat{\mathcal{U}}_q(sl(3))$ have the same linear representations, but are represented by different operators in the tensor products of such representations. However, it can be seen from (49) that the spectrum of $\Delta(A)$ and $\Delta(B)$ is the same as that of $A \otimes A$ and $B \otimes B$, respectively, in the tensor product of any two finite dimensional highest weight representations. Thus the branching rules of $\mathcal{U}_q(sl(3))$ and $\widehat{\mathcal{U}}_q(sl(3))$ are probably very similar, if not identical. We also note that (45) allows for defining the quantum dimension of a finite dimensional representations of $\widehat{\mathcal{U}}_q(sl(3))$ to be the trace of a certain power of C, in the same way as in the standard case [5, 8].

The relation [5, 8] between the branching rules of $\mathcal{U}_q(sl(n))$ for q^2 root of unity, and the fusion rules of the $SU(n)$ WZNW model is one of the celebrated examples for the appearance of quantum group symmetries in conformal field theory. It would be very interesting to know if $\widehat{\mathcal{U}}_q(sl(n))$ can also be used to describe the qualitative, topological features of the $SU(n)$ WZNW and Toda field theories.

3. Consequences of the Hecke relation

We argued that the branching rules of $\mathcal{U}_q(sl(3))$ and those of $\widehat{\mathcal{U}}_q(sl(3))$ are expected to be closely related. For general n, this expectation is also supported by the fact that $\check{R} = q^{1/n}PR$ satisfies the 'Hecke relation'

$$\check{R}^2 = (q - \frac{1}{q})\check{R} + 1 \tag{50}$$

for both exchange matrices (2) and (17). As a consequence of (50), the operators $\check{R}_{i,i+1}$ acting in some multiple tensor product of C^n with itself, generate two representations of the Hecke algebra (see, for example, [2, 5]), corresponding to (2) and (17). The representation of the Hecke algebra belonging to R_{DJ} (2) commutes with the generators of $\mathcal{U}_q(sl(n))$ acting in the multiple tensor product of the defining, n-dimensional representation, while the representation associated to R in (17) commutes with the generators of $\widehat{\mathcal{U}}_q(sl(n))$. The representation theory of the Hecke algebra, which is a deformation of the group algebra of the permutation group, and that of $\mathcal{U}_q(sl(n))$ are connected by a duality relation [2, 5, 18]. Clearly, the same duality connects the Hecke algebra also with $\widehat{\mathcal{U}}_q(sl(n))$, and for this reason the branching of the multiple tensor products of the defining representation is expected to be the same for $\mathcal{U}_q(sl(n))$ and for $\widehat{\mathcal{U}}_q(sl(n))$.

Recently an example of covariant differential calculus on a quantum hyperplane has been worked out by Wess and Zumino [19] (see also [20]). This exterior calculus is based on the fact that consistent exchange relations between the basis objects x^i, dx^i and $\frac{\partial}{\partial x^i}$ can be postulated by using a numerical R-matrix satisfying the Yang-Baxter and the Hecke relations. It would be interesting to compare the variant of the calculus based on R (17) with the example of [19] based on R_{DJ} (2).

We also note that in the same way as for the standard R-matrix (2) (see e.g. [18]), the following expression

$$\mathcal{R}(q,u) = R(q) + \frac{1}{2}q^{-1/n}[1 - i\,\mathrm{ctg}(\pi u)](q^{-1} - q)P \tag{51}$$

provides us a solution of the full Yang-Baxter equation

$$\mathcal{R}_{12}(q,u)\mathcal{R}_{13}(q,u+v)\mathcal{R}_{23}(q,v) = \mathcal{R}_{23}(q,v)\mathcal{R}_{13}(q,u+v)\mathcal{R}_{12}(q,u). \tag{52}$$

In fact the 'Yang-Baxterization' formula (51) is valid for any spectral parameter independent solution of (1), which satisfies the Hecke relation (50). We note that $R(q)$ is recovered from (51) at $i\infty$, and that at $-i\infty$ (51) reduces to a closely related solution of (1) which satisfies (50) with $q \to q^{-1}$. We stress that the $\mathcal{R}(q,u)$ (and its classical limit $r(u)$) should, in principle, give rise to a new series of integrable systems in statistical (classical) mechanics.

4. Concluding remarks

It is obvious that there remain a lot of interesting questions to be investigated. For example, the integrable systems associated to (51) should be explored. It is likely that $\widehat{\mathcal{U}}_q(sl(n))$ is to play an important role there. We think $\widehat{SL}(n)$ and $\widehat{\mathcal{U}}_q(sl(n))$ deserve further attention on their own right as well. We expect that a Chevalley basis can be introduced also in $\widehat{\mathcal{U}}_q(sl(n))$. Like in the special case of $n = 3$, this should be achieved by a Gauss decomposition oredered in a special manner. It seems to us that this ordering has to do with the ordering appearing in the Belavin-Drinfeld description of the classical r-matrix (10).

A theorem of Drinfeld [7] tells us that all quasi-triangular Hopf algebras deforming $\mathcal{U}(sl(n))$ are equivalent as quasi-triangular quasi-Hopf algebras, in a perturbative sense (as formal power series in h). This means that, in a formal sense, the multiplicative stucture of all such Hopf algebras is the same, and the coproduct is obtained from the trivial coproduct of $\mathcal{U}(sl(n))$ by twisting, i. e. by a similarity transformation in $\mathcal{U}(sl(n)) \otimes \mathcal{U}(sl(n))$. It is well known that the formal equivalence between $\mathcal{U}(sl(n))$ and $\mathcal{U}_q(sl(n))$ breaks down for q^2 root of unity. In contrast, we have seen that the equivalence of the product structures of $\mathcal{U}_q(sl(3))$ and $\widehat{\mathcal{U}}_q(sl(3))$ is valid for every value of q. It is possible that the twisting connecting the coproducts of $\mathcal{U}_q$ and $\widehat{\mathcal{U}}_q$ is also regular for every q. If this is true then also the branchings of tensor products of representations of $\mathcal{U}_q$ and $\widehat{\mathcal{U}}_q$ would be the same. We hope to report on some of these issues in a future publication.

Acknowledgement. We would like to thank the Dublin Institute for Advanced Studies for hospitality and L. O'Raifeartaigh for numerous discussions. L. D. would also like to thank the organizers of the XIV$^{\text{th}}$ John Hopkins workshop for providing him the opportunity to talk and for finantial support.

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