RENORMALIZATION GROUP ANALYSIS OF A
SIMPLE HIERARCHICAL FERMION MODEL[1]

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ABSTRACT

A simple hierarchical fermion model is constructed which gives rise to an exact renormalization transformation in a 2-dimensional parameter space. The behaviour of this transformation is studied. It has two hyperbolic fixed points for which the existence of a *global* critical line is proven. The asymptotic behaviour of the transformation is used to prove the existence of the thermodynamic limit in a certain domain in parameter space. Also the existence of a continuum limit for these theories is investigated using information about the asymptotic renormalization behaviour. It turns out that the "trivial" fixed point gives rise to a two-parameter family of continuum limits corresponding to that part of parameter space where the renormalization trajectories originate at this fixed point.

Although the model is not very realistic it serves as a simple example of the application of the renormalization group to proving the existence of the thermodynamic limit and the continuum limit of lattice models. Moreover, it illustrates possible compications that can arrise in global renormalization group behaviour, and that might also be present in other models where no global analysis of the renormalization transformation has yet been achieved.

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1. Introduction.

Hierarchical models were introduced by Dyson [8] before Wilson [19] formulated his theory of the renormalization group. It was Baker [1] who pointed out the simple renormalization group structure of the model. Actually, Bakers's model is different from Dyson's in that it has continuous spins instead of Ising spins. The first mathematical investigation of hierarchical models was carried out by Bleher and Sinai [3] and [4]. It was elaborated by Collet and Eckmann [5] and, more recently, by Gawedzki and Kupiainen [11,12]. Here we construct a fermionic analogue of Baker's model by replacing the spins with elements of a Grassmann algebra. Thus we obtain a model with a very simple renormalization group structure. It is somewhat like a hierarchical version of the Gross-Neveu model, the renormalization group structure of which was studied by Gawedzki and Kupiainen [14] and Feldman et. al.[9]. However our hierarchical model does not satisfy reflection positivity, so that the continuum limit is not a feasible candidate for a quantum field theory. The renormalization group transformation of our model takes place in a two-dimensional space of coupling parameters r and g. It is given by formulas (2.16). The simplicity of the transformation enables us to study the *global* flow of the transformation.

In the next section the model is introduced and its renormalization transformation derived. In Section 3 the main results about the asymptotic behaviour of the transformation are stated and discussed. Theorems 1 and 2 are precise statements about the existence and uniqueness of a global critical line for each of the fixed points $(0,0)$ and $\left(-\frac{4}{3}, \frac{8}{27}\right)$. Theorem 3 is a result about the asymptotic behaviour of points that are not critical. Proofs of these theorems are deferred to Section 7. Figure 1 shows a computer picture of the flow of the transformation. In Section 4 we use the asymptotic renormalization group behaviour to investigate the existence of a thermodynamic limit. Along the same lines, in Section 5, we prove a result about the decay of correlation functions, which is used in the study of the continuum limit in Section 6. The existence of a continuum limit is proven for each point on the trajectories receding from the trivial fixed point $(0,0)$. The existence of a continuum limit for points on the line $g = 0, r > 0$ is the easiest to establish: it is Gaussian. As in the case of the Gross-Neveu model, the other trajectories give rise to non-trivial continuum limits. Section 8 contains some concluding remarks.

2

2. The model.

For a Grassmann algebra $\mathcal{G}_n$ with an even number of generators arranged in conjugate pairs $\{\psi_1, \bar{\psi}_1, \ldots, \psi_n, \bar{\psi}_n\}$, there exists an analogue of a Gaussian integral due to Berezin [2]. It is given by the linear functional ω_C on $\mathcal{G}_n$ with

$$(2\text{-}1) \qquad \omega_C\left(\psi_i\bar{\psi}_j\right) = \frac{\int \psi_i\bar{\psi}_j \exp\left[-\langle\bar{\psi}, B\psi\rangle\right] d\psi d\bar{\psi}}{\int \exp\left[-\langle\bar{\psi}, B\psi\rangle\right] d\psi d\bar{\psi}} = \left(B^{-1}\right)_{ij} = C_{ij}.$$

Here $d\psi d\bar{\psi}$ stands for $d\psi_1 \ldots d\psi_n d\bar{\psi}_n \ldots d\bar{\psi}_1$ and $\int \cdot d\psi_i$ is the usual fermionic integration defined by

$$\int \psi_i \, d\psi_i = 1 \ , \ \int d\psi_i = 0.$$

B is a non-degenerate $n \times n$ matrix and

$$\langle\bar{\psi}, B\bar{\psi}\rangle = \sum_{i,j=1}^n \bar{\psi}_i B_{ij} \psi_j.$$

The exponential is given by its (terminating) Taylor expansion.

ω_C satisfies the usual fermionic Wick formulae,

$$(2\text{-}2) \qquad \omega_C\left(\psi_{i_1}\bar{\psi}_{j_1}\psi_{i_2}\bar{\psi}_{j_2}\right) = C_{i_1 j_1}C_{i_2 j_2} - C_{i_1 j_2}C_{i_2 j_1}, \ \text{etc.}$$

For a degenerate matrix C we can still define ω_C by $\omega_C\left(\psi_i\bar{\psi}_j\right) = C_{ij}$, and the Wick formulae.

We now introduce a hierarchical covariance C as follows. We consider a 2-dimensional square lattice $\Lambda_N \subset \mathbb{Z}^2$ with $\left(2^N\right)^2$ sites and subdivide it into blocks, blocks of blocks and so forth at each level or scale. The blocks B_l at the l-th level will contain 2^{2l} lattice sites each. For $x \in \Lambda_N$ we denote by $\dot{x} \in \Lambda_{N-1}$ the index of the block $B(\dot{x})$ containing x. More generally, $x^{(k)} \in \Lambda_{N-k}$ is defined by $x^{k-1} \in B(x^{(k)})$. We define a matrix $\Gamma = (\Gamma_{xy})_{x,y\in\Lambda_N}$ by $\Gamma_{xy} = (\Gamma_{\dot{x}})_{xy}$ if $\dot{x} = \dot{y}$ and $\Gamma_{xy} = 0$ if $\dot{x} \neq \dot{y}$. $\Gamma_{\dot{x}}$ is defined on a block $B(\dot{x})$ by

$$(2\text{-}3) \qquad \Gamma_{\dot{x}} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

3

with an arbitrary numbering of the sites in a block, fixed once for all. The hierarchical covariance is now defined by the series

$$(2\text{-}4) \qquad (C_N)_{xy} = \sum_{k=0}^{N-1} 2^{-k} \Gamma\left(x^{(k)}, y^{(k)}\right).$$

A slightly different covariance was considered in [6]. Our present choice has the advantage that $M\Gamma = 0$, where the operator M is defined by (2.5).) This defines a Gaussian state on the Grassmann algebra generated by the $2 \cdot 2^{2N}$ spins $\{\psi_x, \bar{\psi}_x\}_{x \in \Lambda_N}$. Introducing an average spin $(M\psi)_{\dot{x}}$ for each block $B(\dot{x})$ by

$$(2\text{-}5) \qquad (M\psi)_{\dot{x}} = L^{-d+\sigma} \sum_{x \in B(\dot{x})} \psi_x \,,$$

with $d = 2, \sigma = \frac{1}{2}, L = 2$ and analogously for $\bar{\psi}$, we find for the renormalized covariance (cf. [7]):

$$(2\text{-}6) \qquad (C_N)' = M\, C_N\, M^t = C_{N-1}.$$

Here M^t is the transposed matrix.

Before being able to introduce a non-trivial *local* interaction we have to double the number of spin components so that the corresponding Grassmann algebra $\mathcal{G}_N$ now has $4 \cdot 2^{2N}$ generators. The covariance of the resulting 2-component lattice field becomes $C_N \oplus C_N$, i.e.

$$(2\text{-}7) \qquad \omega_{C_N \oplus C_N}\left(\psi_{\alpha x} \bar{\psi}_{\beta y}\right) = \delta_{\alpha\beta}\left(C_N\right)_{xy} \,; \quad \alpha, \beta = 1, 2; \, x, y \in \Lambda_N.$$

We consider the general local potentials $V_N(\bar{\psi}, \psi) = \sum_{x \in \Lambda_N} v(\bar{\psi}_x, \psi_x)$ with

$$(2\text{-}8) \qquad v(\bar{\psi}_x, \psi_x) = r(\bar{\psi}_{1x}\psi_{1x} + \bar{\psi}_{2x}\psi_{2x}) + g\, \bar{\psi}_{1x}\psi_{1x}\bar{\psi}_{2x}\psi_{2x}.$$

In the presence of the potential V_N the 'expectation value' of a general polynomial F in the fields is given by

$$(2\text{-}9) \qquad \rho(F) = \frac{\omega_N(F(\bar{\psi}, \psi) \exp\left[-V_N(\bar{\psi}, \psi)\right])}{\omega_N(\exp\left[-V_N(\bar{\psi}, \psi)\right])},$$

where we have written ω_N instead of $\omega_{C_N \oplus C_N}$. The renormalized state ρ' is defined on $\mathcal{G}_{N-1}$ by

$$(2\text{-}10) \qquad \rho'(F) = \rho(F(M\bar{\psi}, M\psi)).$$

4

Due to the fact that

$$(2\text{-}11) \qquad (C_N)_{xy} = \frac{1}{2}(C_{N-1})_{\dot{x}\dot{y}} + \Gamma_{xy},$$

we can split the field ψ into $\psi' = M\psi$ and a fluctuation field ζ with covariance $\Gamma \oplus \Gamma$ so that

$$(2\text{-}12) \qquad \omega_N(F(\bar{\psi}, \psi)) = \omega_{N-1} \otimes \omega_{\Gamma \oplus \Gamma}\left(F\left(\frac{1}{\sqrt{2}}\bar{\psi}'_{\dot{x}} + \bar{\zeta}_x, \frac{1}{\sqrt{2}}\psi'_{\dot{x}} + \zeta_x\right)\right).$$

It then easily follows that

$$(2\text{-}13) \qquad \rho'(F) = \frac{\omega_{N-1}(F(\bar{\psi}', \psi')\exp\left[-V'_{N-1}(\bar{\psi}', \psi')\right])}{\omega_{N-1}(\exp\left[-V'_{N-1}(\bar{\psi}', \psi')\right])},$$

where the effective potential V'_{N-1} is also local, $V'_{N-1}(\bar{\psi}', \psi') = \sum_{\dot{x} \in \Lambda_{N-1}} v'(\bar{\psi}'_{\dot{x}}, \psi'_{\dot{x}})$ with v' given by

(2-14)

$$\exp\left[-v'(\bar{\psi}'_{\dot{x}}, \psi'_{\dot{x}})\right] = \frac{\omega_{\Gamma_{\dot{x}} \oplus \Gamma_{\dot{x}}}\left(\exp\left[-\sum_{x \in B(\dot{x})} v\left(\frac{1}{\sqrt{2}}\bar{\psi}'_{\dot{x}} + \bar{\zeta}_x, \frac{1}{\sqrt{2}}\psi'_{\dot{x}} + \zeta_x\right)\right]\right)}{\omega_{\Gamma_{\dot{x}} \oplus \Gamma_{\dot{x}}}\left(\exp\left[-\sum_{x \in B(\dot{x})} v(\bar{\zeta}_x, \zeta_x)\right]\right)}.$$

Remark. In this expression one has to collect the ζ's before calculating the expectation w.r.t. $\omega_{\Gamma_{\dot{x}} \oplus \Gamma_{\dot{x}}}$. Thus, for instance, $\omega_{\Gamma_0 \oplus \Gamma_0}\left(\psi'_\alpha \zeta_1 \bar{\psi}'_\beta \bar{\zeta}_1\right) = -\psi'_\alpha \bar{\psi}'_\beta$.

The fact that local potentials are conserved under the block-spin transformation is a general property of hierarchical models, much stressed by Gawedzki and Kupiainen [11,12]. It is due to the fact that $\Gamma_{xy} = 0$ if $\dot{x} \neq \dot{y}$, so that $\omega_{\Gamma \oplus \Gamma} = \bigotimes_{\dot{x} \in \Lambda_{N-1}} \omega_{\Gamma_{\dot{x}} \oplus \Gamma_{\dot{x}}}$. Our fermionic hierarchical model has the additional simplifying property that the exponentials in (2.14) break off. Therefore we can calculate the expectations explicitly. Diagonalizing $\Gamma_{\dot{x}}$ and rewriting the result as an exponential, we obtain

$$(2\text{-}15) \qquad \begin{aligned} &\omega_{\Gamma_0 \oplus \Gamma_0}\left(\exp\left[-\sum_{x \in B(\dot{x})} v\left(\frac{1}{\sqrt{2}}\bar{\psi}' + \bar{\zeta}_x, \frac{1}{\sqrt{2}}\psi' + \zeta_x\right)\right]\right) = \\ &\qquad = [(1+r)^2 - \frac{1}{4}g] \cdot \exp\left[-r'(\bar{\psi}'_1\psi'_1 + \bar{\psi}'_2\psi'_2) - g'\bar{\psi}'_1\psi'_1\bar{\psi}'_2\psi'_2\right] \end{aligned}$$

with

$$(2\text{-}16) \qquad \begin{cases} r' &= 2r - \frac{1}{2}\dfrac{g(1+r)}{(1+r)^2 - g/4} \\[2ex] g' &= g\left\{\dfrac{(1+r)^2 - g/2}{(1+r)^2 - g/4}\right\}^2. \end{cases}$$

5

Notice that (2.15) is in fact independent of $\dot{x}$, so that we can omit all indices $\dot{x}$ and write Γ_0 instead of $\Gamma_{\dot{x}}$.

To derive (2.15) we define $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, and introduce new

variables $\eta_{\alpha x} = (Q\zeta_\alpha)_x$ satisfying $\omega_{\Gamma_0 \oplus \Gamma_0}((Q\zeta_\alpha)_x (Q\bar{\zeta}_\beta)_y) = \delta_{\alpha\beta}\delta_{x3}\delta_{y3}$. Clearly only the third component $\eta_\alpha = \eta_{\alpha 3}$ of η matters and we can write

$$\omega_{\Gamma_0 \oplus \Gamma_0}\left(\exp\left[-\sum_x v\left(\frac{1}{\sqrt{2}}\bar{\psi}' + \bar{\zeta}_x, \frac{1}{\sqrt{2}}\psi' + \zeta_x\right)\right]\right) =$$

$$= \omega_1\left(\exp\left[-2r\sum_{\alpha=1,2}\sum_{s=\pm 1}\left(\frac{1}{\sqrt{2}}\bar{\psi}'_\alpha + \frac{1}{2}s\bar{\eta}_\alpha\right)\left(\frac{1}{\sqrt{2}}\psi'_\alpha + \frac{1}{2}s\eta_\alpha\right)\right.\right.$$

$$-2g\sum_{s=\pm 1}\left(\frac{1}{\sqrt{2}}\bar{\psi}'_1 + \frac{1}{2}s\bar{\eta}_1\right)\left(\frac{1}{\sqrt{2}}\psi'_1 + \frac{1}{2}s\eta_1\right)$$

$$\left.\left.\left(\frac{1}{\sqrt{2}}\bar{\psi}'_2 + \frac{1}{2}s\bar{\eta}_2\right)\left(\frac{1}{\sqrt{2}}\psi'_2 + \frac{1}{2}s\eta_2\right)\right]\right)$$

$$= \exp\left[-2r\left(\bar{\psi}'_1\psi'_1 + \bar{\psi}'_2\psi'_2\right) - g\,\bar{\psi}'_1\psi'_1\bar{\psi}'_2\psi'_2\right]\cdot$$

$$\cdot\,\omega_1\left(\exp\left[-r(\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) - \frac{1}{2}g\,\bar{\psi}'_1\psi'_1\bar{\eta}_2\eta_2\right.\right.$$

$$+\frac{1}{2}g\,\bar{\psi}'_1\bar{\psi}'_2\eta_1\eta_2 - \frac{1}{2}g\,\bar{\psi}'_1\psi'_2\eta_1\bar{\eta}_2 - \frac{1}{2}g\,\psi'_1\bar{\psi}'_2\bar{\eta}_1\eta_2$$

$$\left.\left.+\frac{1}{2}g\,\psi'_1\psi'_2\bar{\eta}_1\bar{\eta}_2 - \frac{1}{2}g\,\bar{\psi}'_2\psi'_2\bar{\eta}_1\eta_1 - \frac{1}{4}g\,\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\right]\right)$$

$$= \exp\left[-2r(\bar{\psi}'_1\psi'_1 + \bar{\psi}'_2\psi'_2) - g\,\bar{\psi}'_1\psi'_1\bar{\psi}'_2\psi'_2\right]\cdot$$

$$\left\{1 + 2r + \frac{1}{2}g(1 + r)(\bar{\psi}'_1\psi'_1 + \bar{\psi}'_2\psi'_2) - \frac{1}{4}g + r^2 + \frac{3}{4}g^2\bar{\psi}'_1\psi'_1\bar{\psi}'_2\psi'_2\right\}$$

$$= [(1 + r)^2 - \frac{1}{4}g]\cdot\exp\left[-r'(\bar{\psi}'_1\psi'_1 + \bar{\psi}'_2\psi'_2) - g'\bar{\psi}'_1\psi'_1\bar{\psi}'_2\psi'_2\right].$$

From (2.15) it follows that the denominator in (2.14) equals $(1 + r)^2 - \frac{1}{4}g$, so that v' has the same form as (2.1) but with r and g replaced by r' and g' respectively. Apparently, the renormalization transformation for this model is given by a simple explicit transformation $(r, g) \longmapsto R(r, g)$ in a 2-dimensional parameter space.

3. Analysis of the flow of R.

Figure 1 shows a computer picture of the flow of R. For clarity successive points under the iteration of R have been connected by straight lines. The parabola is the set of singular points of the transformation.
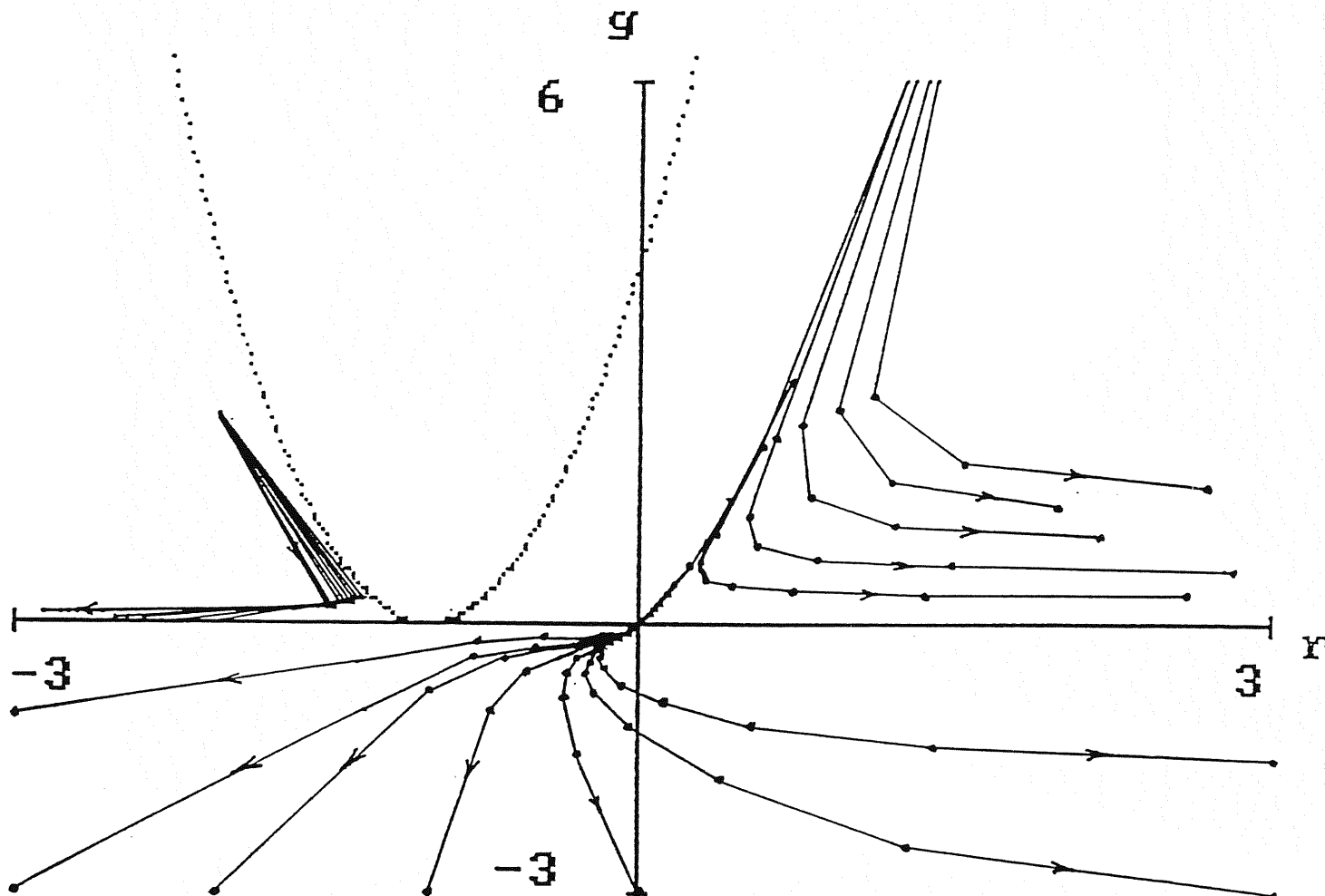

FIGURE 1.

The fixed points of R are $(0,0)$ and $\left(-\frac{4}{3}, \frac{8}{27}\right)$. By construction the linearized transformation about the origin is

$$(3\text{-}1) \qquad \partial R(0,0) = \begin{pmatrix} 2 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

7

The $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigendirection is repulsive; $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a marginal eigendirection. To second order the latter is attracting, which is also clearly visible in Fig.1. The existence and uniqueness of a critical line in this direction is stated in Theorem 1.

The linearization of R about $\left(-\frac{4}{3}, \frac{8}{27}\right)$ is given by

$$(3\text{-}2) \qquad \partial R\left(-\frac{4}{3}, \frac{8}{27}\right) = \begin{pmatrix} 22 & \frac{27}{2} \\ \frac{64}{3} & 13 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = \frac{35}{2} \pm \frac{3}{2}\sqrt{137}$; $\lambda_+ \approx 35$, $\lambda_- \approx -.057$. The corresponding eigendirections are $\begin{pmatrix} 1 \\ -\frac{1}{3} \pm \frac{1}{9}\sqrt{137} \end{pmatrix} = \begin{pmatrix} 1 \\ .97 \end{pmatrix}$ resp. $\begin{pmatrix} 1 \\ -1.63 \end{pmatrix}$. The existence and uniqueness of a critical line for this hyperbolic fixed point is stated in Theorem 2. Finally, the asymptotic behaviour of other parameter values (r, g) under the iteration of R is stated in Theorem 3. Unfortunately, we have not been able to prove any rigorous result about the behaviour of points (r, g) with $g > 0$ between the two critical lines.

For a precise statement of the existence and uniqueness of a critical line associated with the fixed point $(0, 0)$ we define a region $S(g_0)$ with $g_0 > 0$ as follows. Let the curves $r_-(g), r_+(g)$ and $r_0(g)$ be defined by

$$(3\text{-}3) \qquad \begin{cases} r_-(g) & = \frac{1}{2}\left(\sqrt{1+g} - 1\right) \\ r_+(g) & = \frac{1}{2}\left(\sqrt{1+4g} - 1\right) \\ r_0(g) & = \sqrt{g/2} - 1, \end{cases}$$

for $g \geq 0$. Then $S(g_0)$ is the region bounded by these curves and the line $g = g_0$, i.e.

$$(3\text{-}4) \qquad S(g_0) = \{(r, g) \mid 0 \leq g \leq g_0, \ \max\left(r_-(g), r_0(g)\right) \leq r \leq r_+(g)\}.$$

We shall prove

Theorem 1.

For all $g_0 > 0$ there exists a unique critical r-value $r_c(g_0) > 0$ such that $R^n(r_c(g_0), g_0) \in S(g_0)$ for all $n \geq 0$, and $R^n(r_c(g_0), g_0) \to (0, 0)$ as $n \to \infty$.

The existence of a critical line in the neighbourhood of $(0, 0)$ also follows from the Centre Manifold Theorem. We can use it in the form proven by Lanford in [16] (see

8

also [17]) by inverting R. The inverse mapping R^{-1} can be calculated explicitly,

(3-5)
$$\begin{cases} r = & \frac{1}{2}r' + \frac{1}{4}\dfrac{g'(1+r'/2)}{(1+r'/2)^2 - g'/2} \\ g = & g'\left\{\dfrac{(1+r'/2)^2 - g'/4}{(1+r'/2)^2 - g'/2}\right\}^2 . \end{cases}$$

The centre manifold is not unique in general. (See Van Strien [18] for a striking counter example.) Our uniqueness result in Theorem 1 together with the Centre Manifold Theorem imply

Corollary

The critical line $r_c(g)$ is C^∞ for $g \geq 0$.

The inverse mapping has also been used to draw the critical line in Fig.1. It is in fact sufficient to know only a small portion of the critical line to generate the whole line in a finite number of steps. This follows from Lemma 2 below together with the fact that, if $g' > 4\epsilon$ with $\epsilon < \frac{3}{16}$ and $g' > r'(1+r')$ then $g > (1+\epsilon)g'$. A small piece of the critical line extending from $g' = 0$ to $g' > 4\epsilon$ will therefore extend beyond the line $g' = 2(1+\frac{1}{2}r')^2$ after applying R^{-1} a finite number of times. The next time R^{-1} is applied it extends to infinity. Points above the line $g' = 2(1+\frac{1}{2}r')^2$ are mapped into the second quadrant, which therefore also contains critical points ! Applying R^{-1} sufficiently many times critical points may even return to the neighbourhood of $(0,0)$, which is why Theorem 1 contains the clause $R^n(r_c(g_0), g_0) \in \mathcal{S}(g_0)$.

An analogous result holds for the fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$. We define

(3-6)
$$\begin{cases} g_+(r) & = \frac{8}{3}(1+r)^2 \\ g_-(r) & = 2(1+r)^2 + \frac{2}{27}, \end{cases}$$

and

(3-7)
$$\begin{cases} \mathcal{T}_1 & = \{(r,g)\,|\,r \leq -\frac{4}{3},\ g_-(r) \leq g \leq g_+(r)\} \\ \mathcal{T}_2 & = \{(r,g)\,|\,-\frac{4}{3} \leq r \leq r_0,\ g_+ \leq g \leq g_-(r)\}. \end{cases}$$

We have put

(3-8)
$$r_0 = -\frac{2}{3} - \frac{4}{9}\sqrt{2},$$

so that $2(1 + \frac{1}{2}r_0)^2 = g_-(r_0)$.

Theorem 2.

For all $r \leq r_0$ there exists a unique critical value $g_c(r)$ such that $R^n(r, g_c(r)) \in T_1 \cup T_2$ for all $n \geq 0$ and $R^n(r, g_c(r)) \to \left(-\frac{4}{3}, \frac{8}{27}\right)$ as $n \to \infty$.

For the proof of both theorems we make use of a version of the interval argument due to Bleher and Sinai [3]. We need several technical lemmas which are postponed to Section 7.

We now state our result about the asymptotic behaviour of other points (r, g) in the plane. This result will prove to be useful for taking the infinite volume limit $N \to \infty$ and the continuum limit in Sections 4 and 6 respectively.

Theorem 3.

For all $g < 0$ there exists a unique value $r_s(g)$ of r such that

(1.) $\qquad R^n(r_s(g), g) \to \left(-\frac{2}{3}, -\infty\right)$ as $n \to \infty$,

(2.) $\qquad r > r_s(g) \implies R^n(r, g) \to (\infty, g_\infty(r, g))$ as $n \to \infty$,

(3.) $\qquad r < r_s(g) \implies R^n(r, g) \to (-\infty, g_\infty(r, g))$ as $n \to \infty$.

where $-\infty < g_\infty(r, g) < \infty$.

Furthermore, if $g \geq 0$ and $r > r_c(g)$ then $R^n(r, g) \to (\infty, g_\infty(r, g))$, and if $r \leq r_0$ and $0 \leq g < g_c(r)$ or $r \leq -1$ and $0 \leq g < \frac{8}{3}(1 + r)^2$ then $R^n \to (-\infty, g_\infty(r, g))$ as $n \to \infty$, with $0 \leq g_\infty(r, g) < g$.

We postpone the proof to Section 7. Finally we mention a result about the existence of an unstable line for the fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$.

Proposition 1.1.

For all r between -2 and $-4/3$ there exists a unique value $g_u(r)$ such that $(\tilde{r}_n, \tilde{g}_n) \equiv R^{-n}(r, g_u(r))$ satisfies $(4 + \tilde{r}_n)(1 + \frac{1}{2}\tilde{r}_n)^2 \leq \tilde{g}_n \leq \frac{8}{3}(1 + \frac{1}{2}\tilde{r}_n)^2$ for $n \geq 0$, and $(\tilde{r}_n, \tilde{g}_n) \to \left(-\frac{4}{3}, \frac{8}{27}\right)$ as $n \to \infty$.

10

4. The Infinite Volume Limit.

We can use the renormalization transformation to study the existence of the infinite volume limit $N \to \infty$ and the existence of the continuum limit. (Cf. [11] and [15].) We shall find that the infinite volume limit exists for the points (r, g) such that $R^n(r, g) \to (\pm\infty, g_\infty)$, and also for the critical points (r, g) associated with the fixed point $(0, 0)$, but not for the critical points associated with the fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$. An elaboration of the methods used to prove the existence of the thermodynamic limit yields information about the decay of correlation functions. This is shown in Section 5. In Section 6 this information is used to prove the existence of the continuum limit for points (r, g) on the trajectories receding from the 'trivial' fixed point $(0, 0)$.

To study the thermodynamic limit $N \to \infty$ we start by considering the 2-point function $\rho_N(\psi_{1x}\bar\psi_{1y})$ given by (2.9). Assume first that $\dot{x} \neq \dot{y}$. Then we can use the decomposition formula (2.12) to reduce N by 1. Indeed, by symmetry and the fact that $M\Gamma = 0$, it follows that $\rho_N(\psi_{1x}\bar\psi_{1y}) = \frac{1}{2}\rho'_{N-1}(\psi'_{1x}\bar\psi'_{1y})$. Iterating this relation we find

$$(4\text{-}1) \qquad \rho_N(\psi_{1x}\bar\psi_{1y}) = 2^{-s}\rho^{(s)}_{N-s}\left(\psi^{(s)}_{1x^{(s)}}\bar\psi^{(s)}_{1y^{(s)}}\right),$$

where $s = s(x, y)$ is the smallest number $s \geq 0$ such that $y^{(s+1)} = x^{(s+1)}$. We are left with the calculation of $\rho_N(\psi_{1x}\bar\psi_{1y})$ when $\dot{x} = \dot{y}$. Again we apply the decomposition formula (2.12) to find

$$(4\text{-}2) \qquad \rho_N(\psi_{1x}\bar\psi_{1y}) = \rho'_{N-1}(T(\psi_{1x}\bar\psi_{1y})),$$

where, for a general polynomial F in ψ and $\bar\psi$, we define TF by

$(4\text{-}3)$

$$(TF)(\psi', \bar\psi') = \frac{\omega_{\Gamma\oplus\Gamma}\left(F\left(\frac{1}{\sqrt{2}}\psi' + \zeta, \frac{1}{\sqrt{2}}\bar\psi' + \bar\zeta\right)\exp\left[-V\left(\frac{1}{\sqrt{2}}\bar\psi' + \bar\zeta, \frac{1}{\sqrt{2}}\psi' + \zeta\right)\right]\right)}{\omega_{\Gamma\oplus\Gamma}\left(\exp\left[-V\left(\frac{1}{\sqrt{2}}\bar\psi' + \bar\zeta, \frac{1}{\sqrt{2}}\psi' + \zeta\right)\right]\right)}.$$

Let us also introduce a block expectation $\langle\cdot\rangle$ by

$$(4\text{-}4) \qquad \langle F(\zeta, \bar\zeta)\rangle = \frac{\omega_{\Gamma\oplus\Gamma}(\,F(\zeta, \bar\zeta)\exp[-V(\bar\zeta, \zeta)]\,)}{\omega_{\Gamma\oplus\Gamma}(\,\exp[-V(\bar\zeta, \zeta)]\,)}.$$

11

It is easy to see that this is a product state over blocks, and that, for x, y with $\dot{x} = \dot{y}$,

$$(4\text{-}5) \qquad \langle \zeta_{\alpha x} \bar{\zeta}_{\beta y} \rangle = \epsilon_x \epsilon_y \delta_{\alpha \beta} c_1 \quad,$$

with

$$(4\text{-}6) \qquad c_1 = \frac{1}{4} \frac{1 + r}{(1 + r)^2 - g/4}$$

and

$$\epsilon_x = \begin{cases} +1, & \text{if } x = 1, 3; \\ -1, & \text{if } x = 2, 4; \end{cases}$$

i.e. $\epsilon_x \epsilon_y = -1$ if x and y are nearest neighbours, and $=+1$ otherwise. (Assuming that the points in a block are numbered in a circular way.)

Similarly, for $\dot{x}_1 = \dot{x}_2 = \dot{y}_1 = \dot{y}_2$,

$$(4\text{-}7) \qquad \langle \zeta_{\alpha_1 x_1} \bar{\zeta}_{\beta_1 y_1} \zeta_{\alpha_2 x_2} \bar{\zeta}_{\beta_2 y_2} \rangle = \epsilon_{x_1} \epsilon_{x_2} \epsilon_{y_1} \epsilon_{y_2} \gamma(\alpha_1, \alpha_2; \beta_1, \beta_2) c_2 \quad,$$

with

$$(4\text{-}8) \qquad c_2 = \frac{1}{16} \frac{1}{(1 + r)^2 - g/4},$$

and

$$(4\text{-}9) \qquad \begin{aligned} \gamma(\alpha_1, \alpha_2; \beta_1, \beta_2) &= (\delta_{\alpha_1 \beta_1} - \delta_{\alpha_2 \beta_2})(1 - \delta_{\alpha_1 \alpha_2})(1 - \delta_{\beta_1 \beta_2}) \\ &= \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2} - \delta_{\alpha_1 \beta_2} \delta_{\alpha_2 \beta_1}. \end{aligned}$$

All other block expectations are zero. We define truncated block expectations $\langle F_1; \ldots; F_n \rangle = \left\langle \prod_{j=1}^{n} F_j \right\rangle_c$ for monomials $F_1, \ldots, F_n$ by the usual inductive procedure:

$$(4\text{-}10) \qquad \langle F_1 \ldots F_n \rangle = \sum_{\{I_l\}_{l=1}^{p}} (-1)^{\pi(I_1, \ldots, I_p)} \prod_{l=1}^{p} \left\langle \prod_{i \in I_l} F_i \right\rangle_c,$$

where the sum is over all partitions $\{I_l\}_{l=1}^{p}$ of $\{1, \ldots, n\}$, and $\pi(I_1, \ldots, I_p)$ is the number of odd transpositions needed to reorder $(1, \ldots, n)$ according to $I_1, \ldots, I_p$. A transposition of i and j is called odd if the monomials F_i and F_j both have odd degree.

Let us denote a general monomial by ψ_X, where X is a set of pairs (α, x) with

12

$\alpha \in \{1, \bar{1}, 2, \bar{2}\}$. Thus, for example, $\psi_{1x}\bar{\psi}_{2y} = \psi_X$ with $X = \{(1, x), (\bar{2}, y)\}$. We then have:

Proposition 4.1

The coefficient of ψ'_Y in the expansion of $T\psi_X$:

$$(4\text{-}11) \qquad\qquad T\psi_X = \sum_Y 2^{-|Y|/2}\, T_{XY}\, \psi'_Y$$

is given by

$$(4\text{-}12)$$

$$T_{XY} = \sum_{\{X_0 \mid \dot{X}_0 \subset Y\}} \sum_{p=0}^{|Y \setminus \dot{X}_0|} (-1)^p \sum_{\{X_l\}_{l=1}^p} (-1)^{\sigma(X_0, \ldots, X_p)}$$

$$\left\langle \frac{\partial^{|X_0|}\zeta_X}{\partial\zeta_{X_0}} ; \frac{\partial^{|X_1|}V}{\partial\zeta_{X_1}}(\bar{\zeta}, \zeta) ; \cdots ; \frac{\partial^{|X_p|}V}{\partial\zeta_{X_p}}(\bar{\zeta}, \zeta) \right\rangle_c .$$

Here the first sum is over all sets X_0 of pairs (α, x) such that $\dot{X}_0 \equiv \{(\alpha, \dot{x}) \mid (\alpha, x) \in X_0\} \subset Y$ and the third sum is over all collections of sets $X_1, \ldots, X_p$ such that $\{\dot{X}_1, \ldots, \dot{X}_p\}$ is a partition of $Y \setminus \dot{X}_0$. The sets $X_0, \ldots, X_p$ are assumed to be ordered according to Y and the differentiation is to be performed in reverse order. $\sigma(X_0, \ldots, X_p)$ is the number of transpositions needed to reorder X according to $X_0, \ldots, X_p$.

Proof. This kind of formula is standard in perturbation theory: see e.g. formula (5.8) in [11]. However, since we are dealing with Grassmann variables here, we have to be careful about reordering factors. This gives rise to factors $(-1)^{\sigma(X_0, \ldots, X_p)}$ in the expression (4.12). Notice also that $\frac{\partial^{|X_l|}V}{\partial\zeta_{X_l}} = 0$ unless all $\zeta_{\alpha x}$ with $(\alpha, x) \in X_l$ are situated at the same lattice point x. Moreover, looking at the expressions (4.5) and (4.7) for the non-zero block expectations it transpires that changing the lattice point at which X_l is situated can only change the sign of the expectation. Since we are summing over all X_l it follows that $|X_l|$ must be even. This justifies pulling the derivatives $\frac{\partial^{|X_l|}}{\partial\zeta_{X_l}}$ through F. It also means that all $\frac{\partial^{|X_l|}V}{\partial\zeta_{X_l}}$ are also even so that we can restrict the sum to collections $\{X_l\}$ with the same order as $(1, \ldots, n)$ omitting a factor $1/k!$. ∎

Notice that the non-zero terms in (4.12) must satisfy:

1. $X_0 \subset$ ind (F) and, in particular, $|X_0| \le$ deg (F), where ind (F) is the index set of F, and deg (F) is the degree of F.

2. $|X_l| = 2$ for all $l = 1, 2, \ldots, p$ and the points of X_l are equal, because of the factors ϵ_x.

3. $X_0 \cup X_1 \cup \cdots \cup X_p \subset \bigcup_{x \in \text{ supp } (F)} B(\dot{x})$, where supp $(F) = S$ is the set of points x such that $\psi_{\alpha x}$ occurs in $F(\psi, \bar{\psi})$ for some $\alpha = 1, \bar{1}, 2, \bar{2}$. Hence supp $Y \subset S$.

Applying (4.12) to $F(\psi, \bar{\psi}) = \psi_{1x}\bar{\psi}_{1y}$ we find that the only non-zero terms are:
$Y = \emptyset$; $Y = \{(1, \dot{x}), (\bar{1}, \dot{x})\}, X_0 = \emptyset, X_1 = \{(1, u), (\bar{1}, u)\}$ with $\dot{x} = \dot{u}$;
$Y = \{(1, \dot{x}), (\bar{1}, \dot{x})\}, X_0 = \{(1, x), (\bar{1}, y)\}$; $Y = \{(2, \dot{x}), (\bar{2}, \dot{x})\}, X_0 = \emptyset,$
$X_1 = \{(2, u), (\bar{2}, u)\}$ with $\dot{u} = \dot{x}$; $Y = \{(1, \dot{x}), (\bar{1}, \dot{x}), (2, \dot{x}), (\bar{2}, \dot{x})\}, X_0 = \emptyset,$
$X_1 = \{(1, u), (\bar{1}, u)\}, X_2 = \{(2, v), (\bar{2}, v)\}$ or $X_1 = \{(1, u), (2, u)\},$
$X_2 = \{(\bar{1}, v), (\bar{2}, v)\}$ or $X_1 = \{(1, u), (\bar{2}, u)\}, X_2 = \{(\bar{1}, v), (2, v)\}$ with $\dot{u} = \dot{v} = \dot{x}$;
$Y = \{(1, \dot{x}), (\bar{1}, \dot{x}), (2, \dot{x}), (\bar{2}, \dot{x})\}, X_0 = \{(1, x), (\bar{1}, y)\}, X_1 = \{(2, u), (\bar{2}, u)\}$
with $\dot{u} = \dot{x}$.
In all we find:

$$(4\text{-}13) \quad T(\psi_{1x}\bar{\psi}_{1y}) = \epsilon_x \epsilon_y \left\{ c_1 + a_{11}\psi'_{1\dot{x}}\bar{\psi}'_{1\dot{x}} + a_{12}\psi'_{2\dot{x}}\bar{\psi}'_{2\dot{x}} + a_{13}\psi'_{1\dot{x}}\bar{\psi}'_{1\dot{x}}\psi'_{2\dot{x}}\bar{\psi}'_{2\dot{x}} \right\}$$

with

$$(4\text{-}14) \quad \begin{cases} a_{11} & = a_{22} = \tfrac{1}{2} - 2g(c_2 - c_1^2) \\ a_{12} & = a_{21} = 2gc_1^2 \\ a_{13} & = a_{23} = 8g^2 c_1(c_1^2 - 2c_2). \end{cases}$$

Of course, by symmetry, the same formula (4.13) but with the indices 1 and 2 interchanged, holds for $T(\psi_{2x}\bar{\psi}_{2y})$ with $\dot{x} = \dot{y}$. Apparently, in order to be able to iterate these equations as in (4.1) we need to include the 4-point function in our consideration. A straight-forward calculation yields

$$(4\text{-}15) \quad \begin{aligned} T(\psi_{1x}\bar{\psi}_{1x}\psi_{2x}\bar{\psi}_{2x}) = {} & c_2 + a_{31}\psi'_{1\dot{x}}\bar{\psi}'_{1\dot{x}} + a_{32}\psi'_{2\dot{x}}\bar{\psi}'_{2\dot{x}} \\ & + a_{33}\psi'_{1\dot{x}}\bar{\psi}'_{1\dot{x}}\psi'_{2\dot{x}}\bar{\psi}'_{2\dot{x}}, \end{aligned}$$

14

where

$$(4\text{-}16) \qquad \begin{cases} a_{31} & = a_{32} = \frac{1}{2}c_1(1 + 4gc_2) \\ a_{33} & = \frac{1}{4} + 2gc_1^2 - 6gc_2 + 4g^2c_2(2c_1^2 - 3c_2). \end{cases}$$

By inserting the equation (4.13) into (4.2) we have expressed the ρ_N−expectation into ρ'_{N-1}−expectations. We can do the same with $\rho_N(\psi_{1x}\bar{\psi}_{1x}\psi_{2x}\bar{\psi}_{2x})$ and equation (4.15). In fact the ρ_N−expectation of *any* polynomial can be expressed in terms of a finite number of $\rho_{N-k}^{(k)}$−expectations of monomials at a single point $x^{(k)}$. Thus, in order to prove the existence of the thermodynamic limit it suffices to consider the monomials at a single point x:

Theorem 4.

Let $\mathcal{D}$ be the set of points (r, g) satisfying $g < 0$ or $\big(g > 0$ and $r \geq r_c(g)\big)$ or $\big(r \leq r_0$ and $g < g_c(r)\big)$ or $\big(r_0 < r < -1$ and $g < \frac{8}{3}(1 + r)^2\big)$, where r_0 is defined by (3.9) and $r_c(g)$ and $g_c(r)$ have been defined in Theorems 2 and 3. Define the state ρ_N on the Grassmann algebra $\mathcal{G}(E_N)$ over $E_N = (\mathbf{R})^{\Lambda_N}$ by (2.8). Then the thermodynamic limit exists if $(r, g) \in \mathcal{D}$ in the sense that $\rho(F) = \lim_{N \to \infty} \rho_N(F)$ exists for all polynomials F in the fields.

Proof. It is sufficient to consider the cases $F = \psi_{1x}\bar{\psi}_{1x}$, $F = \psi_{2x}\bar{\psi}_{2x}$ and $F = \psi_{1x}\bar{\psi}_{1x}\psi_{2x}\bar{\psi}_{2x}$. By symmetry all other single point expectations vanish. Also, $\rho_N(\psi_{1x}\bar{\psi}_{1x}) = \rho_N(\psi_{2x}\bar{\psi}_{2x})$. Writing

$$(4\text{-}17) \qquad \begin{cases} u_1^{(N)} & = u_2^{(N)} = \rho_N(\psi_{1x}\bar{\psi}_{1x}) \text{ and} \\ u_3^{(N)} & = \rho_N(\psi_{1x}\bar{\psi}_{1x}\psi_{2x}\bar{\psi}_{2x}), \end{cases}$$

and defining the matrices

$$(4\text{-}18) \qquad A^{(n)} = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} & a_{13}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} & a_{23}^{(n)} \\ a_{31}^{(n)} & a_{32}^{(n)} & a_{33}^{(n)} \end{pmatrix},$$

where the $a_{ij}^{(n)}$ are given by (4.14) and (4.16) with r and g replaced by $r^{(n)}$ and $g^{(n)}$, we can rewrite (4.13) and (4.15) in the form of a vector equation:

$$\begin{pmatrix} u_1^{(N)} \\ u_2^{(N)} \\ u_3^{(N)} \end{pmatrix} = \begin{pmatrix} c_1^{(0)} \\ c_2^{(0)} \\ c_3^{(0)} \end{pmatrix} + A^{(0)} \begin{pmatrix} u_1^{(N-1)} \\ u_2^{(N-1)} \\ u_3^{(N-1)} \end{pmatrix},$$

and more generally,

$$(4\text{-}19) \qquad \begin{pmatrix} u_1^{(N-n)} \\ u_2^{(N-n)} \\ u_3^{(N-n)} \end{pmatrix} = \begin{pmatrix} c_1^{(n)} \\ c_2^{(n)} \\ c_3^{(n)} \end{pmatrix} + A^{(n)} \begin{pmatrix} u_1^{(N-n-1)} \\ u_2^{(N-n-1)} \\ u_3^{(N-n-1)} \end{pmatrix}.$$

Iterating this equation we obtain

$$(4\text{-}20) \qquad \mathbf{u}^{(N)} = \sum_{n=0}^{N} A^{(0)} \dots A^{(n-1)} \mathbf{c}^{(n)}.$$

We used the fact that $\mathbf{u}^{(0)} = \mathbf{c}^{(N)}$; the $n = 0-$term is simply $\mathbf{c}^{(0)}$.)

We now want to study the convergence of (4.20) as $N \to \infty$. We use the following lemma:

Lemma 4.1

Suppose that $\limsup_{n \to \infty} \|A^{(n)}\| < 1$ and that $\|\mathbf{c}^{(n)}\|$ is bounded.
Then $\sum_{n=0}^{\infty} A^{(o)} \dots A^{(n-1)} \mathbf{c}^{(n)}$ converges.

Now, for points (r, g) such that $R^n(r, g) \to (\pm\infty, g_\infty)$ we have

$$A^{(n)} \to \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \text{ as } n \to \infty,$$

and

$$c_1^{(n)} = \frac{1}{4} \frac{1 + r^{(n)}}{(1 + r^{(n)})^2 - g^{(n)}/4} \to 0 \text{ and } c_2^{(n)} = \frac{1}{16} \frac{1}{(1 + r^{(n)})^2 - g^{(n)}/4} \to 0,$$

so that the thermodynamic limit exists. For points (r, g) such that $(r^{(n)}, g^{(n)}) \to (0, 0)$, i.e. $r = r_c(g)$, we have

$$A^{(n)} \to \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1/8 & 1/8 & 1/4 \end{pmatrix} \text{ and } \mathbf{c}^{(n)} \to \begin{pmatrix} 1/4 \\ 1/4 \\ 1/8 \end{pmatrix},$$

so that, again, $\lim_{N \to \infty} \mathbf{u}^{(N)}$ exists.
Next consider the points (r, g) with $r = r_s(g)$, so that $R^n(r, g) \to (-\frac{2}{3}, -\infty)$. In that case

$$A^{(n)} \to \begin{pmatrix} 1 & 0 & -4/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{c}^{(n)} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

16

and Lemma 4.1 does not apply. Instead we can use

Lemma 4.2

Suppose that $\frac{||c^{(n+1)}||}{||c^{(n)}||} \to \delta > 0$ and $\limsup_{n\to\infty} ||A^{(n)}|| < \delta^{-1}$.
Then $\sum_{n=0}^{\infty} A^{(0)} \ldots A^{(n-1)} c^{(n)}$ converges absolutely.

In the present case, $g^{(n+1)} \sim 4g^{(n)}$, so that $c^{(n+1)} \sim \frac{1}{4} c^{(n)}$, and $||A^{(n)}||_{1,1} \to$
$\frac{11}{3} < 4$. It follows that $\sum_{n=0}^{\infty} A^{(0)} \ldots A^{(n-1)} c^{(n)}$ still converges.
Finally, we consider the points (r, g) on the critical line belonging to the fixed point
$\left(-\frac{4}{3}, \frac{8}{27}\right)$, i.e. $g = g_c(r)$. We have

$$A^{(n)} \to \begin{pmatrix} 5/2 & 3 & -8/3 \\ 3 & 5/2 & -8/3 \\ -27/8 & -27/8 & 13/4 \end{pmatrix} \text{ and } c^{(n)} \to \begin{pmatrix} -9/4 \\ -9/4 \\ 27/16 \end{pmatrix}$$

and $A^{(\infty)}$ has three eigenvalues: $-\frac{1}{2}$ and $\lambda_\pm = \frac{35}{8} \pm \frac{3}{8}\sqrt{137}$. Since $\left(-\frac{9}{4}, \frac{27}{16}\right)$ has a component in the expanding right-eigendirection, $\left\| \left(A^{(\infty)}\right)^k c^{(\infty)} \right\| \to \infty$, and the thermodynamic limit does not exist. ∎

5. Decay of Correlation Functions.

By an extension of the methods used to prove the existence of the thermodynamic limit we can obtain information about the decay of correlation functions. We shall do this in the region $g < 0$ which is relevant for the continuum limit.

We need a few definitions:

Given a point (r, g) with $g < 0$ we define the trajectory $\mathcal{T}(r, g)$ as the set of points $R^n(r, g)$ with $n = \dots, -2, -1, 0, 1, 2, \dots$. The (truncated) correlation functions

$$\rho^T(\psi_{\alpha_1 x_1} \bar\psi_{\beta_1 y_1} \cdots \psi_{\alpha_n x_n} \bar\psi_{\beta_n y_n}) = \rho^T(\psi_{\alpha_1 x_1} \cdots \psi_{\alpha_n x_n} \bar\psi_{\beta_n y_n} \cdots \bar\psi_{\beta_1 y_1})$$

(5-1)

$$= \rho^T \left(\overrightarrow{\prod_{i=1}^{n}} \psi_{\alpha_i x_i} \overleftarrow{\prod_{j=1}^{n}} \bar\psi_{\beta_j y_j} \right)$$

are defined inductively as in (4.10):

(5-2)

$$\rho(\psi_X \bar\psi_Y) = \sum_{\{X_k\}_{k=1}^p} \sum_{\{Y_k\}_{k=1}^p \, : \, |Y_k| = |X_k|} (-1)^{\pi(\{X_k\}) + \pi(\{Y_k\})} \prod_{k=1}^{p} \rho^T \left(\psi_{X_k} \bar\psi_{Y_k} \right),$$

where $\{X_k\}$ and $\{Y_k\}$ are partitions of the sets

$X = \{(\alpha_i, x_i)\}_{i=1}^n$ and $Y = \{(\beta_j, y_j)\}_{j=1}^n$ respectively.

For two sequences of points $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ we define a quantity $\ell(\mathbf{x}, \mathbf{y})$ as follows.

Let $\mathcal{S}(\mathbf{x}, \mathbf{y})$ be the set of permutations $\sigma \in \mathcal{S}_n$ of $(1, \dots, n)$ such that every block B_s at any level s containing at least one point x_i or y_j but not all x_i and all y_j is connected to another block B'_s. Here two level-s blocks B_s and B'_s are called *connected* if there exists an $i = 1, \dots, n$ such that either $x_i \in B_s$ and $y_{\sigma(i)} \in B'_s$ or $x_i \in B'_s$ and $y_{\sigma(i)} \in B_s$.

Next we define

(5-3)
$$\ell(\mathbf{x}, \mathbf{y}) = \min_{\sigma \in \mathcal{S}(\mathbf{x}, \mathbf{y})} \sum_{i=1}^{n} s(x_i, y_{\sigma(i)})$$

if $\mathcal{S}(\mathbf{x}, \mathbf{y}) \neq \emptyset$ and $\ell(\mathbf{x}, \mathbf{y}) = \infty$ otherwise.

18

The main theorem in this section is:

Theorem 5.

Let (r, g) be a point in the lower half-plane $(g < 0)$ such that $R^n(r, g) \to (\pm\infty, g_\infty)$. For all $n \geq 1$ there exists a constant $C_n(r, g)$ depending only on n and the trajectory $T(r, g)$, such that, for any set of $2n$ points $x_1, \ldots, x_n, y_1, \ldots, y_n$ and indices $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \{1, 2\}$,

$$(5\text{-}4) \qquad |\rho^T(\psi_{\alpha_1 x_1} \bar{\psi}_{\beta_1 y_1} \cdots \psi_{\alpha_n x_n} \bar{\psi}_{\beta_n y_n})| \leq C_n(r, g) \, 2^{-\ell(x, y)}$$

with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

We shall prove this theorem by inductive application of the renormalization transformation. However, unlike the proof of Theorem 4, we do not have a nice iteration formula for ρ^T as we did for ρ in (4.2) We therefore replace the truncated expectation (5.1) by partially truncated expectations of the form

$$(5\text{-}5) \qquad \rho\left(\prod_{k=1}^{m} ; F_k \right) = \rho(F_1; F_2; \ldots; F_m)$$

where each F_k is a polynomial concentrated at a single point z_k, i.e. supp $(F_k) = \{z_k\}$, and $z_k \neq z_{k'}$ if $k \neq k'$. In fact we can subdivide the polynomials F_k into classes which are invariant under renormalization, as follows. For an arbitrary monomial $F = \psi_{\alpha_1 x_1} \cdots \psi_{\alpha_n x_n} \bar{\psi}_{\beta_m y_m} \cdots \bar{\psi}_{\beta_1 y_1}$, we introduce "charges" $q_\alpha(F)$ $(\alpha = 1, 2)$ by

$$(5\text{-}6) \qquad q_\alpha(F) = \sharp\{i \,|\, \alpha_i = \alpha\} - \sharp\{j \,|\, \beta_j = \alpha\}.$$

Next we observe

Lemma 5.1

If F is a monomial then $T(F)$ given by (4.11) is a charge-homogeneous polynomial, and $q_\alpha(T(F)) = q_\alpha(F)$.

Proof. Clearly, for any polynomial G in ζ and $\bar{\zeta}$, $< G >= 0$ unless $q_\alpha(G) = 0$ $(\alpha = 1, 2)$. It follows that, in formula (4.12),

$$q_\alpha(F) - q_\alpha(\psi_{X_0}) - \sum_{j=1}^{p} q_\alpha(\psi_{X_j}) = 0 \text{ if } T_{XY} \neq 0.$$

19

But $Y = \bigcup_{j=0}^{p} \dot{X}_j$ so that

$$q_\alpha(\psi_Y') = q_\alpha(\psi_{X_0}) + \sum_{k=1}^{p} q_\alpha(\psi_{X_k}) = q_\alpha(F).$$

∎

We now restrict our consideration to polynomials of the form $F = \prod_{k=1}^{m} F_k$ where the polynomials F_k are concentrated at distinct points z_k and each F_k is charge-homogeneous, i.e. it belongs to one of the following classes:

$q_1 = +1$	$q_2 = +1$	$F = f\,\psi_1\psi_2$	$d = 0$
$q_1 = -1$	$q_2 = -1$	$F = f\,\bar\psi_1\bar\psi_2$	$d = 0$
$q_1 = +1$	$q_2 = 0$	$F = f_1\,\psi_1 + f_2\,\psi_1\psi_2\bar\psi_2$	$d = 1$
$q_1 = 0$	$q_2 = +1$	$F = f_1\,\psi_2 + f_2\,\psi_1\bar\psi_1\psi_2$	$d = 1$
$q_1 = -1$	$q_2 = 0$	$F = f_1\,\bar\psi_1 + f_2\,\bar\psi_1\psi_2\bar\psi_2$	$d = 1$
$q_1 = 0$	$q_2 = -1$	$F = f_1\,\bar\psi_2 + f_2\,\psi_1\bar\psi_1\bar\psi_2$	$d = 1$
$q_1 = 0$	$q_2 = 0$	$F = f_0 + f_1\,\psi_1\bar\psi_1 + f_2\,\psi_2\bar\psi_2 + f_3\,\psi_1\bar\psi_1\psi_2\bar\psi_2$	$d = 0$
$q_1 = -1$	$q_2 = +1$	$F = f\,\bar\psi_1\psi_2$	$d = 0$
$q_1 = +1$	$q_2 = -1$	$F = f\,\psi_1\bar\psi_2$	$d = 0$

TABLE 5.1

The third column in this table shows the general form of a polynomial F_k belonging to this class. The f's are coefficients, ψ_1 stands for ψ_{1z_k} etc. The number d is defined in formula (5.12).

The collection of polynomials $F = \prod_{k=1}^{m} F_k$ described above we shall denote by $\mathcal{F}$. For $F \in \mathcal{F}$ we write $\rho(;F)$ for $\rho\left(\prod_{k=1}^{m}; F_k\right)$. It is defined analogous to $\langle F_1; \ldots; F_n\rangle$ in Section 4.

Instead of Theorem 5 we shall prove an analogous result for the expectations $\rho(;F)$, from which Theorem 5 then follows. In order to formulate this result we need to extend the definition of $\ell(\mathbf{x}, \mathbf{y})$ to polynomials of this form. This is possible because of the following fact.

20

Lemma 5.2

Suppose that, for two sequences of points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ the following holds:

$\dot{x}_n = \dot{y}_n$ and there exists $i \neq n$ such that $\dot{x}_i = \dot{x}_n$ or $\dot{y}_i = \dot{y}_n$ (or both). Then $\ell(\mathbf{x}, \mathbf{y}) = \ell(\mathbf{x}', \mathbf{y}')$, where $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ and $\mathbf{y}' = (y_1, \ldots, y_{n-1})$.

Proof. Suppose that $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$. Then we can define $\tilde{\sigma} \in \mathcal{S}(\mathbf{x}, \mathbf{y})$ by $\tilde{\sigma}(i) = \sigma(i)$ for $i = 1, \ldots, n-1$, and $\tilde{\sigma}(n) = n$. Obviously $\ell(\tilde{\sigma}) = \ell'(\sigma)$ if we put $\ell'(\sigma) = \sum_{i=1}^{n-1} s(x_i, y_{\sigma(i)})$ and $\ell(\tilde{\sigma}) = \sum_{i=1}^{n} s(x_i, y_{\tilde{\sigma}(i)})$.

Conversely, for $\tau \in \mathcal{S}(\mathbf{x}, \mathbf{y})$ we shall presently construct $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$ with $\ell'(\sigma) \leq \ell(\tau)$. That completes the proof of the lemma.

We may assume that, for some $i \neq n$, $\dot{x}_i = \dot{x}_n = \dot{y}_n$. (The case $\dot{y}_i = \dot{x}_n = \dot{y}_n$ is similar.) There are five separate cases to be considered:

(i) $\dot{x}_n = \dot{y}_{\tau(n)}$ or $\dot{x}_n = \dot{x}_{\tau^{-1}(n)}$ or $\dot{x}_i = \dot{y}_{\tau(i)}$.

(ii) $\dot{x}_{\tau^{-1}(n)} = \dot{y}_{\tau(n)} = \dot{y}_{\tau(i)} \neq \dot{x}_n$.

(iii) $\dot{x}_i \neq \dot{y}_{\tau(n)} = \dot{y}_{\tau(i)} \neq \dot{x}_{\tau^{-1}(n)} \neq \dot{x}_i$ or $\dot{x}_i \neq \dot{x}_{\tau^{-1}(n)} = \dot{y}_{\tau(i)} \neq \dot{y}_{\tau(n)} \neq \dot{x}_i$.

(iv) $\dot{x}_i \neq \dot{x}_{\tau^{-1}(n)} = \dot{y}_{\tau(n)} \neq \dot{y}_{\tau(i)} \neq \dot{x}_i$.

(v) $\dot{x}_i = \dot{x}_n = \dot{y}_n$ and $\dot{x}_{\tau^{-1}(n)}$ and $\dot{y}_{\tau(n)}$ and $\dot{y}_{\tau(i)}$ all different.

<u>Case (i).</u>

If $\dot{y}_{\tau(n)} = \dot{x}_n$ but $\tau(n) \neq n$ then we can put $\sigma(\tau^{-1}(n)) = \tau(n)$. It is clear that $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$ and $\ell'(\sigma) = \ell(\tau)$ since every block containing x_n, y_n or $y_{\tau(n)}$ contains all three.

Remark. Notice that the connectivity of $s = 0-$blocks, i.e. points, need not be checked because if there is a point not connected to another point (that is $y_{\sigma(j)} = x_j$)

21

then the level-1 block containing that point contains another point x_k connected outside the block; we can then simply modify σ as follows: $\sigma'(i) = \sigma(j)$ and $\sigma'(j) = \sigma(i)$.

Similarly, if $\dot{y}_{\tau(i)} = \dot{x}_n$ then we can put $\sigma(i) = \tau(n)$ and $\sigma(\tau^{-1}(n)) = \tau(i)$. If $\dot{x}_{\tau^{-1}(n)} = \dot{x}_n$ we can put $\sigma(\tau^{-1}(n)) = \tau(n)$.

Case (ii).

In this case we can simply define $\sigma(\tau^{-1}(n)) = \tau(n)$. Clearly $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$ because the two blocks are still connected via $i \mapsto \sigma(i) = \tau(i)$. Also $\ell'(\sigma) < \ell(\tau)$ because $s(x_n, y_n) + s(x_{\tau^{-1}(n)}, y_{\tau(n)}) = 0 < s(x_n, y_{\tau(n)}) + s(x_{\tau^{-1}(n)}, y_n)$. This means that τ is not a minimiser of $\ell(\tau)$ since $\ell(\tilde{\sigma}) = \ell'(\sigma) < \ell(\tau)$.

Case (iii)

In both these cases we define $\sigma(\tau^{-1}(n)) = \tau(n)$ as in case (ii). This connects the blocks $B(\dot{x}_{\tau^{-1}(n)})$ and $B(\dot{y}_{\tau(n)})$, while the block $B(\dot{x}_i)$ is still connected to $B(\dot{y}_{\sigma(i)}) = B(\dot{y}_{\tau(n)})$ or $B(\dot{y}_{\sigma(i)}) = B(\dot{x}_{\tau^{-1}(n)})$ respectively. These equalities ensure that the connectivity of higher-level blocks is not destroyed: the three different level-1 blocks cannot be disconnected by higher level blocks. Hence $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$. As in case (ii) we have $\ell'(\sigma) < \ell(\tau)$.

Case (iv).

Here we define $\sigma(i) = \tau(n)$ and $\sigma(\tau^{-1}(n)) = \tau(i)$. The blocks $B(\dot{x}_i)$ and $B(\dot{y}_{\tau(n)})$ are then connected as well as $B(\dot{y}_{\tau(i)})$ and $B(\dot{x}_{\tau^{-1}(n)}) = B(\dot{y}_{\tau(n)})$. The latter equality guarantees higher-level connectedness as in case (iii). Finally, $s(x_i, y_{\sigma(i)}) + s(x_{\tau^{-1}(n)}, y_{\sigma(\tau^{-1}(n))}) \leq s(x_i, y_{\tau(i)}) + s(x_n, y_{\tau(n)}) + s(x_{\tau^{-1}(n)}, y_n)$ because $s(x_i, y_{\sigma(i)}) = s(x_n, y_{\tau(n)})$ and $s(x_{\tau^{-1}(n)}, y_{\tau(n)}) \leq s(x_{\tau^{-1}(n)}, y_n) + s(x_n, y_{\tau(n)})$ by the triangle inequality.

Case (v).

This is the most complicated case. We must distinguish the following two possibilities:

(a) $s(x_i, y_{\tau(n)}) + s(x_{\tau^{-1}(n)}, y_{\tau(i)}) \leq s(x_i, y_{\tau(i)}) + s(x_{\tau^{-1}(n)}, y_{\tau(n)})$,

and

(b) $\qquad s(x_i, y_{\tau(n)}) + s(x_{\tau^{-1}(n)}, y_{\tau(i)}) > s(x_i, y_{\tau(i)}) + s(x_{\tau^{-1}(n)}, y_{\tau(n)}).$

In the first case we put $\sigma(i) = \tau(i)$ and $\sigma(\tau^{-1}(n)) = \tau(n)$; in the second case we put $\sigma(i) = \tau(n)$ and $\sigma(\tau^{-1}(n)) = \tau(i)$. Both instances are similar. We consider only the first.

Let B_s be a level-s block containing x_i. If it does not contain $y_{\tau(i)}$ then B_s is connected to $B_s\left(y_{\sigma(i)}^{(s)}\right) = B_s\left(y_{\tau(i)}^{(s)}\right)$. If $y_{\tau(i)} \in B_s$ then either $x_{\tau^{-1}(n)} \in B_s$ or $y_{\tau(n)} \in B_s$. Indeed, suppose $x_{\tau^{-1}(n)} \notin B_s$ and $y_{\tau(n)} \notin B_s$, and let $B_{s'}$ be the highest-level block containing x_i and $y_{\tau(i)}$ but not $x_{\tau^{-1}(n)}$ and $y_{\tau(n)}$. Then

$$
\begin{aligned}
s(x_i, y_{\tau(i)}) + s(x_{\tau^{-1}(n)}, y_{\tau(n)}) &\le s' - 1 + s(x_{\tau^{-1}(n)}, y_{\tau(n)}) \\
&< s' + \max\{s(x_{\tau^{-1}(n)}, y_{\tau(i)}), s(x_i, y_{\tau(n)})\} \\
&\le s(x_{\tau^{-1}(n)}, y_{\tau(i)}) + s(x_i, y_{\tau(n)})
\end{aligned}
$$

contradicting hypothesis (a).

A similar reasoning applies to blocks B_s containing $y_{\tau(i)}, x_{\tau^{-1}(n)}$ and $y_{\tau(n)}$ respectively. We conclude that $\sigma \in \mathcal{S}(\mathbf{x}', \mathbf{y}')$. Moreover, $\ell'(\sigma) < \ell(\tau)$ because

$$s(x_{\tau^{-1}(n)}, y_{\tau(n)}) < s(x_{\tau^{-1}(n)}, y_n) + s(x_n, y_{\tau(n)}). \qquad \blacksquare$$

From this lemma it follows that, for a polynomial $F = \prod_{k=1}^{m} F_k$ in the class $\mathcal{F}$, we can define

$$(5\text{-}7) \qquad \ell\left(\prod_{k=1}^{m} F_k\right) = \ell\left(\bigcup_{k=1}^{m} X_k\right),$$

where ψ_{X_k} is one of the monomials of F_k. ($X_k \subset \{(1, z_k), (\bar{1}, z_k), (2, z_k), (\bar{2}, z_k)\}$) We can now state the result for $\rho(; F)$ from which Theorem 5 follows:

Lemma 5.3

If $F = \prod_{k=1}^{m} F_k$ is a polynomial in the class $\mathcal{F}$ then there exists a constant $\tilde{C}_m(r, g)$ depending only on m and the trajectory $\mathcal{T}(r, g)$ such that

$$(5\text{-}8) \qquad |\rho(; F)| \le \tilde{C}_m(r, g) \prod_{k=1}^{m} \|F_k\| \, 2^{-\ell(F)}.$$

Proof of Theorem 5 given Lemma 5.3 We use the following relation:

Lemma 5.4

Let $F_k = \psi_{X_k}\bar{\psi}_{Y_k}$. Then

$$(5\text{-}9) \qquad \rho(\,;F) = \sum_{\{X'_l\}_{l=1}^p}^{*} \sum_{\{Y'_l\}_{l=1}^p}^{*} (-1)^{\pi(\{X'_l\})+\pi(\{Y'_l\})} \prod_{l=1}^{p} \rho^T\left(\psi_{X'_l}\bar{\psi}_{Y'_l}\right),$$

where the sum is taken over all partitions $\{X'_l\}_{l=1}^p$ and $\{Y'_l\}_{l=1}^p$ of $\bigcup_{k=1}^m X_k$ and $\bigcup_{k=1}^m Y_k$ respectively, such that $|X'_l| = |Y'_l|$ and for each pair (k,k') with $k, k' \in \{1,2,\ldots,m\}$ there exists a sequence of "links" (k_i, k_{i+1}) $(i = 1,\ldots,r-1)$ such that $k_1 = k, k_r = k'$ and for each $i = 1,\ldots,r-1$ there exists $l_i \in \{1,\ldots,p\}$ such that $X_{k_i} \cap X'_{l_i} \neq \emptyset$ and $Y_{k_{i+1}} \cap Y'_{l_i} \neq \emptyset$ or $X_{k_{i+1}} \cap X'_{l_i} \neq \emptyset$ and $Y_{k_i} \cap Y'_{l_i} \neq \emptyset$.

Now assume that Lemma 5.3 has been proven. We can then make the following rough estimate by moving all the terms in (5.9) with $p > 1$ to the left-hand side of the equation:

(5-10)
$$|\rho^T(\psi_{\alpha_1 x_1}\cdots\psi_{\alpha_n x_n}\bar{\psi}_{\beta_n y_n}\cdots\bar{\psi}_{\beta_1 y_1})| \leq \tilde{C}_m(r,g)2^{-\ell(x,y)} +$$

$$+ \sum_{p=2}^{n} \sum_{\substack{n_1,\ldots,n_p \geq 1 \\ \sum n_l = n}} \sum_{\substack{\{X'_l\}_{l=1}^p \\ |X'_l| = n_l}}^{*} \sum_{\substack{\{Y'_l\}_{l=1}^p \\ |Y'_l| = n_l}}^{*} \prod_{l=1}^{p} C_{n_l}(r,g)\, 2^{-\ell(x'_l,y'_l)}$$

Next we use the following result:

Lemma 5.5

If $\{x'_l\}_{l=1}^p$ and $\{y'_l\}_{l=1}^p$ are partitions of x and y satisfying the link property of Lemma 5.4 then

$$\ell(x,y) \leq \sum_{l=1}^{p} \ell(x'_l, y'_l).$$

Proof. We show that, if $\sigma_l \in \mathcal{S}(x'_l, y'_l)$ then $\prod_{l=1}^p \sigma_l \in \mathcal{S}(x,y)$ where $\left(\prod_{l=1}^p \sigma_l\right)(\alpha,x) = \sigma_l(\alpha,x)$ if $(\alpha,x) \in x'_l$. If a block B_s contains points of $x \cup y$ but not all, then there must exist k, k' such that $z_k \in B_s$, $z_{k'} \notin B_s$. Suppose

24

$x_k \cap x'_l \neq \emptyset$ and $y_{k'} \cap y'_l \neq \emptyset$. Then B_s contains a point (namely z_k) of x'_l but not all points of y'_l. Therefore B_s is connected via σ_l to another block B'_s. ■

Inserting this into the estimate for ρ^T gives:

$$|\rho^T(\psi_{\alpha_1 x_1} \cdots \psi_{\alpha_n x_n} \bar{\psi}_{\beta_n y_n} \cdots \bar{\psi}_{\beta_1 y_1})| \leq C_n(r,g) 2^{-\ell(r,g)}$$

with

(5-11) $$C_n(r,g) = \tilde{C}_m(r,g) + \sum_{p=2}^{n} \sum_{\substack{n_1,\ldots,n_p \geq 1 \\ \sum n_l = n}} \left(\frac{n!}{n_1! \cdots n_p!}\right)^2 \prod_{l=1}^{p} C_{n_l}(r,g)$$

depending only on n and $\mathcal{T}(r,g)$. This provides the induction step in the proof of the theorem, proving the estimate for n given Lemma 5.3 and the same estimate for smaller n. ■

Given $F = \prod_{k=1}^{m} F_k \in \mathcal{F}$ with supp $(F) = \{z_k\}$, we define

(5-12) $$d_k(F) = q_1(F) + q_2(F) \bmod 2.$$

Essential in the proof of Lemma 5.3 is the following iterative evaluation of $\ell(\mathbf{x}, \mathbf{y})$:

Lemma 5.6

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$ and assume that $m' \geq 2$ where $m' = \sharp\{z | \exists i : \dot{x}_i = z \text{ or } \dot{y}_i = z\}$. Assume also that $|q_\alpha(B)| \leq 1$ for every level-1 block B. Then

(5-13) $$\ell(\mathbf{x}, \mathbf{y}) = \ell(\dot{\mathbf{x}}, \dot{\mathbf{y}}) + m' - \frac{1}{2} \sum_{k=1}^{m'} d'_k.$$

Here $\dot{\mathbf{x}} = (\dot{x}_i)_{i=1}^{n}$, $\dot{\mathbf{y}} = (y_i)_{i=1}^{n}$, and

$$d'_k = d_k(\psi_{\dot{\mathbf{x}}} \bar{\psi}_{\dot{\mathbf{y}}}) = \sharp\{i | \dot{x}_i = z_k\} - \sharp\{i | \dot{y}_i = z_k\} \,(\bmod 2).$$

Proof. It is clear that if $\sigma \in \mathcal{S}(\mathbf{x}, \mathbf{y})$ then $\tilde{\sigma} \in \mathcal{S}(\dot{\mathbf{x}}, \dot{\mathbf{y}})$, where $\tilde{\sigma}(\dot{x}_i) = \sigma(x_i)\dot{}$. On the other hand, if $\tilde{\tau} \in \mathcal{S}(\dot{\mathbf{x}}, \dot{\mathbf{y}})$ then we can modify τ to $\sigma \in \mathcal{S}(\mathbf{x}, \mathbf{y})$ so that

25

$\ell(\sigma) = \ell(\tau)$ and $\ell(\tilde{\sigma}) = \ell(\tilde{\tau})$. (See the remark on page 21)

Next we use Lemma 5.2 to conclude that we may assume that a block $B(z_k)$ does not contain an x_i and a y_j. Since $|q_\alpha(B(z_k))| \leq 1$ for each k we have the following possibilities for each block:

$B(z_k)$ contains 2 points $x_i, x_j \in \mathbf{x}$

$B(z_k)$ contains 1 point $x_i \in \mathbf{x}$

$B(z_k)$ contains 1 point $y_i \in \mathbf{y}$

$B(z_k)$ contains 2 points $y_i, y_j \in \mathbf{y}$.

For blocks containing 2 points $q(B) = q_1(B) + q_2(B)$ is even so that $d'_k = 0$; for blocks containing 1 point, $d'_k = 1$. If $x_i \in B(z_k)$ then $s(x_i, y_{\sigma(i)}) = s(\dot{x}_i, \dot{y}_{\sigma(i)}) + 1$. Hence $\ell(\mathbf{x}, \mathbf{y}) - \ell(\dot{\mathbf{x}}, \dot{\mathbf{y}})$ equals n which is $\frac{1}{2}$ the number of blocks $+\frac{1}{2}$ the number of blocks containing two points = the number of blocks $-\frac{1}{2}$ the number of blocks containing 1 point = $m' - \frac{1}{2} \sum_{k=1}^{m'} d'_k$. ∎

Lemma 5.3 now follows from the following lemma:

Lemma 5.7

Let (r, g) be a point in the lower half-plane such that $R^n(r, g) \to (\pm\infty, g_\infty)$. Let $F = \prod_{k=1}^{m} F_k$ be a polynomial in the class $\mathcal{F}$ and suppose that $m' = m$. Then the coefficients of $F'_k = \tilde{T}(F_k)$ are obtained from those of F_k by multiplying with a factor $2^{-1+d_k/2}$ and the transpose M^t of a matrix $M(r, g)$ satisfying the following property:

If $M^{(p)} = M(R^p(r, g))$ then $\| \prod_{p=0}^{q} M^{(p)} \|$ is bounded on the trajectory $\mathcal{T}(r, g)$ by a constant $K(r, g)$ independent of q. Here $\tilde{T} = 2^{1 - d'_k/2} (1 - P_0) T (1 - P_0)$ where P_0 is the projection on the constants in the class $(0,0)$. If $\{e\}_{\nu=1}^{15}$ is the set of basic monomials of Table 5.1, $\psi_1 \psi_2$, $\bar{\psi}_1 \bar{\psi}_2$, ψ_1, $\psi_1 \psi_2 \bar{\psi}_2$, ψ_2, $\psi_1 \bar{\psi}_1 \psi_2$, $\bar{\psi}_1$, $\bar{\psi}_1 \psi_2 \bar{\psi}_2$, $\bar{\psi}_2$, $\psi_1 \bar{\psi}_1 \bar{\psi}_2$,

$\psi_1\bar{\psi}_1,\ \psi_2\bar{\psi}_2,\ \psi_1\bar{\psi}_1\psi_2\bar{\psi}_2,\ \bar{\psi}_1\psi_2,\ \psi_1\bar{\psi}_2$, then we can write

$$(5\text{-}14) \qquad F_k = \sum_{\nu=1}^{15} f_{k\nu}\, e_\nu \text{ and } F'_k = \sum_{\nu=1}^{15} f'_{k\nu}\, e_\nu,$$

with

$$(5\text{-}15) \qquad f'_{k\mu} = \sum_{\nu=1}^{15} M_{\nu\mu} f_{k\nu}.$$

M is the direct sum of the matrices $M_{+1,+1}, M_{-1,-1}, M_{+1,0}, M_{0,+1}, M_{-1,0}, M_{0,-1},$ $\tilde{M}_{0,0}, M_{-1,+1}, M_{+1,-1}$.

Proof of Lemma 5.3 given Lemma 5.7. Denote the points of support of the successive $F^{(p)} = T^p F$ by $z_k^{(p)}$, that is $F^{(p)} = \prod_{k=1}^{m_p} F_k^{(p)}$ with supp $\left(F_k^{(p)}\right) = \left\{z_k^{(p)}\right\}$. We are going to proceed by induction on the number r of p's such that $m_p > m_{p+1} \geq 1$. This number is obviously less than m. Clearly, if $r = 0$ then $\ell(F) = 0$ and $m = 1$ and

$$(5\text{-}16) \qquad |\rho(;F)| = |\rho(F)| \leq |f_0| + (|f_1| + |f_2|)A_2 + |f_3|A_4,$$

where $F = f_0 + f_1\psi_1\bar{\psi}_1 + f_2\psi_2\bar{\psi}_2 + f_3\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2$ and $A_2(r,g)$ and $A_4(r,g)$ are constants bounding $\rho(\psi_1\bar{\psi}_1) = \rho(\psi_2\bar{\psi}_2)$ and $\rho(\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2)$ respectively on the trajectory $\mathcal{T}(r,g)$.

Now consider the induction step. If $m' = m > 1$ then

$$\rho(;F) = 2^{-m+\sum_{k=1}^m d_k/2}\, \rho'\left(\prod_{k=1}^m ; F_k^{(1)}\right),$$

where

$$F_k^{(1)} = \tilde{T}F_k = \sum_{\nu=1}^{15} f_{k\mu}^{(1)}\, e_\mu$$

with

$$f_{k\mu}^{(1)} = \sum_{\nu=1}^{15} f_{k\nu} M_{\nu\mu} = \left(M^t f_k\right)_\mu.$$

More generally, if $m_p = m$,

$$(5\text{-}17) \qquad \rho(;F) = 2^{-pm+p\sum_{k=1}^m d_k/2}\, \rho^{(p)}\left(\prod_{k=1}^m ; F_k^{(p)}\right)$$

27

and

$$(5\text{-}18) \qquad F_k^{(p)} = \tilde{T}^p F_k = \sum_{\nu=1}^{15} f_{k\nu}^{(p)} \, e_\nu$$

with

$$(5\text{-}19) \qquad f_{k\nu}^{(p)} = \left[\left(\prod_{i=0}^{p-1} M^{(i)} \right)^t f_k \right]_\nu .$$

Notice that

$$(5\text{-}20) \qquad pm + \frac{1}{2} p \sum_{k=1}^{m} d_k = \ell(F) - \ell(F^{(p)})$$

by Lemma 5.6. Indeed, we can disregard the monomials in F containing $\psi_{\alpha x}$ and $\psi_{\alpha y}$ or $\bar\psi_{\alpha x}$ and $\bar\psi_{\alpha y}$ with $\dot x = \dot y$, and choose in F' the monomials $\psi_{\dot X} \bar\psi_{\dot Y}$. Since $\| \prod_{i=0}^{p-1} M^{(i)} \| \le K(r,g)$ we find that

$$(5\text{-}21) \qquad \rho(; F) = 2^{-\ell(F)+\ell(F^{(p)})} \rho^{(p)} \left(; F^{(p)} \right)$$

where

$$(5\text{-}22) \qquad \prod_{k=1}^{m} \|F_k^{(p)}\| \le K(r,g)^m \prod_{k=1}^{m} \|F_k\|.$$

Now assume that $m_{p+1} < m_p$. We shall prove that

$$(5\text{-}23) \qquad |\rho^{(p)} \left(; F^{(p)} \right)| \le 2^{-\ell(F^{(p)})} \tilde C_m(r,g) \prod_{k=1}^{m} \|F_k^{(p)}\|,$$

where it is now *given* that

$$(5\text{-}24) \qquad |\rho^{(p+1)} (; G)| \le \tilde C_{m_{p+1}}(r,g) \prod_{k=1}^{m_{p+1}} \|G_k\| \, 2^{-\ell(G)}$$

for all $G = \prod_{k=1}^{m(G)} G_k \in \mathcal{F}$ with $m(G) < m_p$.

Without loss of generality we can put $p = 0$ here. From (5.24) it follows that

$$\left| \rho \left(\prod_{k=1}^{m'}; \prod_{l: \dot z_l = z_k'} F_l \right) \right| \le \tilde K(r,g)^{m'} \tilde C_{m'}(r,g) \prod_{l=1}^{m} \|F_l\| \, 2^{-\ell(F')}$$

28

for all $F = \prod_{k=1}^{m'} \prod_{l:\dot{z}_l = z_k'} F_l \in \mathcal{F}$ and $m' < m$. Let us first remark that it is sufficient to prove (5.23) for a monomial F, since then

$$
\begin{aligned}
|\rho(; F)| &= \left| \sum_{\{\nu_k\}} \prod_{k=1}^{m} f_{k\nu_k} \, \rho \left(\prod_{k=1}^{m}; \, \mathbf{e}_{\nu_k}(z_k) \right) \right| \\
&\leq \sum_{\{\nu_k\}} \prod_{k=1}^{m} |f_{k\nu_k}| \, \tilde{C}_m(r,g) \, 2^{-\ell(F)} \\
&= \tilde{C}_m(r,g) \prod_{k=1}^{m} \|F_k\| \, 2^{-\ell(F)}.
\end{aligned}
$$

(5-25)

For $F = \prod_{l=1}^{m} \mathbf{e}_{\nu_l}(z_l)$ we proceed as on page 24 using the following analogue of Lemma 5.4:

$$
\rho \left(\prod_{k=1}^{m'}; \, \prod_{l:\dot{z}_l = z_k'} \mathbf{e}_{\nu_l}(z_l) \right) = \prod_{\{\Lambda_i\}}^{*} (-1)^{\pi(\{\Lambda_i\})} \prod_i \rho \left(\prod_{l \in \Lambda_i}; \mathbf{e}_{\nu_l}(z_l) \right)
$$

(5-26)

where the sum is over all partitions $\{\Lambda_i\}$ of $\{1, \ldots, m\}$ such that every pair (k, k') is connected by a chain of "links" $(k_1, k_2), \ldots (k_r, k_{r+1})$. A link (k, k') satisfies:

$$
\exists i : \left(\exists l \in \Lambda_i : \dot{z}_l = z_k' \text{ and } \exists l' \in \Lambda_i : \dot{z}_{l'} = z_{k'}' \right).
$$

$\pi(\{\Lambda_i\})$ is the number of odd transpositions needed to reorder $\{1, \ldots, m\}$ according to $\{\Lambda_i\}$. (assuming that the $\mathbf{e}_{\nu_l}$ in the left-hand side of (5.26) are in increasing order.) The required result now follows by moving all terms in the sum with $\Lambda_i \neq \{1, \ldots, m\}$ to the left and using induction on n and Lemma 5.5. Notice that the number of terms in the sum is bounded by m^{m+1} and that $m \leq n \leq 4m$. ∎

It remains to prove Lemma 5.7.

Proof of Lemma 5.7 The classes of Table 5.1 transform according to the matrices M_{q_1, q_2} given by

$$
M_{-1,-1} = M_{+1,+1} = 1 \text{ (i.e. } T(\psi_1 \psi_2) = 2^{-1} \psi_1' \psi_2'),
$$

$$
M_{-1,0} = M_{0,-1} = M_{+1,0} = M_{0,+1} = \begin{pmatrix} 1 & 0 \\ c_1 & 2g(c_1^2 - c_2) \end{pmatrix},
$$

$$
M_{0,0} = 2A^{(0)},
$$

29

where $A^{(0)}$ is the matrix defined in (4.18), and

$$M_{-1,+1} = M_{+1,-1} = \frac{1}{2}(1 - 4gc_2).$$

Now, as $(r,g) \to (0,0)$, $M_{-1,0} \to \begin{pmatrix} 1 & 0 \\ 1/4 & 0 \end{pmatrix}$ and $M_{0,0} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$ and

$M_{-1,+1} \to \frac{1}{2}$, all exponentially fast. Furthermore,

$$\begin{pmatrix} 1 & 0 \\ 1/4 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 1/4 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}(1-2^{-n}) & \frac{1}{2}(1-2^{-n}) & \frac{1}{2} \end{pmatrix}$$

are both bounded. The bound on the norm of $\prod_{p=0}^{q} M^{(p)}$ then follows from

Lemma 5.8

Let $M^{(n)}$ be a sequence of matrices satisfying $M^{(n)} = M_\infty + \gamma^{-n} E^{(n)}$, where $\|M_\infty^n\| \le A_0$, $\|E^{(n)}\| \le c$, and $\gamma > 1$. Then

$$\|\prod_{k=n}^{n+p} M^{(k)}\| \le A_0 \frac{e^{-r(\gamma)}}{1 - c\gamma^{-n}},$$

where $r(\gamma) = \sum_{k=1}^{\infty} \ln(1 - \gamma^{-n}) < \infty$.

Proof.

$$\|\prod_{k=n}^{n+p} M^{(k)}\| =$$

$$= \|(M_\infty + \gamma^{-n} E^{(n)})(M_\infty + \gamma^{-n-1} E^{(n+1)}) \cdots \|$$

$$\le \|M_\infty^{p+1}\| + \gamma^{-n}\|M_\infty^p\| \, \|E^{(n)} + \gamma^{-1} E^{(n+1)} + \cdots + \gamma^{-p} E^{(n+p)}\|$$

$$+ \gamma^{-2n}\|M_\infty^{p-1}\|\{\gamma^{-1}\|E^{(n)}\| \, \|E^{(n+1)}\| + \gamma^{-2}\|E^{(n)}\| \, \|E^{(n+2)}\| + \cdots$$

$$+ \gamma^{-3}\|E^{(n+1)}\| \, \|E^{(n+2)}\| + \cdots \} + \cdots$$

$$\le A_0 \left\{ 1 + c\gamma^{-n} \frac{1}{1 - \gamma^{-1}} + (c\gamma^{-n})^2 \frac{\gamma^{-1}}{(1 - \gamma^{-1})(1 - \gamma^{-2})} + \cdots \right\}$$

$$\le A_0 e^{-r(\gamma)} \left\{ 1 + c\gamma^{-n} + (c\gamma^{-n})^2 + \cdots \right\}$$

$$= A_0 \frac{e^{-r(\gamma)}}{1 - c\gamma^{-n}}.$$

As $(r,g) \to (\pm\infty, g_\infty), M_{-1,0} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_{0,0} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$ and

$M_{-1,+1} \to \frac{1}{2}$. This proves Lemma 5.7 for all trajectories *except* the trajectory going off to $\left(-\frac{2}{3}, -\infty\right)$. For the latter we remark that as $(r,g) \to \left(-\frac{2}{3}, -\infty\right)$,

$M_{-1,0} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$, $M_{0,0} \to \begin{pmatrix} 2 & 0 & -8/3 \\ 0 & 2 & -8/3 \\ 0 & 0 & 2 \end{pmatrix}$ and $M_{-1,+1} \to 1$, so that the

bound on the product of matrices $M^{(p)}$ does not hold. ∎

If we extend the definition of $s(x,y)$ to n–tuples by

(5-27) $$s(\mathbf{x}, \mathbf{y}) = \max_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} s(x_i, y_{\sigma(i)})$$

then we have the following

Corollary

Let (r,g) be a point in the lower half-plane $(g < 0)$ such that $R^n(r,g) \to (-\infty, g_\infty)$. For all $n \geq 1$ there exists a constant $D_n(r,g)$ depending only on n and the trajectory $T(r,g)$ such that

(5-28) $$|\rho(\psi_{\alpha_1 x_1} \bar{\psi}_{\beta_1 y_1} \cdots \psi_{\alpha_n x_n} \bar{\psi}_{\beta_n y_n})| \leq D_n(r,g)\, 2^{-s(x_1, \ldots x_n; y_1, \ldots y_n)}$$

for any set of $2n$ points $x_1, \ldots, x_n, y_1, \ldots y_n$ and indices $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \{1,2\}$.

Proof. Use the formula (5.2) together with the relation

(5-29) $$s(\mathbf{x}, \mathbf{y}) \leq \sum_{k=1}^{m} \ell(\mathbf{x}_k, \mathbf{y}_k).$$

∎

Remark This bound on the n–point functions is also correct in the case of the "critical" trajectories $T(r_s(g), g)$.

31

6. The Continuum Limit.

Using the information about the decay of correlation functions obtained in Section 5 we shall prove the existence of the continuum limit for theories corresponding to points (r, g) on the trajectories receding from the "trivial" fixed point $(0,0)$. This is particularly easy for the unstable line $g = 0, r \geq 0$: these theories are "Gaussian" so that we need only consider the 2-point functions. The theories with $g < 0$ are non-trivial and we have to consider the behaviour of the general $n-$point function. However, it turns out that the result of the Corollary of Theorem 5 is strong enough to ensure the existence of the continuum limit, so that the latter also exists for the theories with $R^n(r, g) \rightarrow \left(-\frac{2}{3}, -\infty\right)$. The existence of an unstable line for the fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$ suggests that there also exist continuum limits associated with this fixed point. In fact this appears not to be the case in the proper sense to be defined below. This pathology can be seen to be connected to the fact that the thermodynamic limit does not exist for this fixed point.

The usual way to proceed in constructing a continuum limit is the following. We choose a sequence (r_n, g_n) converging to a point on either of the two critical lines such that $\mathcal{R}^n v_{m+n}$ converges as $n \rightarrow \infty$ for all m (large enough). Here $v_n = v(r_n, g_n)$ is the potential with coupling constants (r_n, g_n) defined by (2.7) and $\mathcal{R}v_n = v\left(\mathcal{R}(r_n, g_n)\right)$ is the transformed potential. Thus we obtain states ρ_m satisfying

$$(6\text{-}1) \qquad\qquad \mathcal{R}\rho_m = \rho_{m-1}.$$

We then define states $\tilde{\rho}_m$ "living" on the rescaled lattices $2^{-m}\mathbb{Z}^2$ by

$$(6\text{-}2) \qquad\qquad \tilde{\rho}_m(F) = \rho_m\left(F\left(2^{m/2}\psi_{2^m}, 2^{m/2}\bar{\psi}_{2^m}\right)\right),$$

where $F(\psi_m, \bar{\psi}_m)$ is a polynomial in the fields ψ_m living on $2^{-m}\mathbb{Z}^2$. By (6.1) we have

$$(6\text{-}3) \quad \tilde{\rho}_{m-1}(F) = \tilde{\rho}_m\left(F\left(\frac{1}{4}\sum_{x\in B(0)}\psi_m(\cdot + 2^{-m}x), \frac{1}{4}\sum_{x\in B(0)}\bar{\psi}_m(\cdot + 2^{-m}x)\right)\right).$$

We want to consider the fields ψ_m as the means of a putative continuum field ψ over blocks $\square_m$:

$$(6\text{-}4) \qquad\qquad \psi_m(\underline{x}) = 2^{2m}\psi\left(1_{\square_m(\underline{x})}\right) \quad (\underline{x} \in 2^{-m}\mathbb{Z}^2),$$

32

where 1_A is the indicator function of the set $A \subset \mathbb{R}^2$ and

(6-5)
$$\square_m(\underline{x}) = \{\underline{v} \in \mathbb{R}^2 \mid x_i - 2^{-m-1} \le \underline{v}_i \le \underline{x}_i + 2^{-m-1}\}.$$

Approximating the smooth functions f_i and g_i $(i = 1, \ldots, k)$ on $\mathbb{R}^2$ successively by

(6-6)
$$f_i^{(m)} = \sum_{\underline{x} \in 2^{-m}\mathbb{Z}^2} f_i(\underline{x}) \, 1_{\square_m(\underline{x})}$$

and $g_i^{(m)}$ defined analogously, we accordingly define the k-point function of the continuum state $\tilde{\rho}$ on $\mathcal{G}(E)$ with $E = S(\mathbb{R}^2, \mathbb{R})$ by

(6-7)
$$\tilde{\rho}\left(\psi_{\alpha_1}(f_1)\bar{\psi}_{\beta_1}(g_1)\cdots\psi_{\alpha_k}(f_k)\bar{\psi}_{\beta_k}(g_k)\right) =$$
$$= \lim_{m\to\infty} \int d_m\underline{x}_1 \cdots \int d_m\underline{x}_k \int d_m\underline{u}_1 \cdots d_m\underline{u}_k f_1(\underline{x}_1)g_1(\underline{u}_1)\cdots f_k(\underline{x}_k)g_k(\underline{u}_k)\cdot$$
$$2^{mk}\rho_m\left(\psi_{\alpha_1,2^m\underline{x}_1}\bar{\psi}_{\beta_1,2^m\underline{u}_1}\cdots\psi_{\alpha_1,2^m\underline{x}_k}\bar{\psi}_{\beta_k,2^m\underline{u}_k}\right).$$

Here $\int d_m\underline{x} \equiv 2^{-2m}\sum_{\underline{x}\in 2^{-m}\mathbb{Z}^2}$ and $f_1, \ldots, f_k, g_1, \ldots, g_k \in S(\mathbb{R}^2; \mathbb{R})$. One easily checks with the help of (6.3) that the limit (6.7) would be trivial if $f_i = f_i^{(m)}$ and $g_i = g_i^{(m)}$ for some m and all i. We remark that the above procedure is only possible in the infinite volume limit. Hence we must take $(r_n, g_n) \in \mathcal{D}$ for all n. As the transformation R does not depend on N it is unchanged in the infinite volume limit. From Lemma 7.1 below we can deduce that, if we keep $g_n = g_0 \ge 0$ fixed and let $r_\downarrow r_c(g_0)$ then we end up with a Gaussian theory, i.e. $\rho_m = \mathcal{R}^{-m}\rho_0$ where ρ_0 is given by a point $(r_0, 0)$ with $r_0 > 0$. (If we take $r_n = r_c(g_0)$ then $r_0 = 0$.) This means that we might as well start from $\rho_m = \mathcal{R}^{-m}\rho_0$. In the same way, taking $r_n = r_0$ fixed and letting $g_n \uparrow g_c(r_0)$ from below we obtain points $(\tilde{r}_m, \tilde{g}_m)$ on the unstable line of the fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$ tending to this fixed point as $m \to \infty$. Finally, we can take sequences $(r_n, g_n) = R^{-n}(r_0, g_0)$ converging to $(0,0)$ from the lower half-plane, i.e. with $g_0 < 0$. In that case $(\tilde{r}_m, \tilde{g}_m) = R^{-m}(r_0, g_0)$. It remains to verify that the limit (6.7) exists in these three cases. In fact this is only in the cases where $(\tilde{r}_m, \tilde{g}_m) \to (0,0)$:

Theorem 6.

Let ρ_m be the state on $\mathcal{G}(E_\infty)$ with $E_\infty = \left(\mathbf{R}^2\right)^{\mathbf{Z}^2}$ defined by Theorem 4 with $(r_m, g_m) = R^{-m}(r_0, g_0)$ where $g_0 < 0$ or $g_0 = 0$ and $r_0 \geq 0$. Then the continuum limit (6.7) exists for all k and $f_1, \ldots, f_k, g_1, \ldots, g_k \in S\left(\mathbf{R}^2; \mathbf{R}\right)$.

Proof. We estimate the m–th and the $m+n$–th element in the sequence:

(6-8)

$$
\int d_{m+n}\underline{x}_1 \cdots \int d_{m+n}\underline{x}_k \int d_{m+n}\underline{u}_1 \cdots \int d_{m+n}\underline{u}_k \, f_1(\underline{x}_1)g_1(\underline{u}_1) \cdots f_1(\underline{x}_k)g_k(\underline{u}_k)
$$

$$
2^{(m+n)k} \rho_{m+n} \left(\psi_{\alpha_1, 2^{m+n}\underline{x}_1} \bar{\psi}_{\beta_1, 2^{m+n}\underline{u}_1} \cdots \psi_{\alpha_k, 2^{m+n}\underline{x}_k} \bar{\psi}_{\beta_k, 2^{m+n}\underline{u}_k} \right)
$$

$$
- \int d_m\underline{x}_1 \cdots \int d_m\underline{x}_k \int d_m\underline{u}_1 \cdots \int d_m\underline{u}_k \, f_1(\underline{x}_1)g_1(\underline{u}_1) \cdots f_k(\underline{x}_k)g_k(\underline{u}_k)
$$

$$
2^{mk} \rho_m \left(\psi_{\alpha_1, 2^m\underline{x}_1} \bar{\psi}_{\beta_1, 2^m\underline{u}_1} \cdots \psi_{\alpha_k, 2^m\underline{x}_k} \bar{\psi}_{\beta_k, 2^m\underline{u}_k} \right)
$$

$$
= \int d_{m+n}\underline{x}_1 \cdots \int d_{m+n}\underline{x}_k \int d_{m+n}\underline{u}_1 \cdots \int d_{m+n}\underline{u}_k
$$

$$
\left\{ f_1(\underline{x}_1)g_1(\underline{u}_1) \cdots f_k(\underline{x}_k)g_k(\underline{u}_k) - f_1^{(m)}(\underline{x}_1)g_1^{(m)}(\underline{u}_1) \cdots f_k^{(m)}(\underline{x}_k)g_k^{(m)}(\underline{u}_k) \right\}
$$

$$
2^{(m+n)k} \rho_{m+n} \left(\psi_{\alpha_1, 2^{m+n}\underline{x}_1} \bar{\psi}_{\beta_1, 2^{m+n}\underline{u}_1} \cdots \psi_{\alpha_k, 2^{m+n}\underline{x}_k} \bar{\psi}_{\beta_k, 2^{m+n}\underline{u}_k} \right).
$$

Now, given $\epsilon > 0$, we can choose m so large that

$$
|f_i(\underline{x}) - f_i^{(m)}(\underline{x})| < \epsilon(1 + |\underline{x}|^2)^{-2} \quad \forall \underline{x} \in \mathbf{R}^2, i = 1, \ldots, k
$$

and the same for $g_i \quad (i = 1, \ldots, k)$. Assume also that we have the bound

$$
|f_i(\underline{x})| \leq M(1 + |\underline{x}|^2)^{-2} \quad \forall \underline{x} \in \mathbf{R}^2, i = 1, \ldots, k,
$$

and the same for $g_i \quad (i = 1, \ldots, k)$. Then , writing n instead of $m + n$ in the

34

expression (6.8) we find

(6-9)

$$\left| \int d_n \underline{x}_1 \cdots \int d_n \underline{x}_k \int d_n \underline{u}_1 \cdots \int d_n \underline{u}_k \right.$$

$$\left\{ f_1(\underline{x}_1) g_1(\underline{u}_1) \cdots f_k(\underline{x}_k) g_k(\underline{u}_k) - f_1^{(m)}(\underline{x}_1) g_1^{(m)}(\underline{u}_1) \cdots f_1^{(m)}(\underline{x}_k) g_1^{(m)}(\underline{u}_k) \right\}$$

$$\left. 2^{nk} \rho_n \left(\psi_{\alpha_1, 2^n \underline{x}_1} \bar{\psi}_{\beta_1, 2^n \underline{u}_1} \cdots \psi_{\alpha_k, 2^n \underline{x}_k} \bar{\psi}_{\beta_k, 2^n \underline{u}_k} \right) \right|$$

$$\leq \sum_{i=1}^{k} \int d_n \underline{x}_1 \cdots \int d_n \underline{x}_k \int d_n \underline{u}_1 \cdots \int d_n \underline{u}_k$$

$$\left\{ |f_i(\underline{x}_i) - f_i^{(m)}(\underline{x}_i)| \left| \prod_{j<i} f_j(\underline{x}_j) \prod_{j>i} f_j^{(m)}(\underline{x}_j) \prod_{j=1}^{k} g_j(\underline{u}_j) \right| \right.$$

$$\left. |g_i(\underline{u}_j) - g_i^{(m)}(\underline{u}_i)| \left| \prod_{j=1}^{k} f_j^{(m)}(\underline{x}_j) \prod_{j<i} g_j(\underline{u}_j) \prod_{j>i} g_j^{(m)}(\underline{u}_j) \right| \right\}$$

$$2^{kn} \left| \rho_n \left(\rho_n \left(\psi_{\alpha_1, 2^n \underline{x}_1} \bar{\psi}_{\beta_1, 2^n \underline{u}_1} \cdots \psi_{\alpha_k, 2^n \underline{x}_k} \bar{\psi}_{\beta_k, 2^n \underline{u}_k} \right) \right) \right|$$

$$\leq 2k M^{2k-1} \epsilon \int d_n \underline{x}_1 \cdots \int d_n \underline{x}_k \int d_n \underline{u}_1 \cdots \int d_n \underline{u}_k \cdot$$

$$\prod_{i=1}^{k} (1 + |\underline{x}_i|^2)^{-2} (1 + |\underline{u}_i|^2)^{-2} \, 2^{nk} \left| \rho_n \left(\psi_{\alpha_1, 2^n \underline{x}_1} \bar{\psi}_{\beta_1, 2^n \underline{u}_1} \cdots \psi_{\alpha_k, 2^n \underline{x}_k} \bar{\psi}_{\beta_k, 2^n \underline{u}_k} \right) \right|,$$

where by the Corollary of Theorem 6,

$$\left| \rho_n \left(\psi_{\alpha_1, 2^n \underline{x}_1} \bar{\psi}_{\beta_1, 2^n \underline{u}_1} \cdots \psi_{\alpha_k, 2^n \underline{x}_k} \bar{\psi}_{\beta_k, 2^n \underline{u}_k} \right) \right| \leq D_k(r, g) \, 2^{-s(x,y)}$$

with $x_i = 2^n \underline{x}_i$ and $y_i = 2^n \underline{u}_i$. We now use the following simple estimate:

Lemma 6.1

For $x, y \in \mathbf{Z}^2$ the following bound holds:

(6-10)
$$2^{-s(x,y)} \leq \frac{3}{1 + |x - y|}.$$

It follows from this lemma that

(6-11)
$$2^{-s(x,y)} \leq \max_{\sigma \in \mathcal{S}_k} \prod_{i=1}^{k} \frac{3}{1 + 2^n |\underline{x} - \underline{u}|},$$

35

and hence the right-hand expression of (6.9) is bounded by

$$2kM^{2k-1}k!D_k(r,g)\epsilon \left\{ \int d_n\underline{x} \int d_n\underline{u} \, (1+|\underline{x}|^2)^{-2}(1+|\underline{u}|^2)^{-2}\frac{3\cdot 2^n}{1+2^n|\underline{x}-\underline{u}|} \right\}^k.$$

The double "integral" is bounded by a constant independent of n. This concludes the proof of the existence of the continuum limit. ∎

Notice that in the case that $(\tilde{r}_m, \tilde{g}_m) = R^{-m}(r_0, g_0) \to \left(-\frac{4}{3}, \frac{8}{27}\right)$ we have $\rho_n\left(\psi_{\alpha x}\bar{\psi}_{\beta y}\right) \approx \delta_{\alpha\beta}\frac{\lambda_+^n}{1+|x-y|}$, with $\lambda_+ = \frac{35}{8} + \frac{3}{8}\sqrt{137}$, if $s(x,y) \ll 2^n$. This means that the continuum limit does not exist for the same reason that the infinite volume limit does not exist at the fixed point.

7. Proofs of the results in Section 3.

The existence part of Theorem 1 follows from

Lemma 7.1

Let γ be a continuous curve in $\mathcal{S}(g_0)$ with endpoints (r_1, g_1) and (r_2, g_2) at the left- and right-hand boundary respectively, i.e. $r_1 = \max\{r_-(g_1), r_0(g_1)\}$ and $r_2 = r_+(g_2)$. Then $\gamma' = R(\gamma)$ is contained within the region $0 \le g \le g_0' = F_+(g_0)$, where the function F_+ is defined by

$$(7\text{-}1) \qquad\qquad F_+(g) = \frac{g}{(1 + r_+(g)/2)^2}.$$

Furthermore, the endpoints (r_1', g_1') and (r_2', g_1') satisfy $r_1' \le \max\{r_-(g_1'), r_0(g_1')\}$ and $r_2' \ge r_+(g_2')$.

Proof. If (r, g) is a point of γ then $g \le 2(1 + r)^2$, so $\frac{\partial}{\partial r}\left\{\frac{(1+r)^2 - g/2}{(1+r)^2 - g/4}\right\} \ge 0$. Consequently,

$$g' \le g \left\{\frac{(1 + r_+(g))^2 - g/2}{(1 + r_+(g))^2 - g/4}\right\}^2 = F_+(g) \le F_+(g_0) = g_0'.$$

This proves the first statement of the lemma. The second statement follows by a simple calculation. ∎

The existence of the critical line now follows with the interval argument of Sinai and Bleher [3]:

Proposition 7.1

For all $g_0 > 0$ there exists a critical r-value $r_c(g_0)$ such that $R^n(r_c(g_0), g_0) \to (0, 0)$ as $n \to \infty$.

Proof. Let I_0 be the interval of r-values: $I_0 = [\max\{r_-(g_0), r_0(g_0)\}, r_+(g_0)]$, and let γ_0 be the curve of points (r, g_0) where r runs through I_0. Let γ_1 be a maximal connected part of $\gamma_0' \cap \mathcal{S}(g_0')$ and let $I_1 \subset I_0$ be such that the corresponding part of γ_0 is mapped onto γ_1. Proceeding in this way we find $I_n \subset I_{n-1}$ such that R^n

37

maps the corresponding part of γ_0 onto the connected part γ_n of $\gamma'_{n-1} \cap \mathcal{S}(g_n)$ with $g_n = F_+(g_{n-1})$. We take $r_c(g_0) \in \bigcap_{n \geq 0} I_n$. Then, by construction, $R^n(r_c(g_0), g_0) \in \mathcal{S}(g_n)$. The fact that $F^n_+(g_0) \to 0$ implies that $R^n(r_c(g_0), g_0) \to (0, 0)$. ∎

The proof of the uniqueness is rather delicate. Again we make use of a horizontal curve γ and prove by induction that it is stretched in the $r-$direction. The induction works only after the first iteration with which we have to deal separately. Let us first define the variable

$$(7\text{-}2) \qquad\qquad z = \frac{g}{4(1+r)^2}.$$

Then

$$(7\text{-}3) \qquad\qquad 1 + r = \frac{1-z}{1-2z}\left(1 + \frac{1}{2}r'\right)$$

and

$$(7\text{-}4) \qquad\qquad z' = z\left(\frac{1 + r'/2}{1 + r'}\right)^2.$$

The following result is easy:

Lemma 7.2

If $(r, g) \in \mathcal{S}(g_0)$ for some $g_0 > 0$, and $(r', g') \in \mathcal{S}(g_0)$ then $g' \leq 2\left(1 + \frac{1}{2}r'\right)^2$ and $r' \leq \frac{2}{\sqrt{3}}$.

Lemma 7.3

Assume $(r, g) \in \mathcal{S}(g_0)$ and $(r', g') \in \mathcal{S}(g_0)$. Then $0 \leq \frac{\partial g'}{\partial r} < 6\, z'(1 + r')\frac{\partial r'}{\partial r}$.

Proof. We have $\frac{\partial r'}{\partial r} = 2\left\{1 + z\frac{1+z}{(1-z)^2}\right\}$ and $\frac{\partial g'}{\partial r} = \frac{16z^2(1-2z)}{(1-z)^3}(1+r)$. With (7.3) and (7.4) we find

$$
\begin{aligned}
\frac{\partial g'}{\partial r} &= \frac{16z^2}{(1-z)^2}\left(1 + \frac{1}{2}r'\right) = \frac{16zz'}{(1-z)^2}\frac{1+r'}{1+r'/2}(1+r') \\
&= \frac{8zz'}{1 - z + 2z^2}\frac{1+r'}{1+r'/2}(1+r')\frac{\partial r'}{\partial r} < 6z'(1+r')\frac{\partial r'}{\partial r}
\end{aligned}
$$

38

since $z \leq \frac{1}{2} \Rightarrow \frac{2z}{1-z+2z^2} \leq 1$ and $\frac{1+r'}{1+r'/2} < \frac{2+\sqrt{3}}{1+\sqrt{3}} < \frac{2}{3}$ according to Lemma 7.2. ■

For the next iterations we may assume that $r < \frac{2}{\sqrt{3}}$ and $g \leq 2(1 + \frac{1}{2}r)^2 < \frac{8}{3} + \frac{4}{3}\sqrt{3}$. But then $g' \leq F_+\left(\frac{8}{3} + \frac{4}{3}\sqrt{3}\right) = 2$. In addition $g \leq 2(1 + \frac{1}{2}r)^2 \Rightarrow z \leq z_0$, where z_0 is given by $4z_0(1+r)^2 = 4r(1+r) = 2(1 + \frac{1}{2}r)^2$, i.e. $z_0 = \frac{4\sqrt{2}-2}{4\sqrt{2}+5} < \frac{2}{5}$.

Lemma 7.4

Let γ be a curve in $\mathcal{S}(g_0)$ lying entirely below the line $g = 2(1 + \frac{1}{2}r)^2$. Assume that its tangent satisfies

(7-5) $$0 \leq \frac{dg}{dr} < 6z(1+r),$$

and assume also that the endpoints of γ' lie within $\mathcal{S}(g_0)$. Then $0 \leq \frac{dg'}{dr'} < 6z'(1+r')$. In particular γ' lies entirely within $\mathcal{S}(g_0)$.

Proof. One easily shows that the denominator in the expression

$$\frac{dg'}{dr'} = \frac{\dfrac{\partial g'}{\partial r} + \dfrac{\partial g'}{\partial g}\cdot\dfrac{dg}{dr}}{\dfrac{\partial r'}{\partial r} + \dfrac{\partial r'}{\partial g}\cdot\dfrac{dg}{dr}}$$

is positive. Indeed, putting $\frac{dg}{dr} = 4\alpha z(1+r)$ we have

(7-6) $$\frac{\partial r'}{\partial r} + \frac{\partial r'}{\partial g}\frac{dg}{dr} = 2\frac{1-(1+\alpha)z+2z^2}{(1-z)^2} > 0.$$

Furthermore, using (7.3) and (7.4),

(7-7)
$$\frac{\partial g'}{\partial r} + \frac{\partial g'}{\partial g}\frac{dg}{dr} = \frac{16z^2(1-2z)}{(1-z)^3}(1+r) + \frac{(1-2z)(1-5z+2z^2)}{(1-z)^3}\frac{dg}{dr}$$
$$= \frac{4[\alpha-(5\alpha-4)z+2\alpha z^2]}{(1-z)^2}\frac{1+r'}{1+r'/2}z'(1+r').$$

It remains to show that $\dfrac{\alpha-(5\alpha-4)z+2\alpha z^2}{1-(1-\alpha)z+2z^2}\dfrac{1+r'}{1+r'/2} < 3$. This is simple if we use the fact that $g' < 2$ and $g' \geq 2r'(1+r') \Rightarrow \dfrac{1+r'}{1+r'/2} < \dfrac{5}{4}$ and $z \leq \dfrac{2}{5}$. ■

Proposition 7.2

For all $g_0 > 0$ there exists a unique critical value $r_c(g_0) > 0$ such that $R^n(r_c(g_0), g_0) \in \mathcal{S}(g_0)$ for all $n \geq 0$ and $R^n(r_c(g_0), g_0) \to (0,0)$ as $n \to \infty$.

Proof. Suppose that for a certain value of $g_0 > 0$ there exist two values $r_1(g_0)$ and $r_2(g_0)$ with the required properties. Consider the horizontal line γ_0 connecting the points $(r_1(g_0), g_0)$ and $(r_2(g_0), g_0)$, and its iterates γ_n. According to Lemma 7.3, $\gamma_1 \subset \mathcal{S}(g_0)$ and its tangent is less than $6z(1 + r)$. Furthermore, Lemma 7.2 shows that $(r, g) \in \gamma_1 \Rightarrow g < 2(1 + \frac{1}{2}r)^2$. Thus we can use Lemma 7.4 to iterate R and find that (7.5) holds for all γ_n. For large n, z becomes small and from (7.6) we have $\Delta r' > \frac{3}{2}\Delta r$, where Δr is the distance in the r−direction between the endpoints. Clearly, this contradicts the hypothesis that γ_n lies in $\mathcal{S}(g_0)$ for all n. ∎

The proof of Theorem 2 is similar to the above proof of Theorem 1. We simply state the necessary lemmas.

Lemma 7.5

Let γ be a continuous curve in $\mathcal{T}_1$ with endpoints at the upper- and lower boundary respectively, i.e. assume that the points (r_1, g_1) and (r_2, g_2) satisfy $g_1 = g_-(r_1)$ and $g_2 = g_+(r_2)$. Then γ' is contained in the region $2(1 + \frac{1}{2}r')^2 \leq g' \leq \frac{8}{3}(1 + \frac{1}{2}r')^2$ with $r' > -2$, and $g_1' \leq g_+(r_1')$ and $g_2' \geq g_-(r_2')$. If, in addition, γ lies below the line $g = \left(\frac{8}{3} + \epsilon\right)\left(1 + \frac{1}{2}r\right)^2$ then γ' lies above the line $g' = \left(\frac{8}{3} - \frac{1}{4}\epsilon\right)\left(1 + \frac{1}{2}r'\right)^2$ provided that $\epsilon \leq \frac{2}{3}$.

Lemma 7.6

Let $\tilde{\gamma}$ be a continuous curve in $\mathcal{T}_2$ with endpoints (r_1, g_1) and (r_2, g_2) satisfying $g_1 = g_+(r_1)$ and $g_2 = g_-(r_2)$. Then $\tilde{\gamma}'$ is contained in the region $\frac{8}{3}(1 + \frac{1}{2}r')^2 \leq g' \leq \frac{10}{3}(1 + \frac{1}{2}r')^2$ with $r' > -2$, and $g_1' \leq g_-(r_1')$, $g_2' \geq g_+(r_2')$.
If $\tilde{\gamma}$ lies above the line $g = \left(\frac{8}{3} - \epsilon\right)\left(1 + \frac{1}{2}r\right)^2$ with $\epsilon \leq \frac{2}{3}$ then $\tilde{\gamma}'$ lies below $g' = \left(\frac{8}{3} + \frac{2}{3}\epsilon\right)\left(1 + \frac{1}{2}r'\right)^2$.

Only the proof of the final parts of these lemmas is somewhat tricky. In Lemma

40

7.5, for instance, we use the fact that

$$g' > \left(\frac{8}{3} - \frac{1}{4}\epsilon\right)\left(1 + \frac{1}{2}r'\right)^2 \iff g > \left(\frac{8}{3} - \frac{1}{4}\epsilon\right)(1 + r)^2.$$

The right-hand inequality then follows with a little calculation from
$g \geq 2(1 + r)^2 + \frac{2}{27}$ and $r \geq r_-$, where $2(1 + r_-)^2 + \frac{2}{27} = \left(\frac{8}{3} + \epsilon\right)\left(1 + \frac{1}{2}r_-\right)^2$.
The uniqueness part of Theorem 2 is straightforward. One immediately shows that
$\Delta g' > 2\Delta g$ in every iteration provided that $\Delta r \geq 0$.

Proof of Theorem 3.

The proof consists of a succesive reduction to smaller regions.

<u>Case 1.</u> $g > 0, r > r_c(g)$.

The proof of Proposition 7.2 shows that these points eventually end up in the region
$g < 2r(1 + r)$. We can then use the simple

Lemma 7.7

If $g = 2\alpha r(1 + r)$ with $0 < \alpha < 1$ and $r > 0$, then $g' = 2\alpha' r'(1 + r')$ with $\alpha' < \frac{\alpha}{2-\alpha}$.

to reduce the problem to the case $g < r(1 + r)$. But then $r' > \frac{4}{3}r$ so that
$r^{(n)} \to \infty$ in all cases. Since $g' < g$, $g^{(n)}$ stays bounded. In fact $r^{(n)} \to \infty$ so fast
that $g^{(n)}$ does not reach 0 but tends to some finite value $g_\infty(r, g) > 0$.

<u>Case 2.</u> $r < -1$, $0 < g < 2(1 + r)^2$.

We use

Lemma 7.8

If $r < -1$ and $g = 4z(1 + r)^2$ with $\frac{1}{4} < z < \frac{1}{2}$ then $r' < -2$ and $g' < (1 + r')^2$.

But when $z < \frac{1}{4}$ then $r' < \frac{4}{3}r - \frac{2}{3}$ so that $r^{(n)} \to -\infty$. Since $g^{(n)}$ is decreasing
and positive it converges also.

<u>Case 3.</u> $r \leq r_0$, $g < g_c(r)$ or $r < -1$, $g < g_+(r)$.

We can reduce this case to the former by the following argument. First of all we

41

may assume that $g \leq 2(1+r)^2 + \frac{2}{27}$ and $g \leq \frac{8}{3}(1+r)^2$ using the Lemmas 7.5 and 7.6. We may also assume that $g \leq \frac{8}{3}(1+\frac{1}{2}r)^2$ and $r > -2$ since all points with $2(1+r)^2 < g < g_c(r)$ end up in this region anyhow. Thus we have reduced the problem to the region $\frac{1}{2} < z < \frac{2}{3}\left(\frac{1+r/2}{1+r}\right)^2$; $r < -\frac{4}{3}$. This can be dealt with by putting $\delta = -\frac{4}{3} - r$. It follows that $\delta' > 4\delta$.

Case 4. $g < 0$, $r > 0$.

One easily shows that $g = -cr$ with $c > 0$ implies that $g' = -c'r'$ with $c' < c$ and $r' > 2r$. Also $g' \geq \left(1 + \frac{c}{4r}\right)^2 g$ so that $g^{(n)}$ is bounded.

Case 5. $-\frac{2}{3} < r < 0$, $g \leq \frac{4r(1+r)^2}{2+3r}$.

These points end up in the fourth quadrant after a finite number of iterations.

Case 6. $r \leq -1$, $g < 0$.

$g' = \left(1 + \frac{|z|}{1+|z|}\right)^2 g > (1+|z|)^2 g$ and $|1 + r'| > 2\left(1 + \frac{|z|}{1+|z|}\right)|1+r|$, so $|z'| < \frac{1}{4}|z|$. Hence $g^{(n+1)} > (1 + 4^{-n}|z|)^2 g^{(n)}$ from which it follows that $g^{(n)}$ is bounded below and therefore convergent.

Case 7. $-1 < r \leq -\frac{2}{3}$, $g < 0$.

This case can be reduced to the former by remarking that, if $r = -\frac{2}{3}$ then $r' < -\frac{2}{3}$ and if $-1 < r < -\frac{2}{3}$ then $\frac{\partial r'}{\partial r} > \frac{7}{4}$.

Case 8. $-\frac{2}{3} < r < 0$, $0 > g > \frac{4r(1+r)^2}{2+3r}$.

The existence and uniqueness of the line $r_s(g)$ can be proven in the same way as the existence and uniqueness of $r_c(g)$ and $g_c(r)$ was established in the proofs of Theorems 1 and 2.

8. Final Remarks.

Although the model that we have studied in this paper is rather artificial it has the advantage that it can be renormalized easily. The renormalization transformation involves only two coupling parameters r and g and is given by the exact transformation formulas (2.16). Thus many technicalities that appear in other models when studying the asymptotic renormalization group behaviour do not occur here. Other simple renormalization group transformations were considered by Nelson and Fisher [10]. Their aim was different, however, and they did not consider the decay of correlations or the existence of a continuum limit.

Also unlike most other models it is possible to make non-perturbative, i.e. *global* statements about the renormalization group flow. In particular we have seen that there exists, apart from the trivial fixed point (0,0), a non-trivial fixed point $\left(-\frac{4}{3}, \frac{8}{27}\right)$, and we have managed to prove the existence and uniqueness of global critical lines for both fixed points. This is not to say that everything about the flow in parameter space is fully understood. Computer studies show that the behaviour in the region above and between the two critical lines in rather erratic. Iterating the inverse mapping for a small part of the critical line near (0,0) one obtains an array of points that seem to be concentrated on a bundle of curves in this region. (For a picture, see [7].) If this behaviour is genuine all these curves are critical lines for the fixed point (0,0) ! Also, some of the points thus obtained lie in the neighbourhood of (0,0) itself.

Some of the unusual features of the renormalization group flow may be due to the hierarchical structure of the model. However, it cannot be ruled out that similar complications uccur also in other, more realistic models. For instance, the recurrence of critical points to the neighbourhood of the fixed point is a possibility to be kept in mind. The large null space of Γ_0 means that there is no Hamiltonian formulation for this model. Hence the concept of a phase transition is unclear but, judging by the behaviour of the correlation functions, the line $r_s(g)$ in the lower half-plane behaves as a critical line: The decay of correlation functions for points (r, g) on this line is slower than for other points of the lower half-plane.

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